
Contents

Preface to the Second Edition	vii
Preface to the First Edition	ix
A Brief Introduction	1

Chapter 1 Euclidean Spaces

§1 Smooth Functions on a Euclidean Space	3
1.1 C^∞ Versus Analytic Functions	4
1.2 Taylor's Theorem with Remainder	5
Problems	8
§2 Tangent Vectors in \mathbb{R}^n as Derivations	10
2.1 The Directional Derivative	10
2.2 Germs of Functions	11
2.3 Derivations at a Point	13
2.4 Vector Fields	14
2.5 Vector Fields as Derivations	16
Problems	17
§3 The Exterior Algebra of Multivectors	18
3.1 Dual Space	19
3.2 Permutations	20
3.3 Multilinear Functions	22
3.4 The Permutation Action on Multilinear Functions	23
3.5 The Symmetrizing and Alternating Operators	24
3.6 The Tensor Product	25
3.7 The Wedge Product	26
3.8 Anticommutativity of the Wedge Product	27
3.9 Associativity of the Wedge Product	28
3.10 A Basis for k -Covectors	31

Problems	32
§4 Differential Forms on \mathbb{R}^n	34
4.1 Differential 1-Forms and the Differential of a Function	34
4.2 Differential k -Forms	36
4.3 Differential Forms as Multilinear Functions on Vector Fields	37
4.4 The Exterior Derivative	38
4.5 Closed Forms and Exact Forms	40
4.6 Applications to Vector Calculus	41
4.7 Convention on Subscripts and Superscripts	44
Problems	44

Chapter 2 Manifolds

§5 Manifolds	48
5.1 Topological Manifolds	48
5.2 Compatible Charts	49
5.3 Smooth Manifolds	52
5.4 Examples of Smooth Manifolds	53
Problems	57
§6 Smooth Maps on a Manifold	59
6.1 Smooth Functions on a Manifold	59
6.2 Smooth Maps Between Manifolds	61
6.3 Diffeomorphisms	63
6.4 Smoothness in Terms of Components	63
6.5 Examples of Smooth Maps	65
6.6 Partial Derivatives	67
6.7 The Inverse Function Theorem	68
Problems	70
§7 Quotients	71
7.1 The Quotient Topology	71
7.2 Continuity of a Map on a Quotient	72
7.3 Identification of a Subset to a Point	73
7.4 A Necessary Condition for a Hausdorff Quotient	73
7.5 Open Equivalence Relations	74
7.6 Real Projective Space	76
7.7 The Standard C^∞ Atlas on a Real Projective Space	79
Problems	81

Chapter 3 The Tangent Space

§8 The Tangent Space	86
-----------------------------------	----

8.1	The Tangent Space at a Point	86
8.2	The Differential of a Map	87
8.3	The Chain Rule	88
8.4	Bases for the Tangent Space at a Point	89
8.5	A Local Expression for the Differential	91
8.6	Curves in a Manifold	92
8.7	Computing the Differential Using Curves	95
8.8	Immersions and Submersions	96
8.9	Rank, and Critical and Regular Points	96
	Problems	98
§9	Submanifolds	100
9.1	Submanifolds	100
9.2	Level Sets of a Function	103
9.3	The Regular Level Set Theorem	105
9.4	Examples of Regular Submanifolds	106
	Problems	108
§10	Categories and Functors	110
10.1	Categories	110
10.2	Functors	111
10.3	The Dual Functor and the Multicovector Functor	113
	Problems	114
§11	The Rank of a Smooth Map	115
11.1	Constant Rank Theorem	115
11.2	The Immersion and Submersion Theorems	118
11.3	Images of Smooth Maps	120
11.4	Smooth Maps into a Submanifold	124
11.5	The Tangent Plane to a Surface in \mathbb{R}^3	125
	Problems	127
§12	The Tangent Bundle	129
12.1	The Topology of the Tangent Bundle	129
12.2	The Manifold Structure on the Tangent Bundle	132
12.3	Vector Bundles	133
12.4	Smooth Sections	136
12.5	Smooth Frames	137
	Problems	139
§13	Bump Functions and Partitions of Unity	140
13.1	C^∞ Bump Functions	140
13.2	Partitions of Unity	145
13.3	Existence of a Partition of Unity	146
	Problems	147
§14	Vector Fields	149
14.1	Smoothness of a Vector Field	149

14.2	Integral Curves	152
14.3	Local Flows	154
14.4	The Lie Bracket	157
14.5	The Pushforward of Vector Fields	159
14.6	Related Vector Fields	159
	Problems	161

Chapter 4 Lie Groups and Lie Algebras

§15	Lie Groups	164
15.1	Examples of Lie Groups	164
15.2	Lie Subgroups	167
15.3	The Matrix Exponential	169
15.4	The Trace of a Matrix	171
15.5	The Differential of \det at the Identity	174
	Problems	174
§16	Lie Algebras	178
16.1	Tangent Space at the Identity of a Lie Group	178
16.2	Left-Invariant Vector Fields on a Lie Group	180
16.3	The Lie Algebra of a Lie Group	182
16.4	The Lie Bracket on $\mathfrak{gl}(n, \mathbb{R})$	183
16.5	The Pushforward of Left-Invariant Vector Fields	184
16.6	The Differential as a Lie Algebra Homomorphism	185
	Problems	187

Chapter 5 Differential Forms

§17	Differential 1-Forms	190
17.1	The Differential of a Function	191
17.2	Local Expression for a Differential 1-Form	191
17.3	The Cotangent Bundle	192
17.4	Characterization of C^∞ 1-Forms	193
17.5	Pullback of 1-Forms	195
17.6	Restriction of 1-Forms to an Immersed Submanifold	197
	Problems	199
§18	Differential k-Forms	200
18.1	Differential Forms	200
18.2	Local Expression for a k -Form	202
18.3	The Bundle Point of View	203
18.4	Smooth k -Forms	203
18.5	Pullback of k -Forms	204
18.6	The Wedge Product	205
18.7	Differential Forms on a Circle	206

18.8	Invariant Forms on a Lie Group	207
	Problems	208
§19	The Exterior Derivative	210
19.1	Exterior Derivative on a Coordinate Chart	211
19.2	Local Operators	211
19.3	Existence of an Exterior Derivative on a Manifold	212
19.4	Uniqueness of the Exterior Derivative	213
19.5	Exterior Differentiation Under a Pullback	214
19.6	Restriction of k -Forms to a Submanifold	216
19.7	A Nowhere-Vanishing 1-Form on the Circle	216
	Problems	218
§20	The Lie Derivative and Interior Multiplication	221
20.1	Families of Vector Fields and Differential Forms	221
20.2	The Lie Derivative of a Vector Field	223
20.3	The Lie Derivative of a Differential Form	226
20.4	Interior Multiplication	227
20.5	Properties of the Lie Derivative	229
20.6	Global Formulas for the Lie and Exterior Derivatives	232
	Problems	233

Chapter 6 Integration

§21	Orientations	236
21.1	Orientations of a Vector Space	236
21.2	Orientations and n -Covectors	238
21.3	Orientations on a Manifold	240
21.4	Orientations and Differential Forms	242
21.5	Orientations and Atlases	245
	Problems	246
§22	Manifolds with Boundary	248
22.1	Smooth Invariance of Domain in \mathbb{R}^n	248
22.2	Manifolds with Boundary	250
22.3	The Boundary of a Manifold with Boundary	253
22.4	Tangent Vectors, Differential Forms, and Orientations	253
22.5	Outward-Pointing Vector Fields	254
22.6	Boundary Orientation	255
	Problems	256
§23	Integration on Manifolds	260
23.1	The Riemann Integral of a Function on \mathbb{R}^n	260
23.2	Integrability Conditions	262
23.3	The Integral of an n -Form on \mathbb{R}^n	263
23.4	Integral of a Differential Form over a Manifold	265

23.5	Stokes's Theorem	269
23.6	Line Integrals and Green's Theorem	271
	Problems	272

Chapter 7 De Rham Theory

§24	De Rham Cohomology	274
24.1	De Rham Cohomology	274
24.2	Examples of de Rham Cohomology	276
24.3	Diffeomorphism Invariance	278
24.4	The Ring Structure on de Rham Cohomology	279
	Problems	280
§25	The Long Exact Sequence in Cohomology	281
25.1	Exact Sequences	281
25.2	Cohomology of Cochain Complexes	283
25.3	The Connecting Homomorphism	284
25.4	The Zig-Zag Lemma	285
	Problems	287
§26	The Mayer–Vietoris Sequence	288
26.1	The Mayer–Vietoris Sequence	288
26.2	The Cohomology of the Circle	292
26.3	The Euler Characteristic	295
	Problems	295
§27	Homotopy Invariance	296
27.1	Smooth Homotopy	296
27.2	Homotopy Type	297
27.3	Deformation Retractions	299
27.4	The Homotopy Axiom for de Rham Cohomology	300
	Problems	301
§28	Computation of de Rham Cohomology	302
28.1	Cohomology Vector Space of a Torus	302
28.2	The Cohomology Ring of a Torus	303
28.3	The Cohomology of a Surface of Genus g	306
	Problems	310
§29	Proof of Homotopy Invariance	311
29.1	Reduction to Two Sections	311
29.2	Cochain Homotopies	312
29.3	Differential Forms on $M \times \mathbb{R}$	312
29.4	A Cochain Homotopy Between i_0^* and i_1^*	314
29.5	Verification of Cochain Homotopy	315
	Problems	316

Appendices

§A	Point-Set Topology	317
	A.1 Topological Spaces	317
	A.2 Subspace Topology	320
	A.3 Bases	321
	A.4 First and Second Countability	323
	A.5 Separation Axioms	324
	A.6 Product Topology	326
	A.7 Continuity	327
	A.8 Compactness	329
	A.9 Boundedness in \mathbb{R}^n	332
	A.10 Connectedness	332
	A.11 Connected Components	333
	A.12 Closure	334
	A.13 Convergence	336
	Problems	337
§B	The Inverse Function Theorem on \mathbb{R}^n and Related Results	339
	B.1 The Inverse Function Theorem	339
	B.2 The Implicit Function Theorem	339
	B.3 Constant Rank Theorem	343
	Problems	344
§C	Existence of a Partition of Unity in General	346
§D	Linear Algebra	349
	D.1 Quotient Vector Spaces	349
	D.2 Linear Transformations	350
	D.3 Direct Product and Direct Sum	351
	Problems	352
§E	Quaternions and the Symplectic Group	353
	E.1 Representation of Linear Maps by Matrices	354
	E.2 Quaternionic Conjugation	355
	E.3 Quaternionic Inner Product	356
	E.4 Representations of Quaternions by Complex Numbers	356
	E.5 Quaternionic Inner Product in Terms of Complex Components	357
	E.6 \mathbb{H} -Linearity in Terms of Complex Numbers	357
	E.7 Symplectic Group	358
	Problems	359
	Solutions to Selected Exercises Within the Text	361
	Hints and Solutions to Selected End-of-Section Problems	367

List of Notations	387
References	395
Index	397



<http://www.springer.com/978-1-4419-7399-3>

An Introduction to Manifolds

Tu, L.W.

2011, XVIII, 410 p. 124 illus., 1 illus. in color., Softcover

ISBN: 978-1-4419-7399-3