Preface

Partial differential equations is a many-faceted subject. Created to describe the mechanical behavior of objects such as vibrating strings and blowing winds, it has developed into a body of material that interacts with many branches of mathematics, such as differential geometry, complex analysis, and harmonic analysis, as well as a ubiquitous factor in the description and elucidation of problems in mathematical physics.

This work is intended to provide a course of study of some of the major aspects of PDE. It is addressed to readers with a background in the basic introductory graduate mathematics courses in American universities: elementary real and complex analysis, differential geometry, and measure theory.

Chapter 1 provides background material on the theory of ordinary differential equations (ODE). This includes both very basic material–on topics such as the existence and uniqueness of solutions to ODE and explicit solutions to equations with constant coefficients and relations to linear algebra–and more sophisticated results–on flows generated by vector fields, connections with differential geometry, the calculus of differential forms, stationary action principles in mechanics, and their relation to Hamiltonian systems. We discuss equations of relativistic motion as well as equations of classical Newtonian mechanics. There are also applications to topological results, such as degree theory, the Brouwer fixed-point theorem, and the Jordan-Brouwer separation theorem. In this chapter we also treat scalar first-order PDE, via Hamilton–Jacobi theory.

Chapters 2–6 constitute a survey of basic linear PDE. Chapter 2 begins with the derivation of some equations of continuum mechanics in a fashion similar to the derivation of ODE in mechanics in Chap. 1, via variational principles. We obtain equations for vibrating strings and membranes; these equations are not necessarily linear, and hence they will also provide sources of problems later, when nonlinear PDE is taken up. Further material in Chap. 2 centers around the Laplace operator, which on Euclidean space $\mathbb{R}^n$ is

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \tag{1}$$

and the linear wave equation,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0. \tag{2}$$

We also consider the Laplace operator on a general Riemannian manifold and the wave equation on a general Lorentz manifold. We discuss basic consequences of Green's formula, including energy conservation and finite propagation speed for solutions to linear wave equations. We also discuss Maxwell's equations for electromagnetic fields and their relation with special relativity. Before we can establish general results on the solvability of these equations, it is necessary to develop some analytical techniques. This is done in the next couple of chapters.

Chapter 3 is devoted to Fourier analysis and the theory of distributions. These topics are crucial for the study of linear PDE. We give a number of basic applications to the study of linear PDE with constant coefficients. Among these applications are results on harmonic and holomorphic functions in the plane, including a short treatment of elementary complex function theory. We derive explicit formulas for solutions to Laplace and wave equations on Euclidean space, and also the heat equation,

$$\frac{\partial u}{\partial t} - \Delta u = 0. \tag{3}$$

We also produce solutions on certain subsets, such as rectangular regions, using the method of images. We include material on the discrete Fourier transform, germane to the discrete approximation of PDE, and on the fast evaluation of this transform, the FFT. Chapter 3 is the first chapter to make extensive use of functional analysis. Basic results on this topic are compiled in Appendix A, Outline of Functional Analysis.

Sobolev spaces have proven to be a very effective tool in the existence theory of PDE, and in the study of regularity of solutions. In Chap. 4 we introduce Sobolev spaces and study some of their basic properties. We restrict attention to L^2-Sobolev spaces, such as $H^k(\mathbb{R}^n)$, which consists of L^2 functions whose derivatives of order $\leq k$ (defined in a distributional sense, in Chap. 3) belong to $L^2(\mathbb{R}^n)$, when k is a positive integer. We also replace k by a general real number s. The L^p-Sobolev spaces, which are very useful for nonlinear PDE, are treated later, in Chap. 13.

Chapter 5 is devoted to the study of the existence and regularity of solutions to linear elliptic PDE, on bounded regions. We begin with the Dirichlet problem for the Laplace operator,

$$\Delta u = f \text{ on } \Omega, \qquad u = g \text{ on } \partial\Omega, \tag{4}$$

and then treat the Neumann problem and various other boundary problems, including some that apply to electromagnetic fields. We also study general boundary problems for linear elliptic operators, giving a condition that guarantees regularity and solvability (perhaps given a finite number of linear conditions on the data). Also in Chap. 5 are some applications to other areas, such as a proof of the Riemann mapping theorem, first for smooth simply connected domains in the complex plane $\mathbb{C}$, then, after a treatment of the Dirichlet problem for the Laplace operator on domains with rough boundary, for general simply connected domains in $\mathbb{C}$. We also develop Hodge theory and apply it to DeRham cohomology, extending the study of topological applications of differential forms begun in Chap. 1.

In Chap. 6 we study linear evolution equations, in which there is a "time" variable t, and initial data are given at $t = 0$. We discuss the heat and wave equations. We also treat Maxwell's equations, for an electromagnetic field, and more general hyperbolic systems. We prove the Cauchy–Kowalewsky theorem, in the linear case, establishing local solvability of the Cauchy initial value problem for general linear PDE with analytic coefficients, and analytic data, as long as the initial surface is "noncharacteristic." The nonlinear case is treated in Chap. 16. Also in Chap. 6 we treat geometrical optics, providing approximations to solutions of wave equations whose initial data either are highly oscillatory or possess simple singularities, such as a jump across a smooth hypersurface.

Chapters 1–6, together with Appendix A and Appendix B, Manifolds, Vector Bundles, and Lie Groups, make up the first volume of this work. The second volume consists of Chaps. 7–12, covering a selection of more advanced topics in linear PDE, together with Appendix C, Connections and Curvature.

Chapter 7 deals with pseudodifferential operators (ψDOs). This class of operators includes both differential operators and parametrices of elliptic operators, that is, inverses modulo smoothing operators. There is a "symbol calculus" allowing one to analyze products of ψDOs, useful for such a parametrix construction. The L^2-boundedness of operators of order zero and the Gårding inequality for elliptic ψDOs with positive symbol provide very useful tools in linear PDE, which will be used in many subsequent chapters.

Chapter 8 is devoted to spectral theory, particularly for self-adjoint elliptic operators. First we give a proof of the spectral theorem for general self-adjoint operators on Hilbert space. Then we discuss conditions under which a differential operator yields a self-adjoint operator. We then discuss the asymptotic distribution of eigenvalues of the Laplace operator on a bounded domain, making use of a construction of a parametrix for the heat equation from Chap. 7. In the next four sections of Chap. 8 we consider the spectral behavior of various specific differential operators: the Laplace operator on a sphere, and on hyperbolic space, the "harmonic oscillator"

$$-\Delta + |x|^2, \tag{5}$$

and the operator

$$-\Delta - \frac{K}{|x|}, \tag{6}$$

which arises in the simplest quantum mechanical model of the hydrogen atom. Finally, we consider the Laplace operator on cones.

In Chap. 9 we study the scattering of waves by a compact obstacle K in $\mathbb{R}^3$. This scattering theory is to some degree an extension of the spectral theory of the

Laplace operator on $\mathbb{R}^3 \setminus K$, with the Dirichlet boundary condition. In addition to studying how a given obstacle scatters waves, we consider the *inverse* problem: how to determine an obstacle given data on how it scatters waves.

Chapter 10 is devoted to the Atiyah–Singer index theorem. This gives a formula for the index of an elliptic operator D on a compact manifold M, defined by

$$\text{Index}\, D = \dim\ker D - \dim\ker D^*. \tag{7}$$

We establish this formula, which is an integral over M of a certain differential form defined by a pair of "curvatures," when D is a first order differential operator of "Dirac type," a class that contains many important operators arising from differential geometry and complex analysis. Special cases of such a formula include the Chern–Gauss–Bonnet formula and the Riemann–Roch formula. We also discuss the significance of the latter formula in the study of Riemann surfaces.

In Chap. 11 we study Brownian motion, described mathematically by Wiener measure on the space of continuous paths in $\mathbb{R}^n$. This provides a probabilistic approach to diffusion and it both uses and provides new tools for the analysis of the heat equation and variants, such as

$$\frac{\partial u}{\partial t} = -\Delta u + Vu, \tag{8}$$

where V is a real-valued function. There is an integral formula for solutions to (8), known as the Feynman–Kac formula; it is an integral over path space with respect to Wiener measure, of a fairly explicit integrand. We also derive an analogous integral formula for solutions to

$$\frac{\partial u}{\partial t} = -\Delta u + Xu, \tag{9}$$

where X is a vector field. In this case, another tool is involved in constructing the integrand, the stochastic integral. We also study stochastic differential equations and applications to more general diffusion equations.

In Chap. 12 we tackle the $\overline{\partial}$-Neumann problem, a boundary problem for an elliptic operator (essentially the Laplace operator) on a domain $\Omega \subset \mathbb{C}^n$, which is very important in the theory of functions of several complex variables. From a technical point of view, it is of particular interest that this boundary problem does not satisfy the regularity criteria investigated in Chap. 5. If Ω is "strongly pseudoconvex," one has instead certain "subelliptic estimates," which are established in Chap. 12.

The third and final volume of this work contains Chaps. 13–18. It is here that we study nonlinear PDE.

We prepare the way in Chap. 13 with a further development of function space and operator theory, for use in nonlinear analysis. This includes the theory of L^p-Sobolev spaces and Hölder spaces. We derive estimates in these spaces on nonlinear functions $F(u)$, known as "Moser estimates," which are very useful.

We extend the theory of pseudodifferential operators to cases where the symbols have limited smoothness, and also develop a variant of ψDO theory, the theory of "paradifferential operators," which has had a significant impact on nonlinear PDE since about 1980. We also estimate these operators, acting on the function spaces mentioned above. Other topics treated in Chap. 13 include Hardy spaces, compensated compactness, and "fuzzy functions."

Chapter 14 is devoted to nonlinear elliptic PDE, with an emphasis on second order equations. There are three successive degrees of nonlinearity: semilinear equations, such as

$$\Delta u = F(x, u, \nabla u), \tag{10}$$

quasi-linear equations, such as

$$\sum a^{jk}(x, u, \nabla u)\partial_j \partial_k u = F(x, u, \nabla u), \tag{11}$$

and completely nonlinear equations, of the form

$$G(x, D^2 u) = 0. \tag{12}$$

Differential geometry provides a rich source of such PDE, and Chap. 14 contains a number of geometrical applications. For example, to deform conformally a metric on a surface so its Gauss curvature changes from $k(x)$ to $K(x)$, one needs to solve the semilinear equation

$$\Delta u = k(x) - K(x)e^{2u}. \tag{13}$$

As another example, the graph of a function $y = u(x)$ is a minimal submanifold of Euclidean space provided u solves the quasilinear equation

$$\big(1 + |\nabla u|^2\big)\Delta u + (\nabla u) \cdot H(u)(\nabla u) = 0, \tag{14}$$

called the minimal surface equation. Here, $H(u) = (\partial_j \partial_k u)$ is the Hessian matrix of u. On the other hand, this graph has Gauss curvature $K(x)$ provided u solves the completely nonlinear equation

$$\det H(u) = K(x)\big(1 + |\nabla u|^2\big)^{(n+2)/2}, \tag{15}$$

a Monge-Ampère equation. Equations (13)–(15) are all scalar, and the maximum principle plays a useful role in the analysis, together with a number of other tools. Chapter 14 also treats nonlinear systems. Important physical examples arise in studies of elastic bodies, as well as in other areas, such as the theory of liquid crystals. Geometric examples of systems considered in Chap. 14 include equations for harmonic maps and equations for isometric imbeddings of a Riemannian manifold in Euclidean space.

In Chap. 15, we treat nonlinear parabolic equations. Partly echoing Chap. 14, we progress from a treatment of semilinear equations,

$$\frac{\partial u}{\partial t} = Lu + F(x, u, \nabla u), \tag{16}$$

where L is a linear operator, such as $L = \Delta$, to a treatment of quasi-linear equations, such as

$$\frac{\partial u}{\partial t} = \sum \partial_j a^{jk}(t, x, u)\partial_k u + X(u). \tag{17}$$

(We do very little with completely nonlinear equations in this chapter.) We study systems as well as scalar equations. The first application of (16) we consider is to the parabolic equation method of constructing harmonic maps. We also consider "reaction-diffusion" equations, $\ell \times \ell$ systems of the form (16), in which $F(x, u, \nabla u) = X(u)$, where X is a vector field on $\mathbb{R}^\ell$, and L is a diagonal operator, with diagonal elements $a_j \Delta,\ a_j \geq 0$. These equations arise in mathematical models in biology and in chemistry. For example, $u = (u_1, \dots, u_\ell)$ might represent the population densities of each of ℓ species of living creatures, distributed over an area of land, interacting in a manner described by X and diffusing in a manner described by $a_j \Delta$. If there is a nonlinear (density-dependent) diffusion, one might have a system of the form (17).

Another problem considered in Chap. 15 models the melting of ice; one has a linear heat equation in a region (filled with water) whose boundary (where the water touches the ice) is moving (as the ice melts). The nonlinearity in the problem involves the description of the boundary. We confine our analysis to a relatively simple one-dimensional case.

Nonlinear hyperbolic equations are studied in Chap. 16. Here continuum mechanics is the major source of examples, and most of them are systems, rather than scalar equations. We establish local existence for solutions to first order hyperbolic systems, which are either "symmetric" or "symmetrizable." An example of the latter class is the following system describing compressible fluid flow:

$$\frac{\partial v}{\partial t} + \nabla_v v + \frac{1}{\rho} \operatorname{grad} p = 0, \quad \frac{\partial \rho}{\partial t} + \nabla_v \rho + \rho \operatorname{div} v = 0, \tag{18}$$

for a fluid with velocity v, density ρ, and pressure p, assumed to satisfy a relation $p = p(\rho)$, called an "equation of state." Solutions to such nonlinear systems tend to break down, due to shock formation. We devote a bit of attention to the study of weak solutions to nonlinear hyperbolic systems, with shocks.

We also study second-order hyperbolic systems, such as systems for a k-dimensional membrane vibrating in $\mathbb{R}^n$, derived in Chap. 2. Another topic covered in Chap. 16 is the Cauchy–Kowalewsky theorem, in the nonlinear case. We use a method introduced by P. Garabedian to transform the Cauchy problem for an analytic equation into a symmetric hyperbolic system.

In Chap. 17 we study incompressible fluid flow. This is governed by the Euler equation

$$\frac{\partial v}{\partial t} + \nabla_v v = -\operatorname{grad} p, \qquad \operatorname{div} v = 0, \tag{19}$$

in the absence of viscosity, and by the Navier–Stokes equation

$$\frac{\partial v}{\partial t} + \nabla_v v = \nu \mathcal{L} v - \operatorname{grad} p, \qquad \operatorname{div} v = 0, \tag{20}$$

in the presence of viscosity. Here $\mathcal{L}$ is a second-order operator, the Laplace operator for a flow on flat space; the "viscosity" ν is a positive quantity. The (19) shares some features with quasilinear hyperbolic systems, though there are also significant differences. Similarly, (20) has a lot in common with semilinear parabolic systems.

Chapter 18, the last chapter in this work, is devoted to Einstein's gravitational equations:

$$G_{jk} = 8\pi\kappa T_{jk}. \tag{21}$$

Here G_{jk} is the Einstein tensor, given by $G_{jk} = \mathrm{Ric}_{jk} - (1/2)Sg_{jk}$, where Ric_{jk} is the Ricci tensor and S the scalar curvature, of a Lorentz manifold (or "spacetime") with metric tensor g_{jk}. On the right side of (21), T_{jk} is the stress-energy tensor of the matter in the spacetime, and κ is a positive constant, which can be identified with the gravitational constant of the Newtonian theory of gravity. In local coordinates, G_{jk} has a nonlinear expression in terms of g_{jk} and its second order derivatives. In the empty-space case, where $T_{jk} = 0$, (21) is a quasilinear second order system for g_{jk}. The freedom to change coordinates provides an obstruction to this equation being hyperbolic, but one can impose the use of "harmonic" coordinates as a constraint and transform (21) into a hyperbolic system. In the presence of matter one couples (21) to other systems, obtaining more elaborate PDE. We treat this in two cases, in the presence of an electromagnetic field, and in the presence of a relativistic fluid.

In addition to the 18 chapters just described, there are three appendices, already mentioned above. Appendix A gives definitions and basic properties of Banach and Hilbert spaces (of which L^p-spaces and Sobolev spaces are examples), Fréchet spaces (such as $C^\infty(\mathbb{R}^n)$), and other locally convex spaces (such as spaces of distributions). It discusses some basic facts about bounded linear operators, including some special properties of compact operators, and also considers certain classes of unbounded linear operators. This functional analytic material plays a major role in the development of PDE from Chap. 3 onward.

Appendix B gives definitions and basic properties of manifolds and vector bundles. It also discusses some elementary properties of Lie groups, including a little representation theory, useful in Chap. 8, on spectral theory, as well as in the Chern–Weil construction.

Appendix C, Connections and Curvature, contains material of a differential geometric nature, crucial for understanding many things done in Chaps. 10–18. We consider connections on general vector bundles, and their curvature. We discuss in detail special properties of the primary case: the Levi–Civita connection and Riemann curvature tensor on a Riemannian manifold. We discuss basic properties of the geometry of submanifolds, relating the second fundamental form to curvature via the Gauss–Codazzi equations. We describe how vector bundles arise from principal bundles, which themselves carry various connections and curvature forms. We then discuss the Chern–Weil construction, yielding certain closed differential forms associated to curvatures of connections on principal bundles. We give several proofs of the classical Gauss–Bonnet theorem and some related results on two-dimensional surfaces, which are useful particularly in Chaps. 10 and 14. We also give a geometrical proof of the Chern–Gauss–Bonnet theorem, which can be contrasted with the proof in Chap. 10, as a consequence of the Atiyah–Singer index theorem.

We mention that, in addition to these "global" appendices, there are appendices to some chapters. For example, Chap. 3 has an appendix on the gamma function. Chapter 6 has two appendices; Appendix A has some results on Banach spaces of harmonic functions useful for the proof of the linear Cauchy–Kowalewsky theorem, and Appendix B deals with the stationary phase formula, useful for the study of geometrical optics in Chap. 6 and also for results later, in Chap. 9. There are other chapters with such "local" appendices. Furthermore, there are two *sections*, both in Chap. 14, with appendices. Section 6, on minimal surfaces, has a companion, §6B, on the second variation of area and consequences, and §12, on nonlinear elliptic systems, has a companion, §12B, with complementary material.

Having described the scope of this work, we find it necessary to mention a number of topics in PDE that are not covered here, or are touched on only very briefly.

For example, we devote little attention to the real analytic theory of PDE. We note that harmonic functions on domains in $\mathbb{R}^n$ are real analytic, but we do not discuss analyticity of solutions to more general elliptic equations. We do prove the Cauchy–Kowalewsky theorem, on analytic PDE with analytic Cauchy data. We derive some simple results on unique continuation from these few analyticity results, but there is a large body of lore on unique continuation, for solutions to nonanalytic PDE, neglected here.

There is little material on numerical methods. There are a few references to applications of the FFT and of "splitting methods." Difference schemes for PDE are mentioned just once, in a set of exercises on scalar conservation laws. Finite element methods are neglected, as are many other numerical techiques.

There is a large body of work on free boundary problems, but the only one considered here is a simple one space dimensional problem, in Chap. 15.

While we have considered a variety of equations arising from classical physics and from relativity, we have devoted relatively little attention to quantum mechanics. We have considered one quantum mechanical operator, given in formula (6) above. Also, there are some exercises on potential scattering mentioned in Chap. 9. However, the physical theories behind these equations are not discussed here.

There are a number of nonlinear evolution equations, such as the Korteweg–deVries equation, that have been perceived to provide infinite dimensional analogues of completely integrable Hamiltonian systems, and to arise "universally" in asymptotic analyses of solutions to various nonlinear wave equations. They are not here. Nor is there a treatment of the Yang–Mills equations for gauge fields, with their wonderful applications to the geometry and topology of four dimensional manifolds.

Of course, this is not a complete list of omitted material. One can go on and on listing important topics in this vast subject. The author can at best hope that the reader will find it easier to understand many of these topics with this book, than without it.

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Introduction to the Second Edition

In addition to making numerous small corrections to this work, collected over the past dozen years, I have taken the opportunity to make some very significant changes, some of which broaden the scope of the work, some of which clarify previous presentations, and a few of which correct errors that have come to my attention.

There are seven additional sections in this edition, two in Volume 1, two in Volume 2, and three in Volume 3. Chapter 4 has a new section, "Sobolev spaces on rough domains," which serves to clarify the treatment of the Dirichlet problem on rough domains in Chap. 5. Chapter 6 has a new section, "Boundary layer phenomena for the heat equation," which will prove useful in one of the new sections in Chap. 17. Chapter 7 has a new section, "Operators of harmonic oscillator type," and Chap. 10 has a section that presents an index formula for elliptic systems of operators of harmonic oscillator type. Chapter 13 has a new appendix, "Variations on complex interpolation," which has material that is useful in the study of Zygmund spaces. Finally, Chap. 17 has two new sections, "Vanishing viscosity limits" and "From velocity convergence to flow convergance."

In addition, several other sections have been substantially rewritten, and numerous others polished to reflect insights gained through the use of these books over time.

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