Chapter 2
Basic properties of the solutions

In this chapter, the basic concepts of second order linear differential equations are introduced, such as regular and singular points — both located in the finite complex plane or at infinity. Moreover, the general properties of the solutions at such points are analyzed.

2.1 ODE of second order

2.1.1 Standard forms

The standard form of a second order linear differential equation (ODE) is, cf. (1.1)

$$\frac{d^2 u(z)}{dz^2} + p(z) \frac{du(z)}{dz} + q(z) u(z) = 0$$

(2.1)

where \( p(z) \) and \( q(z) \) are analytic functions in a domain \( S \subset \mathbb{C} \), or analytic in \( S \) except at a finite number of isolated points, i.e., meromorphic functions in \( S \). The domain \( S \) can be the entire complex plane including the \( \infty \)-point — the extended complex \( z \)-plane.\(^1\) We seek solutions \( u(z) \) to this equation that are analytic in at least some parts of the domain \( S \).

The equation can be transformed to a reduced form

$$\frac{d^2 u_1(z)}{dz^2} + q_1(z) u_1(z) = 0$$

(2.2)

with the following change of dependent variable\(^2\)

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\(^1\) A function \( f(z) \) is analytic at infinity, \( z = \infty \), provided the function \( f(1/\zeta) \) is analytic at \( \zeta = 0 \). The meromorphic properties at infinity are defined in the same way.

\(^2\) If the domain \( S \) is not simply connected, multi-valued functions occur, e.g., if \( S = \mathbb{C} \setminus \{0\} \) and \( p(z) = c/z \). To avoid this situation, assume \( S \) is simply connected.
\[ \begin{cases} u(z) = u_1(z) \exp \left\{ -\frac{1}{2} \int_b^z p(z') \, dz' \right\} \\ q_1(z) = q(z) - \frac{1}{2} \frac{dp(z)}{dz} - \frac{1}{4} \left( p(z) \right)^2 \end{cases} \]

where \( b \in S \). It is easy to prove that if \( u_1(z) \) is a solution to (2.2), then \( u(z) \) is a solution to (2.1), and, conversely, if \( u(z) \) is a solution to (2.1), then \( u_1(z) \) is a solution to (2.2), see Problem 2.2.

In fact, the reduced form of the differential equation, (2.2), is a special case of a more general transformation of variables, given by the following theorem:

**Theorem 2.1.** Let \( u(z) \) be a solution of

\[ \frac{d^2u(z)}{dz^2} + p(z) \frac{du(z)}{dz} + q(z)u(z) = 0 \]

where \( p(z) \) and \( q(z) \) are analytic in a domain \( S \).

a) If \( z = g(t) \), where \( g(t) \) is analytic in a domain \( T \), such that the image is contained in \( S \), i.e., \( g(T) \subset S \), \( g'(t) \neq 0 \), \( t \in T \), then \( v(t) = u(g(t)) \) satisfies

\[ \frac{d^2v(t)}{dt^2} + \left( p(g(t))g'(t) - \frac{g''(t)}{g'(t)} \right) \frac{dv(t)}{dt} + \frac{q(g(t))}{g'(t)^2} v(t) = 0 \]

b) If the function \( h(z) \) is analytic and nonzero in \( S \), then \( u(z) = h(z)v(z) \) defines a function \( v(z) \), which satisfies

\[ \frac{d^2v(z)}{dz^2} + \left( p(z) + 2f(z) \right) \frac{dv(z)}{dz} + \left\{ (f(z))^2 + f'(z) + p(z)f(z) + q(z) \right\} v(z) = 0 \]

where \( f(z) = h'(z)/h(z) \).

This theorem is straightforward to prove and left as an exercise, see Problem 2.1.

### 2.1.2 Classification of points

A point \( z = c \) in the finite complex plane is classified depending on the analytic properties of the functions \( p(z) \) and \( q(z) \) at \( z = c \). The following definition is essential for the analysis of the ordinary differential equations treated in this book.

**Definition 2.1.** A point \( z = c \in \mathbb{C} \) (\( |c| < \infty \)) is called a **regular point** of the differential equation (2.1) if \( p(z) \) and \( q(z) \) are analytic in a neighborhood of \( z = c \). A point \( z = c \in \mathbb{C} \) (\( |c| < \infty \)) is called a **singular point** of the differential equation (2.1) if \( p(z) \) or \( q(z) \) have a singularity at \( z = c \).

Similarly, a point \( z = c \in \mathbb{C} \) (\( |c| < \infty \)) to the reduced form (2.2) is classified as a singular or a regular point depending on whether \( q_1(z) \) is singular or analytic (regular) at \( z = c \), respectively.
2.2 The Wronskian

We start with a definition and a lemma.

**Definition 2.2.** Let \( u_1(z) \) and \( u_2(z) \) be two meromorphic functions in a domain \( S \). The Wronskian\(^3\) of the functions \( u_1(z) \) and \( u_2(z) \) is then defined as

\[
W(u_1, u_2; z) = u_1(z)u_2'(z) - u_1'(z)u_2(z), \quad z \in S
\]

From this definition, the Wronskian depends on the functions \( u_1(z) \) and \( u_2(z) \). The following lemma shows that if \( u_1(z) \) and \( u_2(z) \) are solutions to (2.1), \( W(z) \) depends only on the variable \( z \) and the function \( p(z) \).

**Lemma 2.1.** Let \( S \) be a connected domain \( S \) in the complex plane containing only regular points of the differential equation (2.1). Then the Wronskian of two solutions, \( u_1(z) \) and \( u_2(z) \), to (2.1), i.e.,

\[
W(z) = u_1(z)u_2'(z) - u_1'(z)u_2(z)
\]

in \( S \) satisfies

\[
W(z) = W(a) \exp \left\{ - \int_a^z p(z') \, dz' \right\}
\]

where \( a \) and \( z \) are points in \( S \).

**Proof.** Derivation and use of the differential equation give

\[
\frac{dW(z)}{dz} = u_1(z)u_2''(z) - u_1''(z)u_2(z)
\]

\[
= -u_1(z) \left[ p(z)u_2'(z) + q(z)u_2(z) \right] + \left[ p(z)u_1'(z) + q(z)u_1(z) \right] u_2(z)
\]

\[
= p(z) \left( u_1'(z)u_2(z) - u_1(z)u_2'(z) \right) = -p(z)W(z)
\]

The Wronskian therefore becomes\(^4\) (\( a \) and \( z \) are regular points of the differential equation (2.1) in \( S \))

\[
W(z) = W(a) \exp \left\{ - \int_a^z p(z') \, dz' \right\}
\]

by integration. \( \square \)

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\(^3\) Józef Maria Hoëne-Wroński (1778–1853), Polish mathematician.

\(^4\) See also the comment made in footnote 2 on page 3.
From the result in this lemma, we see that if the Wronskian of two solutions vanishes at one point in $S$, it vanishes everywhere in $S$. Moreover, if the Wronskian is non-zero at one point in $S$, it is non-zero everywhere in $S$. We also observe that $W(z)$ does not depend on the explicit functions $u_1(z)$ and $u_2(z)$, but only on $p(z)$. However, the functions $u_1(z)$ and $u_2(z)$ affect the constant $W(a)$.

**Definition 2.3.** Two functions, $u_1(z)$ and $u_2(z)$, are **linearly dependent** in a domain $S$ (or an interval if we deal with functions defined on the real axis), if there exist constants $c_1$ and $c_2$, not both zero, such that

$$c_1u_1(z) + c_2u_2(z) = 0, \quad z \in S$$

If no such constants can be found, the functions are **linearly independent** in $S$.

The following lemma gives a check on linear independence.

**Lemma 2.2.** Let $u_1(z)$ and $u_2(z)$ be two meromorphic functions in a domain $S$. A necessary and sufficient condition that $u_1(z)$ and $u_2(z)$ are linearly dependent is

$$W(z) = 0, \quad z \in S$$

**Proof.** Without loss of generality, assume $u_1(z)$ not identically zero, and define $f(z) = u_2(z)/u_1(z)$. Then $f(z)$ is meromorphic in $S$, and $u_1(z)$ and $u_2(z)$ are linearly dependent if and only if $f(z)$ is constant. However, since $f(z)$ is meromorphic and

$$f'(z) = \frac{u_1(z)u_2'(z) - u_1'(z)u_2(z)}{(u_1(z))^2} = \frac{W(z)}{(u_1(z))^2}$$

$f(z)$ is constant if and only if $W(z) = 0$. □

We apply the results to the differential equation (2.1) and assume we have two solutions, $u_1(z)$ and $u_2(z)$, to this differential equation in a domain $S$ in the complex plane containing only regular points. If the functions $u_1(z)$ and $u_2(z)$ are linearly dependent, the Wronskian vanishes identically in the domain $S$, and if the Wronskian is non-zero at one point, the solutions $u_1(z)$ and $u_2(z)$ are independent in $S$, see Lemma 2.1.

### 2.3 Solution at a regular point

In a neighborhood of a regular point, a solution to the differential equation (2.1) can always be found. This section contains the details of such an explicit construction.

Let $z = b \in \mathbb{C}$ be a regular point, and let $S_b$ be a circular neighborhood to $b$ (radius $r_b > 0$) so that every point in $S_b$ is a regular point, see Figure 2.1, i.e., take $r_b$ smaller than the distance between $b$ and the closest singular point $P_1$. We also assume $p(z) = 0$ — this is no restriction (transform to the reduced form).
2.3 Solution at a regular point

For given $a_0, a_1 \in \mathbb{C}$, define a sequence of functions $u_n(z), n \in \mathbb{N}$, by the iteration scheme

\[
\begin{aligned}
&\begin{cases}
  u_0(z) = a_0 + a_1(z - b), & a_0, a_1 \in \mathbb{C}
  \\
  u_n(z) = \int_b^z (\zeta - z)q(\zeta)u_{n-1}(\zeta) \, d\zeta, & n \in \mathbb{Z}_+
\end{cases}
\end{aligned}
\]  

(2.3)

where the integration path is a straight line from $b$ to $z$. We have the following fundamental theorem:

**Theorem 2.2.** In a circular neighborhood, $S_b$, to the regular point $z = b \in \mathbb{C}$, the series

\[
  u(z) = \sum_{n=0}^{\infty} u_n(z), \quad z \in S_b
\]

(2.4)

obtained from the recursion formula in (2.3) is uniformly convergent and represents an analytic function in $S_b$. Specifically, $u(z)$ satisfies the differential equation

\[
  \frac{d^2 u(z)}{dz^2} + q(z)u(z) = 0
\]

(2.5)

with the following initial conditions at $z = b$:

\[
\begin{aligned}
  &u(b) = a_0 \\
  &u'(b) = a_1
\end{aligned}
\]

(2.6)

Moreover, there is only one analytic function satisfying the differential equation (2.5) and the initial conditions (2.6), i.e., the function $u(z)$ in (2.4) is the unique analytic solution to the ODE, (2.5), with the given boundary conditions.

**Proof.** We prove this theorem in four lemmas, Lemmas 2.3–2.6. □
Lemma 2.3. The sequence in (2.3) satisfies

\[ |u_n(z)| \leq \mu M^n \frac{|z - b|^{2n}}{n!}, \quad \text{for all } z \in S_b, \text{ and } n \in \mathbb{Z}_+ \]

where

\[
\begin{align*}
|u_0(z)| & \leq \mu \\
|q(z)| & \leq M 
\end{align*}
\]

for all \( z \in S_b \).

Proof. We prove the lemma by induction. For \( n = 0 \) it is trivial. Let \( n \geq 1 \). We have, by the induction assumption, the estimate

\[
|u_n(z)| \leq \int_b^z |\zeta - z| |\zeta - b|^{2n-2} \frac{2n-1}{(n-1)!} |d\zeta|, \quad z \in S_b
\]

The integration path is: \( \zeta(t) = (z - b)t + b, t \in [0, 1] \). Therefore, since \( |\zeta - z| \leq |z - b| \) and \( 2n - 1 \geq n \), we get

\[
|u_n(z)| \leq \mu M^n |z - b| \int_0^1 \frac{|\zeta(t) - b|^{2n-2}}{(n-1)!} |d\zeta(t)| dt
\]

\[
= \mu M^n |z - b|^2 \int_0^1 \frac{|z - b|^{2n-2} t^{2n-2}}{(n-1)!} dt = \mu M^n |z - b|^{2n} \int_0^1 \frac{t^{2n-2}}{(n-1)!} dt
\]

\[
= \mu M^n |z - b|^{2n} \frac{1}{(2n-1)(n-1)!} \leq \mu M^n \frac{|z - b|^{2n}}{n!}
\]

and the lemma is proved. \( \square \)

Lemma 2.4. The series

\[
u(z) = \sum_{n=0}^{\infty} u_n(z), \quad z \in S_b
\]

(2.7)

is uniformly convergent in \( S_b \), and, therefore, analytic in \( S_b \).

Proof. From Lemma 2.3, we have

\[
|u_n(z)| \leq \mu M^n \frac{|z - b|^{2n}}{n!} \leq \mu M^n \frac{r_b^{2n}}{n!}
\]

Since the series

\[
\sum_{n=0}^{\infty} \mu M^n \frac{r_b^{2n}}{n!} < \infty, \quad \text{for } r_b < \infty
\]

the series (2.7) is uniformly convergent in \( S_b \) by the M-test of Weierstrass\(^5\) [13, p. 107], and thus analytic in \( S_b \). \( \square \)

Lemma 2.5. The analytic function \( u(z) \) in Lemma 2.4 satisfies

\[^5\text{Karl Weierstrass (1815–1897), German mathematician.}\]
2.3 Solution at a regular point

\[ \frac{d^2 u(z)}{dz^2} + q(z)u(z) = 0 \]

with the following initial conditions at \( z = b \):

\[
\begin{align*}
    u(b) &= a_0 \\
    u'(b) &= a_1
\end{align*}
\]

Proof. We easily get

\[
\begin{align*}
    u'_n(z) &= -\int_b^z q(\zeta)u_{n-1}(\zeta) \, d\zeta \\
    u''_n(z) &= -q(z)u_{n-1}(z)
\end{align*}
\]

and, since the series (2.7) can be differentiated inside the sum, we get

\[
\frac{d^2 u(z)}{dz^2} = \sum_{n=1}^\infty \frac{d^2 u_n(z)}{dz^2} = -q(z) \sum_{n=1}^\infty u_{n-1}(z) = -q(z)u(z)
\]

Moreover, the sequence takes at \( z = b \) the values

\[
\begin{align*}
    u_0(b) &= a_0 \\
    u_n(b) &= 0, \quad n \in \mathbb{Z}_+ \\
    u'_0(b) &= a_1 \\
    u'_n(b) &= 0, \quad n \in \mathbb{Z}_+
\end{align*}
\]

Therefore

\[
\begin{align*}
    u(b) &= \sum_{n=0}^\infty u_n(b) = a_0 \\
    u'(b) &= \sum_{n=0}^\infty u'_n(b) = a_1
\end{align*}
\]

and the lemma is proved. □

Lemma 2.6. There is only one analytic function \( u(z) \) that satisfies

\[ \frac{d^2 u(z)}{dz^2} + q(z)u(z) = 0 \quad \begin{align*}
    u(b) &= a_0 \\
    u'(b) &= a_1
\end{align*} \]

Proof. Assume there are two analytic solutions, \( u_1 \) and \( u_2 \), and form \( v = u_1 - u_2 \). The function \( v \) is an analytic function and it satisfies

\[ \frac{d^2 v(z)}{dz^2} + q(z)v(z) = 0 \quad \begin{align*}
    v(b) &= 0 \\
    v'(b) &= 0
\end{align*} \]

in \( S_b \). By evaluating the differential equation at \( z = b \), we obtain \( v''(b) = 0 \), since the point \( b \) is regular, i.e., \( |q(b)| < \infty \). If we differentiate the equation, we get
Two analytic continuation paths that do not necessarily give the same value at $z$

Fig. 2.2 The analytic continuation of the solution of the differential equation in the complex $z$-plane. The points $P_i$, $i = 1, 2, 3, 4$, denote the singular points of the differential equation.

\[ v'''(z) + q'(z)v(z) + q(z)v'(z) = 0 \]

and evaluating this equation at $z = b$, we obtain by the same argument as above

\[ v'''(b) = 0 \]

Continuing the argument, we see that $v^{(n)}(b) = 0$ for all $n \in \mathbb{N}$. Since the solution $v(z)$ is analytic at $z = b$, we conclude that $v(z) = 0$ for all $z \in S_b$. 

As a consequence of the Theorem 2.2, we can construct an analytic solution at any regular point to the ordinary differential equation in the complex $z$-plane. This means that we can construct a solution at all points in the complex $z$-plane, except at the singular points of the differential equation, where the solution shows singular behavior of some kind.

From a given initial condition, a unique solution is obtained by analytic continuation in the complex plane. The solution is unique at each regular point in the complex $z$-plane, but might depend on the way the analytic continuation is made, see Figure 2.2.

Any solution to (2.2), and therefore to (2.1), can be obtained as a linear combination of two linearly independent solutions to the equation. Two linearly independent
solutions\(^6\) \(u_1(z)\) and \(u_2(z)\) can be constructed by the initial conditions

\[
\begin{align*}
S_1 : & \quad u_1(b) = a_0 = 0, \quad u'_1(b) = a_1 = 1 \\
S_2 : & \quad u_2(b) = a_0 = 1, \quad u'_2(b) = a_1 = 0
\end{align*}
\]

From the analysis in this section, we see that two linearly independent solutions can be constructed by analytic continuation everywhere in the complex plane, except at the singular points of the differential equation. The behavior near a singular point is, however, not determined by the analysis. In Section 2.4, we investigate the behavior of the solution near a singular point in more detail, but we finish this section with the construction of a second linearly independent solution if one solution is known.

A series solution of the solution at a regular point is investigated in Problem 2.6. This technique is exploited in more detail in Section 2.4.

### 2.3.1 The second solution

Now assume we have obtained one solution \(u_1(z)\) to the differential equation — either by the explicit construction in Theorem 2.2 or by guessing or by some other way. In this section we explicitly construct a second linearly independent solution \(u_2(z)\) from the solution \(u_1(z)\). We proceed by writing the Wronskian in Definition 2.2 as

\[
W(z) = u_1(z)u'_2(z) - u'_1(z)u_2(z) = (u_1(z))^2 \frac{d}{dz} \left( \frac{u_2(z)}{u_1(z)} \right)
\]

Combine the equation with the result obtained in Lemma 2.1 into

\[
\frac{d}{dz} \left( \frac{u_2(z)}{u_1(z)} \right) = \frac{W(a)}{(u_1(z))^2} \exp \left\{ - \int_a^z p(z'') \, dz'' \right\}
\]

and integrate in \(z\) from \(a\) to \(z\). We get

\[
\frac{u_2(z)}{u_1(z)} = W(a) \int_a^z \frac{1}{(u_1(z'))^2} \exp \left\{ - \int_a^{z'} p(z'') \, dz'' \right\} \, dz' + \frac{u_2(a)}{u_1(a)}
\]

A second linearly independent solution to the differential equation then is

\[
u_2(z) = u_1(z) \int_a^z \frac{1}{(u_1(z'))^2} \exp \left\{ - \int_a^{z'} p(z'') \, dz'' \right\} \, dz' \quad (2.8)
\]

where we have dropped the lower limits (and the last constant term), which only give a term that is a proportional to the first solution \(u_1(z)\), and, therefore, adds nothing.

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\(^6\) The check of linear dependence is developed in Section 2.2.
new. In particular, for the reduced differential equation (2.2), where \( p(z) = 0 \), we have a particularly simple form of the second solution, viz.,

\[
 u_2(z) = u_1(z) \int^z \frac{dz'}{(u_1(z'))^2}
\]

The method presented in this section has been employed to find a second solution to the differential equation in a neighborhood of a regular point, but the method may also have potential finding a second solution in the neighborhood of a singular point.

**Example 2.1.** We illustrate the result above with a very simple example. The differential equation

\[
 u''(z) - k^2 u(z) = 0
\]

has a solution \( u_1(z) = e^{kz} \). The second solution then is, \( k \neq 0 \)

\[
 u_2(z) = u_1(z) \int^z \frac{dz'}{(u_1(z'))^2} = e^{kz} \int^z e^{-2kz'} \, dz' = -\frac{1}{2k} e^{kz} e^{-2kz} = -\frac{1}{2k} e^{-kz}
\]

which, of course, is a second solution to the differential equation, which is linearly independent from the first one.

**Comment 2.1.** The analysis in this section proved the existence of two linearly independent solutions \( u_1(z) \) and \( u_2(z) \) in the neighborhood of a regular point \( z = b \). In Section 2.4, an explicit algorithm to find the power series solution is presented, in particular, see Comment 2.2 on page 23.

### 2.4 Solution at a regular singular point

We have seen that, in general, two linearly independent solutions to the second order differential equation can be constructed at all points in \( \mathbb{C} \) except at the singular points of the differential equation. In this section, we investigate the behavior of the solution near these singular points in detail. We start with a definition.

**Definition 2.4.** Assume \( z = c \in \mathbb{C} \) (\( |c| < \infty \)) is a singular point of the differential equation (2.1). The point \( z = c \) is called a **regular singular point** of the differential equation (2.1) if \( p(z) \) and \( q(z) \) have the form

\[
 p(z) = \frac{P(z)}{z - c}, \quad q(z) = \frac{Q(z)}{(z - c)^2}
\]

where \( P(z) \) and \( Q(z) \) are analytic functions in a neighborhood of \( z = c \). In all other cases, the singular point \( z = c \) is called an **irregular singular point** of the differential equation (2.1).

\[\text{We use uppercase letters to denote functions that are analytic functions, and lowercase letters are used to denote meromorphic functions.}\]
The point at infinity is special, and the classification of this point as a regular or an irregular singular point is postponed to Section 2.5.

### 2.4.1 The indicial equation

The appropriate differential equation for a regular singular point at $z = c$ therefore is

$$\frac{d^2 u(z)}{dz^2} + \frac{P(z)}{z-c} \frac{du(z)}{dz} + \frac{Q(z)}{(z-c)^2} u(z) = 0$$  \hspace{1cm} (2.9)

Since the functions $P(z)$ and $Q(z)$ are analytic in a neighborhood of $z = c$, i.e., they have power series expansions

\[
\begin{align*}
P(z) &= \sum_{n=0}^{\infty} p_n (z-c)^n = p_0 + p_1 (z-c) + p_2 (z-c)^2 + \ldots \\
Q(z) &= \sum_{n=0}^{\infty} q_n (z-c)^n = q_0 + q_1 (z-c) + q_2 (z-c)^2 + \ldots
\end{align*}
\]  \hspace{1cm} (2.10)

which are convergent for all $z \in S_c$, where the set $S_c$ is an open circle, radius $r_c > 0$, centered at $z = c$. The set $S_c$ contains no other singular points of (2.9) than $z = c$.

We know that the solution $u(z)$ shows a singular behavior near the regular singular point $z = c \in \mathbb{C}$. To investigate this behavior, we make an ansatz for the solution

$$u(z) = (z-c)^{\alpha} \left(1 + \sum_{n=1}^{\infty} a_n (z-c)^n\right)$$  \hspace{1cm} (2.11)

where, due to linearity and homogeneity of the differential equation, the first coefficient is set to 1, i.e., $a_0 = 1$. Our goal is to determine the unknown complex coefficients $a_n, n \in \mathbb{Z}_+$, provided the complex constants $p_n$ and $q_n, n \in \mathbb{N}$, are known.

The differential equation (2.9) is employed, i.e.,

$$(z-c)^2 u''(z) + (z-c) P(z) u'(z) + Q(z) u(z) = 0$$

and we insert the solution $u(z)$. We get

$$(z-c)^{\alpha} \left\{ \alpha (\alpha - 1) + \sum_{n=1}^{\infty} a_n (\alpha + n)(\alpha + n - 1)(z-c)^n
\right. \right.$$

\begin{align*}
&+ P(z) \left( \alpha + \sum_{n=1}^{\infty} a_n (\alpha + n)(z-c)^n \right) + Q(z) \left( 1 + \sum_{n=1}^{\infty} a_n (z-c)^n \right) \right\} = 0
\end{align*}

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*Footnote:* This is the method of Frobenius named after the German mathematician Ferdinand Georg Frobenius (1849–1917).
Introduce the power series expansions of $P(z)$ and $Q(z)$ in (2.10), and identify the coefficient in front of each power of $z - c$ within the braces. Each of these coefficients must be zero if $u$ is a solution to (2.9).

The result for each coefficient is

\[
\begin{align*}
0) & \quad \alpha^2 + (p_0 - 1)\alpha + q_0 = 0 \\
1) & \quad a_1 \left\{ (\alpha + 1)^2 + (p_0 - 1)(\alpha + 1) + q_0 \right\} + p_1 \alpha + q_1 = 0 \\
2) & \quad a_2 \left\{ (\alpha + 2)^2 + (p_0 - 1)(\alpha + 2) + q_0 \right\} + a_1 \left\{ p_1(\alpha + 1) + q_1 \right\} + p_2 \alpha + q_2 = 0 \\
& \quad \cdots \\
n) & \quad a_n \left\{ (\alpha + n)^2 + (p_0 - 1)(\alpha + n) + q_0 \right\} \\
& \quad + \sum_{m=1}^{n-1} a_{n-m} \left\{ p_m(\alpha + n - m) + q_m \right\} + p_n \alpha + q_n = 0
\end{align*}
\]

We define the indicial equation

\[I(\alpha) = \alpha^2 + (p_0 - 1)\alpha + q_0 = 0 \quad (2.12)\]

and write the relations above as ($a_0 = 1$)

\[
\begin{align*}
I(\alpha) &= 0 \\
a_1 I(\alpha + 1) + p_1 \alpha + q_1 &= 0 \\
a_2 I(\alpha + 2) + a_1 \left\{ p_1(\alpha + 1) + q_1 \right\} + p_2 \alpha + q_2 &= 0 \\
& \quad \cdots \\
a_n I(\alpha + n) + \sum_{m=1}^{n-1} a_{n-m} \left\{ p_m(\alpha + n - m) + q_m \right\} &= 0
\end{align*}
\]

(2.13)

The coefficient in front of the lowest power is

\[I(\alpha) = 0\]

This indicial equation is a quadratic equation in $\alpha$, and we denote the two roots of the equation by $\alpha_1$ and $\alpha_2$, respectively. Notice that the two roots satisfy

\[\alpha_1 + \alpha_2 = 1 - p_0, \quad \alpha_1 \alpha_2 = q_0 \quad (2.14)\]

so that the values of $P(z)$ and $Q(z)$ at $z = c$ determine the roots of the indicial equation.

The coefficient in front of the second lowest power is

\[a_1 I(\alpha + 1) + p_1 \alpha + q_1 = 0\]

and, provided $I(\alpha + 1) \neq 0$, the coefficient $a_1$ has a unique solution in terms of the expansion coefficients of $P(z)$ and $Q(z)$. If we continue this argument, the coefficients $a_n$, $n \in \mathbb{Z}_+$, are uniquely soluble in terms of the expansion coefficients of
2.4 Solution at a regular singular point

$P(z)$ and $Q(z)$, provided $I(\alpha + i) \neq 0$, $i = 1, 2, \ldots, n$. If this condition is met, the formal series in (2.11) can be constructed by the iterative scheme obtained from (2.13). We have

$$
\begin{align*}
&\begin{cases}
a_0 = 1 \\
a_n = -\sum_{m=1}^{n} a_{n-m} \left\{ p_m(\alpha + n - m) + q_m \right\} / I(\alpha + n), \quad n = 1, 2, 3, \ldots 
\end{cases} \\
& \text{(2.15)}
\end{align*}
$$

The following lemma quantifies when $I(\alpha + n)$ vanishes:

**Lemma 2.7.** Denote by $\alpha_1$ and $\alpha_2$ the two roots of the indicial equation (2.12). Then

$$
I(\alpha_1 + n) = n(\alpha_1 - \alpha_2 + n)
$$

**Proof.** Let $s = \alpha_1 - \alpha_2$. The lemma is then easily proved by the following observations:

$$
I(\alpha_1 + n) = (s + \alpha_2 + n)^2 + (p_0 - 1)(s + \alpha_2 + n) + q_0
$$

$$
= \alpha_2^2 + (p_0 - 1)\alpha_2 + q_0 + (s + n)^2 + 2(s + n)\alpha_2 + (p_0 - 1)(s + n)
$$

$$
= (s + n)(s + n + 2\alpha_2 + (p_0 - 1)) = n(s + n) = n(\alpha_1 - \alpha_2 + n)
$$

due to (2.14), and the lemma is proved. \(\square\)

From Lemma 2.7, we see that if the two roots of the indicial equation (2.12), $\alpha_1$ and $\alpha_2$, do not differ by an integer, then $I(\alpha_1 + n) \neq 0$, $n \in \mathbb{Z}_+$. The main investigation of the convergence of this power series, with coefficients obtained in (2.15), is postponed to Section 2.4.2. Meanwhile, we conclude that either of the following two cases can occur:

Case 1. $\alpha_1 - \alpha_2$ is not an integer. Then the iterative procedure in (2.15) has the potential of constructing two linearly independent solutions corresponding to the two roots $\alpha = \alpha_1$ and $\alpha = \alpha_2$, respectively.

Case 2. $\alpha_1 - \alpha_2$ is an integer. Then it is unclear whether the procedure gives a solution or not.

The growth rate of the coefficients $a_n$ obtained from (2.15) is estimated in the next two lemmas.

**Lemma 2.8.** Denote by $\alpha_1$ and $\alpha_2$ the two roots of the indicial equation (2.12), and let $\alpha_1$ be the root with the largest real part, i.e., $\text{Re} \alpha_1 \geq \text{Re} \alpha_2$. Furthermore, assume

$$
|p_n| \leq \frac{M}{r^n}, \quad |p_n \alpha_1 + q_n| \leq \frac{M}{r^n}, \quad n \in \mathbb{Z}_+
$$

where $M > 1$. Then with $\alpha = \alpha_1$ in (2.15), the coefficients $a_n$ satisfy

$$
|a_n| \leq \frac{M^n}{r^n}, \quad n \in \mathbb{Z}_+
$$

\(\square\)
Proof. We prove the lemma by induction, and use Lemma 2.7, which with the notation $s = \alpha_1 - \alpha_2$ reads

$$I(\alpha_1 + n) = n(s + n) \quad (2.17)$$

The statement is true for $n = 1$, since, taking $\alpha = \alpha_1$ in (2.15), we get

$$|a_1| = \left| \frac{p_1 \alpha_1 + q_1}{I(\alpha_1 + 1)} \right| = \frac{|p_1 \alpha_1 + q_1|}{|s + 1|} \leq \frac{M}{r|s + 1|} \leq \frac{M}{r}$$

due to (2.16), (2.17), and $|s + 1| \geq 1$ (remember $\text{Re}s \geq 0$, due to the assumption in the lemma).

Assume the induction statement is true for $k = 1, 2, \ldots, n - 1$. Then with $\alpha = \alpha_1$ in (2.15), we get for $n \geq 2$

$$|a_n| = \left| \sum_{m=1}^{n} a_{n-m} \{ p_m(\alpha_1 + n - m) + q_m \} I(\alpha_1 + n) \right|$$

$$\leq \sum_{m=1}^{n} |a_{n-m}| |p_m\alpha_1 + q_m| + \sum_{m=1}^{n} |a_{n-m}| |p_m|(n - m)$$

$$\leq \sum_{m=1}^{n} M^{n-m} r^{m-n} M r^{-m} + \sum_{m=1}^{n} M^{n-m} r^{m-n} M r^{-n}(n - m)$$

by (2.16), (2.17), and the induction assumption. Further estimates give (remember $M > 1$)

$$|a_n| \leq \frac{M^n n + \sum_{m=1}^{n} (n - m)}{r^n n^2 |1 + s/n|} = \frac{M^n n + n(n - 1)/2}{r^n n^2 |1 + s/n|} = \frac{M^n n + 1}{r^n 2n |1 + s/n|}$$

and since $|1 + s/n| \geq 1$ (remember $\text{Re}s \geq 0$), and $n + 1 \leq 2n$, we have

$$|a_n| \leq \frac{M^n}{r^n}, \quad n \geq 2$$

and the induction proof is finished. \qed

The growth rate of the coefficients $a_n$ corresponding to the other root is more complex.

**Lemma 2.9.** Denote by $\alpha_1$ and $\alpha_2$ the two roots of the indicial equation (2.12), and let $\alpha_2$ be the root with the smallest real part, i.e., $\text{Re}\alpha_1 \geq \text{Re}\alpha_2$. Moreover, assume $s = \alpha_1 - \alpha_2 \notin \mathbb{Z}_+$, and

$$|p_n| \leq \frac{M}{r^n}, \quad |p_n\alpha_2 + q_n| \leq \frac{M}{r^n}, \quad n \in \mathbb{Z}_+ \quad (2.18)$$

Then with $\alpha = \alpha_2$ in (2.15), the coefficients $a_n$ satisfy

$$|a_n| \leq \frac{M^n}{r^n}, \quad n \in \mathbb{Z}_+$$
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where \( M' = M / \kappa \) and \( M' > 1 \). Here \( \kappa = \inf\{|1-s|, |1-s/2|, |1-s/3|, \ldots\} \) is a positive number, due to the assumptions on \( s \).

**Proof.** Again, we prove by induction, and use Lemma 2.7, which with the notion \( s = \alpha_1 - \alpha_2 \) reads

\[
I(\alpha_2 + n) = n(n-s) \tag{2.19}
\]

We conclude that the statement is true for \( n = 1 \), since, taking \( \alpha = \alpha_2 \) in (2.15), we get

\[
|a_1| = \frac{|p_1\alpha_2 + q_1|}{I(\alpha_2 + 1)} = \frac{|p_1\alpha_2 + q_1|}{|1-s|} \leq \frac{M}{r|1-s|} \leq \frac{M}{\kappa r} = \frac{M'}{r}
\]
due to (2.18), (2.19), and \( |1-s| \geq \kappa \).

Assume the induction statement is true for \( k = 1, 2, \ldots, n-1 \). Then with \( \alpha = \alpha_2 \) in (2.15), we get for \( n \geq 2 \)

\[
|a_n| = \left| \sum_{m=1}^{n} a_{n-m} \left\{ p_m(\alpha_2 + n - m) + q_m \right\} \right| I(\alpha_2 + n)
\]

\[
\leq \sum_{m=1}^{n} |a_{n-m}| |p_m\alpha_2 + q_m| + \sum_{m=1}^{n} |a_{n-m}| |p_m|(n-m) / n|n-s|
\]

\[
\leq \sum_{m=1}^{n} M^{m-m} r^{m-n} M r^{-m} + \sum_{m=1}^{n} M^{m-m} r^{m-n} M r^{-m} (n-m) / n^2|1-s/n|
\]

by (2.18), (2.19), and the induction assumption. Further estimates give (remember \( M' > 1 \))

\[
|a_n| \leq \frac{MM^{n-1}}{r^n} n + \frac{n(n-1)/2}{n^2|1-s/n|} = \frac{MM^{n-1}}{r^n} \frac{n(n-1)}{2n|1-s/n|}
\]

and since \( |1-s/n| \geq \kappa \) and \( n+1 \leq 2n \), we have

\[
|a_n| \leq \frac{M^n}{\kappa^n r^n} \leq \frac{M^n}{r^n}
\]

and the induction proof is finished. \( \square \)

### 2.4.2 Convergence of the solution

We now address the question of convergence of the power series

\[
\sum_{n=1}^{\infty} a_n (z-c)^n
\]
with the coefficients $a_n$, $n \in \mathbb{Z}_+$, explicitly constructed by the algorithm in Section 2.4.1 above. The main theorem of this section is:

**Theorem 2.3.** Denote by $\alpha_1$ and $\alpha_2$ the two roots of the indicial equation (2.12), and number the roots such that $\text{Re} \alpha_1 \geq \text{Re} \alpha_2$. Then the power series

$$\sum_{n=1}^{\infty} a_n(z-c)^n$$

with coefficients, $a_n$, obtained by the iteration scheme in (2.15), starting from the root $\alpha_1$, represents an analytic function in a neighborhood of the regular singular point $z = c$.

Moreover, if the two roots $\alpha_1$ and $\alpha_2$ of the indicial equation do not differ by a positive integer, the power series

$$\sum_{n=1}^{\infty} a'_n(z-c)^n$$

with coefficients, $a'_n$, obtained by the iteration scheme in (2.15), starting from the root $\alpha_2$, also represents an analytic function in a neighborhood of the regular singular point $z = c$.

Theorem guarantees that there exists at least one solution to the differential equation, (2.9), in a neighborhood of the regular singular point, $z = c$. If the the roots of the indicial equations do not differ by a positive integer, we can also construct a second solution. In this case, the solutions are

$$u_1(z) = (z-c)^{\alpha_1} \left(1 + \sum_{n=1}^{\infty} a_n(z-c)^n\right), \quad u_2(z) = (z-c)^{\alpha_2} \left(1 + \sum_{n=1}^{\infty} a'_n(z-c)^n\right)$$

**Proof.** The explicit coefficients $a_n$, obtained by the algorithm in Section 2.4.1, is given by (2.15), i.e., $I(\alpha) = 0$ and

$$a_0 = 1, \quad a_n = -\frac{\sum_{m=1}^{n} a_{n-m} \{p_m(\alpha+n-m) + q_m\}}{I(\alpha+n)}, \quad n = 1, 2, 3, \ldots$$

(2.20)

To analyze the convergence, denote the two roots of the indicial equation by $\alpha_1$ and $\alpha_2$, respectively, and let $s = \alpha_1 - \alpha_2$. Number the roots such that $\text{Re} s \geq 0$, i.e., $\text{Re} \alpha_1 \geq \text{Re} \alpha_2$. The assumption of the second part of the theorem implies that $s \notin \mathbb{Z}_+$.

The coefficients in the power series expansions of the functions $P(z)$ and $Q(z)$ in (2.10), i.e.,

---

9 If both roots are identical, $\alpha_1 = \alpha_2$, only one solution is obtained.
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\[
\begin{aligned}
P(z) &= \sum_{n=0}^{\infty} p_n (z - c)^n \\
Q(z) &= \sum_{n=0}^{\infty} q_n (z - c)^n
\end{aligned}
\]

are determined by\(^{10}\)

\[
p_n = \frac{1}{2\pi i} \oint_C \frac{P(t) \, dt}{(t - c)^{n+1}}, \quad q_n = \frac{1}{2\pi i} \oint_C \frac{Q(t) \, dt}{(t - c)^{n+1}}
\]

where \(C\) is a circle with radius \(r_c\), centered at \(z = c\), and contained in the common domain of analyticity of \(P(z)\) and \(Q(z)\). Let the constant \(M_p\) and \(M_q\) be the maximum value of \(|P(z)|\) and \(|Q(z)|\), respectively, on the circle \(C\). Then

\[
|p_n| \leq \frac{M_p}{r_c^n}, \quad |q_n| \leq \frac{M_q}{r_c^n}, \quad n \in \mathbb{Z}_+
\]

The coefficients \(p_0\) and \(q_0\) satisfy similar inequalities, but these are not used in the proof.

Denote by \(M = \max\{M_p, |\alpha_1| M_p + M_q, |\alpha_2| M_p + M_q\}\), which is a finite number, due to the assumptions made on \(P(z)\) and \(Q(z)\). Then, we have for \(n \in \mathbb{Z}_+\)

\[
|p_n| \leq \frac{M}{r_c^n}, \quad |p_n \alpha_1 + q_n| \leq \frac{M}{r_c^n}, \quad |p_n \alpha_2 + q_n| \leq \frac{M}{r_c^n}, \quad (2.21)
\]

There is no loss of generality to assume that the constant \(M > 1\) — increase the value of \(M\) if necessary.

From Lemma 2.8, we then obtain

\[
|a_n| \leq \frac{M^n}{r_c^n}, \quad n \in \mathbb{Z}_+
\]

if we are using the root \(\alpha_1\) — the one with the largest real part. The power series

\[
\sum_{n=1}^{\infty} a_n (z - c)^n
\]

is then uniformly convergent inside the circle \(|z - c| < \rho = r_c/M\) since

---

\(^{10}\) The expansion coefficients of an analytic function \(f(z)\) in a power series expansion satisfy [13, p. 128]

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (z - c)^n
\]

where [13, p. 180]

\[
f^{(n)}(c) = \frac{n!}{2\pi i} \oint_C \frac{f(t) \, dt}{(t - c)^{n+1}}, \quad n \in \mathbb{Z}_+
\]
20

2 Basic properties of the solutions

\[ \sum_{n=1}^{\infty} |a_n| |z - c|^n \leq \sum_{n=1}^{\infty} \frac{M^n |z - c|^n}{r^n c} = \sum_{n=1}^{\infty} \frac{|z - c|^n}{\rho^n} < \infty \]

and the series represents an analytic function inside the circle \(|z - c| < \rho\), and the first part of the theorem is proved.

Using the second root of the indicial equation, \(\alpha_2\), and Lemma 2.9, we obtain

\[ |a_n'| \leq \frac{M^n}{r^n c} = \frac{M^n}{\kappa^n r^n c}, \quad n \in \mathbb{Z}_+ \]

where \(\kappa = \inf\{1 - s, |1 - s/2|, |1 - s/3|, \ldots\}\), which is a positive number, due to the assumption that \(s\) is not a positive integer, and \(M' = M/\kappa\). The power series

\[ \sum_{n=1}^{\infty} a_n'(z - c)^n \]

is then uniformly convergent inside the circle \(|z - c| < \rho' = r_c/M' = \kappa r_c/M\) since

\[ \sum_{n=1}^{\infty} |a_n' |z - c|^n \leq \sum_{n=1}^{\infty} \frac{M^n |z - c|^n}{\kappa^n r^n c} = \sum_{n=1}^{\infty} \frac{|z - c|^n}{\rho'^n} < \infty \]

and the series represents an analytic function inside the circle \(|z - c| < \rho'\). This completes the second part of the proof of the theorem. \(\Box\)

Indeed, if \(\alpha_1 - \alpha_2\) is not a positive integer or zero, we have constructed two linearly independent\(^{11}\) solutions of the differential equation, viz.,

\[
\begin{align*}
&\left\{ u_1(z) = (z - c)^{\alpha_1} \left(1 + \sum_{n=1}^{\infty} d_n(z - c)^n\right) \right. \\
&\left. u_2(z) = (z - c)^{\alpha_2} \left(1 + \sum_{n=1}^{\infty} d_n'(z - c)^n\right) \right. \\
&\quad \tag{2.22}
\end{align*}
\]

2.4.3 The second solution — exceptional case

The existence of one solution to the differential equation of the form, see (2.22),

\[
\begin{align*}
&u_1(z) = (z - c)^{\alpha_1} \left(1 + \sum_{n=1}^{\infty} d_n(z - c)^n\right) = (z - c)^{\alpha_1} f(z) \\
&u_2(z) \quad \tag{2.23}
\end{align*}
\]

where \(f(z)\) is analytic in a neighborhood of the regular singular point \(z = c\), is guaranteed by Theorem 2.3. It is constructed from the root of the indicial equation with the largest real part.

\(^{11}\) They are linearly independent since they have different analytic properties at \(z = c\).
In this section, we construct a second, linearly independent solution, \( u_2(z) \), to the differential equation, in a neighborhood of the regular singular point \( z = c \), when \( s = \alpha_1 - \alpha_2 \in \mathbb{N}, \) i.e., it is a positive integer or zero. This is exactly the case when the construction of the power series of second solution, \( u_2(z) \), in the proof of Theorem 2.3 breaks down or only gives one solution. The reason for the failure is that the quantity \( \kappa \) in the proof then is zero, and the constructions fail due to division by zero. The first solution \( u_1(z) \) in (2.23) is used to construct the second solution.

In Section 2.3.1, we presented a method to find a second solution, \( u_2(z) \), if one solution, \( u_1(z) \), is known. Here, we prefer to employ a variation of this method, which is more pertinent for the analysis in this section. The method is also called “variation of the constant,” and we proceed, by making a formal ansatz

\[
 u_2(z) = C(z)u_1(z)
\]

Our aim is to find the conditions on \( C(z) \) that make \( u_2(z) \) a solution to the differential equation (2.9), if \( u_1(z) \) is a known solution of the same differential equation.

Differentiate the solution \( u_2(z) \) and insert in the differential equation (2.9). This implies

\[
 C''(z)u_1(z) + 2C'(z)u_1'(z) + C(z)u_1''(z) + \frac{P(z)}{z-c}C'(z)u_1(z) + \frac{Q(z)}{(z-c)^2}C(z)u_1(z) = 0
\]

If \( u_1(z) \) is the solution given in (2.23), then it satisfies (2.9), and the expression above simplifies to

\[
 C''(z)u_1(z) + 2C'(z)u_1'(z) + \frac{P(z)}{z-c}C'(z)u_1(z) = 0
\]

or

\[
 \frac{C''(z)}{C'(z)} + 2\frac{u_1'(z)}{u_1(z)} + \frac{P(z)}{z-c} = 0
\]

We easily integrate this expression and get

\[
 \ln C'(z) + 2\ln u_1(z) + \int_{z'}^{z} \frac{P(z')}{z' - c} \, dz' = A
\]

where \( A \) is a complex constant. The integral becomes

\[
 \int_{z'}^{z} \frac{P(z')}{z' - c} \, dz' = \int_{z'}^{z} \frac{p_0 + p_1(z' - c) + p_2(z' - c)^2 + \cdots}{z' - c} \, dz' = p_0 \ln(z - c) + F(z)
\]

where \( F(z) \) is analytic\(^{12} \) in a neighborhood of \( z = c \). The expression above then is

\(^{12} \) Note that

\[
 \int_{z}^{z} \sum_{i=0}^{\infty} p_{i+1} (z-c)^i \, dz' = \sum_{i=0}^{\infty} \frac{p_{i+1}}{i+1} (z-c)^{i+1} + C
\]
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\[ \ln C'(z) + 2 \ln u_1(z) + \ln(z - c)^{p_0} + F(z) = A \]

or\(^{13}\)

\[ C(z) = \int^z \frac{G(z')}{(u_1(z'))^2(z' - c)^{p_0}} \, dz' = \int^z \frac{H(z')}{(z' - c)^{2\alpha_1 + p_0}} \, dz' \]

where \( G(z) = \exp\{A - F(z)\} \) and \( H(z) = G(z)/(f(z))^2 \) are analytic functions in a neighborhood of \( z = c \) (remember \( f(z) \) is non-zero in a neighborhood of \( z = c \)).

The roots of the indicial equations satisfy \( \alpha_1 + \alpha_2 = 1 - p_0 \), see (2.14). The function \( C(z) \) then is \((s = \alpha_1 - \alpha_2 \) is a positive integer or zero\):

\[ C(z) = \int^z H(z') \left( \frac{z' - c}{s+1} \right) \, dz' = \int^z h_0 + h_1(z' - c) + h_2(z' - c)^2 + \ldots \, dz' \]

where the power series expansion of \( H(z) \)

\[ H(z) = \sum_{n=0}^{\infty} h_n(z - c)^n \]

converges in a neighborhood of \( z = c \). Two different cases appear.

Case 1. First, if \( s = 0 \), i.e., the two roots of the indicial equations coincide, then

\[ C(z) = \int^z \frac{h_0 + h_1(z' - c) + h_2(z' - c)^2 + \ldots}{z' - c} \, dz' = h_0 \ln(z - c) + K_1(z) \]

where \( K_1(z) \) is analytic\(^{14}\) in a neighborhood of \( z = c \). The second solution \( u_2(z) \) can then be written as

\[ u_2(z) = C(z)u_1(z) = (h_0 \ln(z - c) + K_1(z)) (z - c)^{\alpha_1} f(z) \]

where \( f(z) \) and \( K_1(z) \) are analytic functions in a neighborhood of \( z = c \).

Case 2. The second case appears when \( s \) is a positive integer, i.e., \( s = n \in \mathbb{Z}_+ \). Then we get

\[ C(z) = \int^z \frac{h_0 + h_1(z' - c) + h_2(z' - c)^2 + \ldots}{(z' - c)^{n+1}} \, dz' \]

\[ = \int^z \left( \sum_{i=0}^{n-1} \frac{h_i}{(z - c)^{n+1-i}} + \frac{h_n}{z - c} + \sum_{i=0}^{\infty} h_{n+1+i}(z - c)^i \right) \, dz' \]

which we simplify to

13 Notice the resemblance with the result in (2.8) in Section 2.3.1.

14 The change of order between summation and integration, and the radius of convergence of the power series of \( H(z) \), see footnote 12 above.
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\[ C(z) = - \sum_{i=0}^{n-1} \frac{h_i(z-c)^{i-n}}{n-i} + h_n \ln(z-c) + K_2(z) \]

\[ = (z-c)^{-n}K_3(z) + h_n \ln(z-c) \]

where \( K_2(z) \) and \( K_2(z) \) are analytic in a neighborhood of \( z = c \). The second solution \( u_2(z) \) can then be written as

\[ u_2(z) = C(z)u_1(z) = \left( (z-c)^{-n}K_3(z) + h_n \ln(z-c) \right) (z-c)^{\alpha_1} f(z) \]

where \( f(z) \) and \( K_3(z) \) are analytic functions in a neighborhood of \( z = c \).

We conclude that, in both cases \((s = 0 \text{ and } s = n \in \mathbb{Z}_+)\), a logarithmic term appears, and the second solution can be written (remember \( s = \alpha_1 - \alpha_2 \))

\[ u_2(z) = (z-c)^{\alpha_2} g(z) + h_n \ln(z-c)(z-c)^{\alpha_1} f(z) \]

where

\[ f(z) = 1 + \sum_{n=1}^{\infty} a_n(z-c)^n \]

and the coefficients \( a_n \) are determined above in Theorem 2.3, and \( g(z) \) is an analytic function in a neighborhood of \( z = c \).

For convenience, we summarize the results in this section in a theorem.

**Theorem 2.4.** Denote by \( \alpha_1 \) and \( \alpha_2 \) the two roots of the indicial equation (2.12) to the differential equation (2.9), and number the roots such that \( \text{Re} \alpha_1 \geq \text{Re} \alpha_2 \). Then the two linearly independent solutions to (2.9) are:

1. If \( s = \alpha_1 - \alpha_2 \notin \mathbb{N} \)
   \[
   \begin{cases}
   u_1(z) = (z-c)^{\alpha_1} f_1(z) \\
   u_2(z) = (z-c)^{\alpha_2} f_2(z)
   \end{cases}
   \]
   where \( f_1(z) \) and \( f_2(z) \) are analytic functions in a neighborhood of \( z = c \). The explicit constructions of these analytic functions are made in Theorem 2.3.

2. If \( s = \alpha_1 - \alpha_2 \in \mathbb{N} \)
   \[
   \begin{cases}
   u_1(z) = (z-c)^{\alpha_1} f_1(z) \\
   u_2(z) = (z-c)^{\alpha_2} g(z) + h_n \ln(z-c)(z-c)^{\alpha_1} f_1(z)
   \end{cases}
   \]
   where \( f_1(z) \) and \( g(z) \) are analytic functions in a neighborhood of \( z = c \). The explicit construction of \( f_1(z) \) is made in Theorem 2.3.

**Comment 2.2.** The construction of the solutions \( u_1(z) \) and \( u_2(z) \) in a neighborhood of a regular singular point \( z = c \), see Theorem 2.4, can, of course, also be applied if the point \( z = c \) is regular. This becomes a special case of the above, simply by letting \( p_0 = q_0 = q_1 = 0 \). We then apply the result of the theorem with the roots of the indicial equation \( \alpha_1 = 1 \) and \( \alpha_2 = 0 \). The iterative scheme in (2.15) corresponding to \( \alpha_1 = 1 \) then becomes (Lemma 2.7 implies \( I(\alpha_1 + n) = n(n+1) \))
Due to Theorem 2.4, the power series solution $u_1(z)$ given by

$$u_1(z) = (z - c) \left( 1 + \sum_{n=1}^{\infty} a_n (z - c)^n \right)$$

is convergent in a neighborhood of the regular point $z = c$. Also, compare this result with the result of Problem 2.6.

### 2.5 Solution at a regular singular point at infinity

The point at infinity is different from all points in the finite complex plane, and it has to be analyzed as a special case. To find out the behavior of the differential equation at the point at infinity, introduce a new variable $\zeta$, defined as

$$\zeta = \frac{1}{z} \quad \Rightarrow \quad \frac{d}{d\zeta} = -\zeta^2 \frac{d}{dz} \quad \text{and} \quad \frac{d^2}{d\zeta^2} = \zeta^2 \frac{d}{d\zeta} \frac{d}{d\zeta} \zeta^2 \frac{d}{d\zeta} = \zeta^4 \frac{d^2}{d\zeta^2} + 2\zeta^3 \frac{d}{d\zeta}$$

The point at $z = \infty$ is then mapped to $\zeta = 0$, and in this new variable, $\zeta$, we can apply the results already obtained in the sections above to the origin $\zeta = 0$.

In fact, the differential equation (2.1) becomes

$$\zeta^4 \frac{d^2u}{d\zeta^2} + \left( 2\zeta^3 - \zeta^2 p(1/\zeta) \right) \frac{du}{d\zeta} + q(1/\zeta)u = 0$$

or

$$\frac{d^2u}{d\zeta^2} + \frac{1}{\zeta} \left( 2 - \frac{1}{\zeta} p(1/\zeta) \right) \frac{du}{d\zeta} + \frac{1}{\zeta^4} q(1/\zeta)u = 0 \quad (2.24)$$

and the behavior of the original equation at $z = \infty$ is transformed to an ordinary differential equation at a finite point $\zeta = 0$, which we know how to handle and classify.

We proceed by investigating the conditions that the coefficients have to satisfy in order to have a regular point, a regular singular point, or an irregular singular point at infinity, respectively. We state the result as a definition.

**Definition 2.5.** The point at infinity is classified as:

1. The point $z = \infty$ is a **regular point** of the differential equation (2.1) provided
2.5 Solution at a regular singular point at infinity

\[ \frac{2}{\zeta} - \frac{1}{\zeta^2} p(1/\zeta) \] is analytic at \( \zeta = 0 \) \iff \[ 2z - z^2 p(z) \] is analytic at \( z = \infty \)

which means that \( p(z) = 2/z + O(z^{-2}) \) as \( z \to \infty \),

and

\[ \frac{1}{\zeta^4} q(1/\zeta) \] is analytic at \( \zeta = 0 \) \iff \[ z^4 q(z) \] is analytic at \( z = \infty \)

which means that \( q(z) = O(z^{-4}) \) as \( z \to \infty \).

2. The point \( z = \infty \) is a regular singular point of the differential equation (2.1) provided

\[ 2 - \frac{1}{\zeta} p(1/\zeta) \] is analytic at \( \zeta = 0 \) \iff \[ z p(z) \] is analytic at \( z = \infty \)

which means that \( p(z) = O(z^{-1}) \) as \( z \to \infty \),

and

\[ \frac{1}{\zeta^2} q(1/\zeta) \] is analytic at \( \zeta = 0 \) \iff \[ z^2 q(z) \] is analytic at \( z = \infty \)

which means that \( q(z) = O(z^{-2}) \) as \( z \to \infty \).

3. The point \( z = \infty \) is an irregular singular point of the differential equation (2.1) in all other cases.

The analysis of a regular or singular point at infinity therefore is transformed to an investigation of the properties at the origin in the \( \zeta \) variable.

**Example 2.2.** The differential equation

\[ \frac{d^2 u(z)}{dz^2} + \frac{2}{z} \frac{du(z)}{dz} + \frac{1}{z^2} u(z) = 0 \]

has a singular point at \( z = \infty \). This point is a regular singular point.

**Example 2.3.** The differential equation

\[ \frac{d^2 u(z)}{dz^2} + \frac{2}{z} \frac{du(z)}{dz} + u(z) = 0 \]

has a singular point at \( z = \infty \). This point is an irregular singular point.
Problems

2.1. Prove the result of Theorem 2.1.

2.2. Prove, using the result of Problem 2.1, that if $u_1(z)$ is a solution (2.2)

$$\frac{d^2u_1(z)}{dz^2} + q_1(z)u_1(z) = 0$$

where $q_1(z)$ is analytic in a domain $S$, then ($b$ is a regular point of the differential equation)

$$u(z) = u_1(z) \exp\left\{-\frac{1}{2} \int_b^z p(z') \, dz'\right\}$$

solves (2.1)

$$\frac{d^2u(z)}{dz^2} + p(z) \frac{du(z)}{dz} + q(z)u(z) = 0$$

and vice versa, provided

$$q_1(z) = q(z) - \frac{1}{2} \frac{dp(z)}{dz} - \frac{1}{4} (p(z))^2$$

2.3. Check that

$$u_1(z) = \frac{\sin z}{z}$$

solves

$$\frac{d^2u_1(z)}{dz^2} + \frac{2}{z} \frac{du_1(z)}{dz} + u_1(z) = 0$$

Find a second, linearly independent solution to the differential equation.

2.4. One form of Lamé’s
differential equation reads ($a \neq b$ and $a, b \neq 0$)

$$\frac{d^2u(z)}{dz^2} + \left(\frac{z}{z^2-a^2} + \frac{z}{z^2-b^2}\right) \frac{du(z)}{dz} + \frac{k-m(m+1)z^2}{(z^2-a^2)(z^2-b^2)} u(z) = 0$$

Find its singular points and classify them, and determine the roots of the indicial equation. Moreover, find the Wronskian $W(z)$ of the two linearly independent solutions, $u_1(z)$ and $u_2(z)$, that satisfy

$$\begin{cases}
    u_1(0) = 1 \\
    u_1'(0) = 0
\end{cases} \quad \begin{cases}
    u_2(0) = 0 \\
    u_2'(0) = 1
\end{cases}$$

15 Gabriel Lamé (1795–1870), French mathematician.
16 Lamé’s differential equation is also briefly mentioned in Section 8.6.1.
17 A way of constructing these solutions is to use the result of Problem 2.6.
2.5 Electromagnetic scattering by a radially inhomogeneous sphere leads to a differential equation of the form
\[
\frac{d^2 u(z)}{dz^2} + \left( q_1(z) - \frac{l(l+1)}{z^2} \right) u(z) = 0
\]
where \( q_1(z) \) is an analytic function everywhere in the finite complex plane, and \( q_1(z) \to \text{constant} \) as \( |z| \to \infty \) and \( l \in \mathbb{Z}_+ \). Classify the points to the differential equation, and find the functional behavior of its solutions at the origin.

2.6. Let \( z = b \) be a regular point to the differential equation
\[
\frac{d^2 u(z)}{dz^2} + p(z) \frac{du(z)}{dz} + q(z) u(z) = 0
\]
where \( p(z) \) and \( q(z) \) have power series expansions
\[
\begin{align*}
p(z) &= \sum_{n=0}^{\infty} p_n (z-b)^n = p_0 + p_1 (z-b) + p_2 (z-b)^2 + \ldots \\
q(z) &= \sum_{n=0}^{\infty} q_n (z-b)^n = q_0 + q_1 (z-b) + q_2 (z-b)^2 + \ldots
\end{align*}
\]
which are convergent in a neighborhood of \( z = b \). Assume a power series expansion of the solution \( u(z) \)
\[
u(z) = \sum_{n=0}^{\infty} a_n (z-b)^n = a_0 + a_1 (z-b) + a_2 (z-b)^2 + \ldots
\]
and write the solution as
\[
u(z) = a_0 u_1(z) + a_1 u_2(z)
\]
Find the power series of \( u_1(z) \) and \( u_2(z) \), and prove that the series converge, and that the solutions \( u_1(z) \) and \( u_2(z) \) are linearly independent. **Hint:** Copy relevant parts of Section 2.4, and use the result of Theorem 2.2.
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