Chapter 2
Uniqueness and Uncertainty

We return to the basic problem \((P_0)\), which is at the core of our discussion,

\[
(P_0): \min_x \|x\|_0 \quad \text{subject to} \quad b = Ax.
\]

While we shall refer hereafter to this problem as our main goal, we stress that we are quite aware of its two major shortcomings in leading to any practical tool:

1. The equality requirement \(b = Ax\) is too strict, as there are small chances for any vector \(b\) to be represented by a few columns from \(A\). A better requirement would be one that allows for small deviation.
2. The sparsity measure is too sensitive to very small entries in \(x\), and a better measure would adopt a more forgiving approach towards such small entries.

Both these considerations will be included in later analysis, but for this to succeed, we must start with the stylized version of the problem as indeed posed by \((P_0)\).

For the underdetermined linear system of equations, \(Ax = b\) (a full-rank matrix \(A \in \mathbb{R}^{n \times m}\) with \(n < m\)), we pose the following questions:

Q1: When can uniqueness of the sparsest solution be claimed?
Q2: Can a candidate solution be tested to verify its (global) optimality?

This section addresses these questions, and some of their extensions. Rather than answering the above questions directly, we first consider special matrices \(A\) for which the analysis seems to be easier, and then extend our answers to the general \(A\). In doing so, we also follow the path taken by researchers who originally addressed these questions.

2.1 Treating the Two-Ortho Case

We shall first discuss the \((P_0)\) problem defined above in a concrete setting: the case where \(A\) is the concatenation of two orthogonal matrices, \(\Psi\) and \(\Phi\). As a classic
example, we can consider the amalgam of the identity and the Fourier matrices $A = [I, F]$. In such a setting, the fact that the system $b = Ax$ is underdetermined means, concretely, that there are many ways of representing a given signal $b$ as a superposition of spikes (i.e., columns from the identity matrix) and sinusoids (i.e., columns from the Fourier matrix). A sparse solution of such a system is a representation of said signal as a superposition of a few sinusoids and a few spikes. The uniqueness of such a sparse solution seemed surprising when first noticed.

### 2.1.1 An Uncertainty Principle

Before addressing sparse solutions for the linear system $[\Psi, \Phi] x = b$, we shall consider a (seemingly) different problem, inspired by the setting of classical uncertainty principles. As the reader no doubt knows, the classical uncertainty principle states that two conjugate variables (e.g., position and momentum, or any other pair coupled by the Fourier transform) cannot both be known with arbitrary precision. Turning to its mathematical formulation, it states that any function $f(x)$ and its Fourier transform $F(\omega)$ must obey the inequality\footnote{The bound 0.5 stated here depends on a specific definition of the Fourier transform.}

$$\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \cdot \int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2 d\omega \geq \frac{1}{2}, \quad (2.1)$$

where we have assumed that these functions are $\ell_2$-normalized,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 1.$$ 

This claim says that a signal cannot be tightly concentrated both in time and in frequency, and there is a lower bound on the product of the spread in time and the spread in frequency.

A comparable claim in our terminology would be that a signal cannot be sparsely represented both in time and in frequency. We now turn to develop this exact viewpoint, as it will be helpful for understanding some of the discussion that follows. Suppose we have a non-zero vector $b \in \mathbb{R}^n$ (a signal, say) and two orthobases $\Psi$ and $\Phi$. Then $b$ can be represented either as a linear combination of columns of $\Psi$ or as a linear combination of columns of $\Phi$:

$$b = \Psi \alpha = \Phi \beta. \quad (2.2)$$

Clearly, $\alpha$ and $\beta$ are uniquely defined. In a particularly important case, $\Psi$ is simply the identity matrix, and $\Phi$ is the matrix of the Fourier transform. Then $\alpha$ is the time-domain representation of $b$ while $\beta$ is the frequency-domain representation.

For an arbitrary pair of bases $\Psi, \Phi$, an interesting phenomenon occurs: either $\alpha$ can be sparse, or $\beta$ can be sparse, but not both! However, this claim is clearly
dependent on the distance between \( \Psi \) and \( \Phi \), since if the two are the same, we can easily construct \( b \) to be one of the columns in \( \Psi \) and get the smallest possible cardinality (being 1) in both \( \alpha \) and \( \beta \). Thus, we turn now to define the proximity between two bases by their mutual-coherence.

**Definition 2.1.** For an arbitrary pair of orthogonal bases \( \Psi, \Phi \) that construct the matrix \( A \), we define the mutual-coherence \( \mu(A) \) as the maximal inner product between columns from these two bases,

\[
\text{proximity}(\Psi, \Phi) = \mu(A) = \max_{1 \leq i, j \leq n} |\psi_i^T \phi_j|.
\]

(2.3)

The mutual-coherence of such two-ortho matrices satisfies \( \frac{1}{\sqrt{n}} \leq \mu(A) \leq 1 \), where the lower bound is achievable for certain pairs of orthogonal bases, such as the identity and the Fourier, the identity and the Hadamard, and more. To see that this is indeed the lower bound on the possible coherence, one simply notices that \( \Psi^T \Phi \) is an orthonormal matrix, having the sum of squares of its entries in each column equal to 1. All entries cannot therefore be less than \( \frac{1}{\sqrt{n}} \) since then we would have that the sum of all squared entries is less than 1. Using Definition 2.3 above, we have the following inequality result:

**Theorem 2.1.** For an arbitrary pair of orthogonal bases \( \Psi, \Phi \) with mutual-coherence \( \mu(A) \), and for an arbitrary non-zero vector \( b \in \mathbb{R}^n \) with representations \( \alpha \) and \( \beta \) correspondingly, the following inequality holds true:

\[
\text{Uncertainty Principle 1: } \|\alpha\|_0 + \|\beta\|_0 \geq \frac{2}{\mu(A)}.
\]

(2.4)

**Proof:** We shall assume hereafter, without loss of generality, that \( \|b\|_2 = 1 \). Since \( b = \Psi\alpha = \Phi\beta \), and \( b^T b = 1 \), we have that

\[
1 = b^T b = \alpha^T \Psi^T \Phi \beta
\]

(2.5)

\[
= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j \psi_i^T \phi_j \leq \mu(A) \cdot \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| \cdot |\beta_j|.
\]

Here we have exploited the definition of the coherence between the two bases. This inequality leads to

\[
1 \leq \mu(A) \cdot \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| \cdot |\beta_j| = \mu(A) \cdot \|\alpha\|_1 \|\beta\|_1.
\]

(2.6)

This could be interpreted as yet another uncertainty principle for the \( \ell_1 \)-norm case, suggesting that two representations cannot be jointly \( \ell_1 \)-short. Indeed, using the relation between the geometric and the algebraic means (\( \forall a, b \geq 0, \sqrt[4]{ab} \leq (a + b)/2 \)), we have
\[ \|\alpha\|_1 \|\beta\|_1 \geq \frac{1}{\mu(A)} \Rightarrow \|\alpha\|_1 + \|\beta\|_1 \geq \frac{2}{\sqrt[4]{\mu(A)}}. \]  

(2.7)

However, this is a distraction from our goal, and we return to an \( \ell_0 \)-based uncertainty principle.

Let us consider the following problem: Among all possible representations \( \alpha \) that satisfy \( \|\alpha\|_2 = 1 \) and have \( A \) non-zeros (i.e., \( \|\alpha\|_0 = A \)), what would be the one giving the longest \( \ell_1 \) length? This defines an optimization problem of the form

\[ \max_{\alpha} \|\alpha\|_1 \quad \text{subject to} \quad \|\alpha\|_2^2 = 1 \quad \text{and} \quad \|\alpha\|_0 = A. \]  

(2.8)

Suppose that this problem leads to a solution \( g(A) = g(\|\alpha\|_0) \). Similarly, a parallel definition for \( \beta \) with \( B \) non-zeros gives a result \( g(\|\beta\|_0) \). This means that using Equation (2.6) we have an inequality of the form

\[ \frac{1}{\mu(A)} \leq \|\alpha\|_1 \|\beta\|_1 \leq g(\|\alpha\|_0) \cdot g(\|\beta\|_0), \]  

(2.9)

since each \( \ell_1 \) length is replaced with its upper-bound. Such an inequality is our target, and thus solution of the problem posed in Equation (2.8) is needed.

We can assume, without loss of generality, that the \( A \) non-zeros in \( \alpha \) are its first entries, the rest being zeros. We further assume that all of these entries are strictly positive (since only absolute values are used in this problem). Using Lagrange multipliers, the \( \ell_0 \) constraint vanishes, and we obtain

\[ L(\alpha) = \sum_{i=1}^{A} \alpha_i + \lambda \left( 1 - \sum_{i=1}^{A} \alpha_i^2 \right). \]  

(2.10)

The derivative of this Lagrangian is

\[ \frac{\partial L(\alpha)}{\partial \alpha_i} = 1 - 2\lambda \alpha_i = 0, \]  

(2.11)

implying that the optimal values are given by \( \alpha_i = 1/2\lambda \), and are all the same. This means that the optimal value (due to the \( \ell_2 \) constraint) is \( \alpha_i = 1/\sqrt{A} \), and thus \( g(A) = A/\sqrt{A} = \sqrt{A} \) is the value of the maximal \( \ell_1 \)-norm of the vector \( \alpha \). Using this and a parallel result for \( \beta \), plugged into (2.9) leads to

\[ \frac{1}{\mu(A)} \leq \|\alpha\|_1 \|\beta\|_1 \leq g(\|\alpha\|_0) \cdot g(\|\beta\|_0) = \sqrt{\|\alpha\|_0 \cdot \|\beta\|_0}. \]  

(2.12)

Using again the geometric-algebraic mean relationship we obtain

\[ \frac{1}{\mu(A)} \leq \sqrt{\|\alpha\|_0 \cdot \|\beta\|_0} \leq \frac{1}{2} \left( \|\alpha\|_0 + \|\beta\|_0 \right), \]  

(2.13)

as claimed. \( \Box \)
An alternative and simpler proof (by Allan Pinkus) is the following: Since $\Phi$ and $\Psi$ are unitary matrices, we have that $\|b\|_2 = \|\alpha\|_2 = \|\beta\|_2$. Let us denote the support of $\alpha$ by $I$. From $b = \Psi\alpha = \sum_{i \in I} \alpha_i \psi_i$ we have

\[ |\beta_j|^2 = |b^T \phi_j|^2 = \left| \sum_{i \in I} \alpha_i \psi_i^T \phi_j \right|^2 \]

\[ \leq \|\alpha\|_2^2 \cdot \left| \sum_{i \in I} (\psi_i^T \phi_j)^2 \right| \]

\[ \leq \|b\|_2^2 \cdot |I| \cdot \mu(A)^2, \]

where we have used the Cauchy-Schwartz inequality,\(^2\) and the definition of the mutual-coherence. Summing the above over all $j \in J$, the support of $\beta$, we obtain

\[ \sum_{j \in J} |\beta_j|^2 = \|b\|_2^2 \leq \|b\|_2^2 \cdot |I| \cdot |J| \cdot \mu(A)^2. \] (2.15)

This leads to the inequality we have posed in Equation (2.13), thus proving the theorem. □

This result suggests that if the mutual-coherence of two bases is small, then $\alpha$ and $\beta$ cannot both be very sparse. For example, if, as above, $\Psi$ is the identity and $\Phi$ is the Fourier matrix, then $\mu([\Psi, \Phi]) = 1/\sqrt{n}$. It follows that a signal cannot have fewer than $2\sqrt{n}$ nonzeros in both the time and frequency-domains. In such a case, we also know that this is a tight relationship, since the picket-fence signal with a uniform spacing of $\sqrt{n}$ (assuming it is an integer) is converted by the Fourier transform (due to Poisson formula) to the same signal, thus accumulating $2\sqrt{n}$ non-zeros. Figure 2.1 shows this signal.

Heisenberg’s classical uncertainty principle, in the discrete setting, says that, if we view $\alpha$ and $\beta$ as probability distributions (by taking the absolute value of the entries and normalizing) then the product of their variances satisfies $\sigma_\alpha^2 \sigma_\beta^2 \geq \text{const.}$ In contrast, (2.4) gives a lower bound on the sum of the nonzeros, regardless of their locations.

### 2.1.2 Uncertainty of Redundant Solutions

We now make a connection to the uniqueness problem. Consider the problem of finding a solution to $Ax = [\Psi, \Phi]x = b$ in light of the uncertainty principle (2.4). Suppose there are two solutions, $x_1, x_2$ for the underlying linear system, and that one is very sparse. We will see that the other cannot be very sparse as well. Necessarily, the difference $e = x_1 - x_2$ must be in the null-space of $A$. Partition $e$ into sub-vectors

\[^2\] The Cauchy-Schwartz inequality is given by: $|x^T y|^2 \leq \|x\|_2^2 \cdot |y|^2$. Equality is obtained if and only if $x$ and $y$ are linearly-dependent.
e_\psi and e_\phi of the first n entries and last n entries of e, respectively. We have
\[ \Psi e_\psi = -\Phi e_\phi = y \neq 0. \quad (2.16) \]

The vector y is nonzero because e is nonzero, and both \( \Psi \) and \( \Phi \) are nonsingular. Now invoke (2.4):
\[ \|e\|_0 = \|e_\psi\|_0 + \|e_\phi\|_0 \geq \frac{2}{\mu(A)}. \quad (2.17) \]

Since \( e = x_1 - x_2 \), we have
\[ (\text{Uncertainty Principle 2}) : \quad \|x_1\|_0 + \|x_2\|_0 \geq \|e\|_0 \geq \frac{2}{\mu(A)}. \quad (2.18) \]

Here we have used the triangle inequality for the \( \ell_0 \) norm, \( \|x_1\|_0 + \|x_2\|_0 \geq \|x_1 - x_2\|_0 \), which is trivially verified, by counting the non-zeros of the two vectors and considering no overlap of supports (which leads to equality) and an overlap (which gives this inequality). To summarize, we have proven the following result:

**Theorem 2.2.** Any two distinct solutions \( x_1, x_2 \) of the linear system \( [\Psi, \Phi]x = b \) cannot both be very sparse, governed by the following uncertainty principle:
\[ (\text{Uncertainty Principle 2}) : \quad \|x_1\|_0 + \|x_2\|_0 \geq \frac{2}{\mu(A)}. \]

We refer to this result as an uncertainty of redundant solutions, as we discuss here solutions to the underdetermined system.
2.2 Uniqueness Analysis for the General Case

2.1.3 From Uncertainty to Uniqueness

A direct consequence of inequality (2.18) is a uniqueness result:

**Theorem 2.3.** If a candidate solution for $[Ψ, Φ]x = b$ has fewer than $1/µ(A)$ non-zeros, then it is necessarily the sparsest one possible, and any other solution must be “denser.”

This seemingly simple claim is wonderful and powerful. At least for the special case discussed here, $A = [Ψ, Φ]$, we have a complete answer for the two questions we have posed at the beginning of this chapter. Namely, we can certainly claim uniqueness for sparse enough solutions, and once such a sufficiently sparse solution is given to us, we can immediately claim its global optimality. Notice that in general non-convex optimization problems, a given solution can at best be verified as being locally optimal, and here we have the ability to verify its optimality globally.

Since the discussion so far concentrated on the two-ortho case, it is now time to dare and address general matrices $A$, using similar treatment. Clearly, though, an uncertainty result of the kind posed in Theorem 1 would be impossible to imitate, and we will have to bypass it somehow.

2.2 Uniqueness Analysis for the General Case

2.2.1 Uniqueness via the Spark

A key property that is crucial for the study of uniqueness is the *spark* of the matrix $A$, a term coined and defined by Donoho and Elad in 2003. The spark is a way of characterizing the null-space of a matrix $A$ using the $ℓ_0$-norm. We start with the following definition:

**Definition 2.2.** The spark of a given matrix $A$ is the smallest number of columns from $A$ that are linearly-dependent.

Recall that the rank of a matrix is defined as the *largest* number of columns from $A$ that are linearly independent. Clearly, there is a resemblance between these two definitions – replace largest with smallest, and independent by dependent, and you return to the definition of the spark. Nevertheless, the spark of a matrix is far more difficult to obtain, compared to its rank, as it calls for a combinatorial search over all possible subsets of columns from $A$.

The importance of this property of matrices for the study of the uniqueness of sparse solutions has been unraveled already by Rao and Gorodnitsky in 1998. Interestingly, this property has previously appeared in the literature of psychometrics (termed *Kruskal rank*), used in the context of studying uniqueness of tensor decomposition. The spark is also related to established notions in matroid theory; formally it is precisely the girth of the linear matroidmatroid defined by $A$, i.e., the length of
the shortest cycle in that matroid. Finally, if we consider the same definition where the arithmetic underlying the matrix product is performed not over the fields of real or complex numbers but instead over the ring of integers mod \( q \), the same quantity arises in coding theory, where it allows to compute the minimum distance of a code. The resemblance between all these concepts is striking and instructive.

The spark gives a simple criterion for uniqueness of sparse solutions. By definition, the vectors in the null-space of the matrix \( Ax = 0 \) must satisfy \( \|x\|_0 \geq \text{spark}(A) \), since these vectors combine linearly columns from \( A \) to give the zero vector, and at least \( \text{spark} \) such columns are necessary by definition. Using the \( \text{spark} \) we obtain the following result:

**Theorem 2.4. (Uniqueness – Spark):** If a system of linear equations \( Ax = b \) has a solution \( x \) obeying \( \|x\|_0 < \text{spark}(A)/2 \), this solution is necessarily the sparsest possible.

**Proof:** Consider an alternative solution \( y \) that satisfies the same linear system \( Ay = b \). This implies that \( x - y \) must be in the null-space of \( A \), i.e., \( A(x - y) = 0 \). By definition of \( \text{spark} \),

\[
\|x\|_0 + \|y\|_0 \geq \|x - y\|_0 \geq \text{spark}(A). \tag{2.19}
\]

The left-most term in the above inequality simply states that the number of non-zeros in the difference vector \( x - y \) cannot exceed the sum of the number of non-zeros within each of the vectors \( x \) and \( y \) separately – this is the triangle inequality mentioned before. Since we have a solution satisfying \( \|x\|_0 < \text{spark}(A)/2 \), we conclude that any alternative solution necessarily has more than \( \text{spark}(A)/2 \) non-zeros, as claimed. \( \square \)

Again, this result is very elementary and yet quite surprising, bearing in mind that \( (P_0) \) is a highly complicated optimization task of combinatorial flavor. In general combinatorial optimization problems, when considering a proposed solution, one hopes only to check local optimality – i.e., that no simple modification gives a better result. Here, we find that simply checking the solution sparsity, and comparing that with the \( \text{spark} \), lets us check global optimality.

Clearly, the value of \( \text{spark} \) can be very informative, and large values of \( \text{spark} \) are evidently very useful. How large can \( \text{spark} \) be? By definition, \( \text{spark} \) must be in the range \( 2 \leq \text{spark}(A) \leq n + 1 \). For example, if \( A \) comprises random independent and identically distributed entries (say Gaussian), then with probability 1 we have \( \text{spark}(A) = n + 1 \), implying that no \( n \) columns are linearly-dependent. The same spark is obtained for Vandermonde matrices constructed from distinct \( m \) scalars. In these cases, uniqueness is ensured for every solution with \( n/2 \) or fewer non-zero entries. Similarly, the \( \text{spark} \) of the two-ortho identity-Fourier pair is \( 4 \sqrt{n} \), using

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\(^3\) The \( \text{spark} \) can be as low as 1 if there exists a zero-column in \( A \), but we do not consider such a case as relevant to our analysis.

\(^4\) If \( n \) is prime, the \( \text{spark} \) of the identity-Fourier pair becomes \( n + 1 \), as the picket-fence signals are no longer relevant.
Poisson formula – a concatenation of two picket-fence signals, each with $\sqrt{n}$ peaks evenly spread is the sparsest possible vector in the null-space of the matrix $[\textbf{I}, \textbf{F}]$.

### 2.2.2 Uniqueness via the Mutual-Coherence

The *spark* is at least as difficult to evaluate as solving $(P_0)$. Thus, simpler ways to guarantee uniqueness are of interest. A very simple way exploits the mutual-coherence of the matrix $\textbf{A}$, defined by generalizing the definition given for the two-ortho case. In the two ortho case, computation of the Gram matrix $\textbf{A}^T\textbf{A}$ leads to

$$
\textbf{A}^T\textbf{A} = \begin{bmatrix}
\textbf{I} & \Psi^T\Phi \\
\Phi^T\Psi & \textbf{I}
\end{bmatrix}.
$$

(2.20)

Thus, the above-defined mutual-coherence for this case is obtained as the maximal off-diagonal entry (in absolute value) in this Gram matrix. Similarly, we propose a generalization of this definition, as follows:

**Definition 2.3.** The mutual-coherence of a given matrix $\textbf{A}$ is the largest absolute normalized inner product between different columns from $\textbf{A}$. Denoting the $k$-th column in $\textbf{A}$ by $a_k$, the mutual-coherence is given by

$$
\mu(\textbf{A}) = \max_{1 \leq i, j \leq m, i \neq j} \frac{|a_i^T a_j|}{\|a_i\|_2 \cdot \|a_j\|_2}.
$$

(2.21)

The mutual-coherence is a way to characterize the dependence between columns of the matrix $\textbf{A}$. For a unitary matrix, columns are pairwise orthogonal, and so the mutual-coherence is zero. For general matrices with more columns than rows, $m > n$, $\mu$ is necessarily strictly positive, and we desire the smallest possible value so as to get as close as possible to the behavior exhibited by unitary matrices.

We have seen that for structured two-ortho matrices $\textbf{A} = [\Psi, \Phi]$ the mutual-coherence satisfies $1/\sqrt{n} \leq \mu(\textbf{A}) \leq 1$. When considering random orthogonal matrices of size $n \times m$, the work in Donoho and Huo has shown that they tend to be incoherent, implying that $\mu(\textbf{A}_{n,m})$ is typically proportional to $\sqrt{\log(nm)/n}$ for $n \to \infty$. It has been shown that for full-rank matrices of size $n \times m$ the mutual-coherence is bounded from below by

$$
\mu \geq \sqrt{\frac{m-n}{n(m-1)}}.
$$

Equality is obtained for a family of matrices named *Grassmannian Frames*. The set of columns in such matrices are called equiangular lines. Indeed, this family of matrices has $\text{spark}(\textbf{A}) = n+1$, the highest value possible. Numerical construction of such matrices has been addressed by Tropp et al. using an iterative projection method.
onto (sometimes not so) convex sets. We will return to this topic towards the end of this chapter.

We also mention work by Calderbank in quantum information theory, constructing error-correcting-codes using a collection of orthogonal bases with minimal coherence, obtaining similar bounds on the mutual-coherence for amalgams of orthogonal bases. Also related to this line of activity is the more recent contribution by Sochen, Gurevitz, and Hadani, on constructions of signal sets such that their shifts exhibit low coherence.

Mutual-coherence is relatively easy to compute, and as such, it allows us to lower-bound the \( \text{spark} \), which is often hard to compute.

**Lemma 2.1.** For any matrix \( A \in \mathbb{R}^{n \times m} \), the following relationship holds:

\[
\text{spark}(A) \geq 1 + \frac{1}{\mu(A)}.
\]

**(Proof):** First, modify the matrix \( A \) by normalizing its columns to be of unit \( \ell_2 \)-norm, obtaining \( \tilde{A} \). This operation preserves both the \( \text{spark} \) and the mutual-coherence. The entries of the resulting Gram matrix \( G = \tilde{A}^T \tilde{A} \) satisfy the following properties:

\[
\{ G_{k,k} : 1 \leq k \leq m \} \quad \text{and} \quad \{ G_{k,j} \leq \mu(A) : 1 \leq k, j \leq m, k \neq j \},
\]

Consider an arbitrary leading minor from \( G \) of size \( p \times p \), built by choosing a sub-group of \( p \) columns from \( \tilde{A} \) and computing their sub-Gram matrix. From the Gershgorin disk theorem,\(^5\) if this minor is diagonally-dominant — i.e., if \( \sum_{j \neq i} |G_{i,j}| < |G_{i,i}| \) for every \( i \) — then this sub-matrix of \( G \) is positive-definite, and so those \( p \) columns from \( \tilde{A} \) are linearly-independent. The condition \( 1 > (p - 1)\mu \rightarrow p < 1 + 1/\mu \) implies positive-definiteness of every \( p \times p \) minor. Thus, \( p = 1 + 1/\mu \) is the smallest possible number of columns that might lead to linear dependence, and thus \( \text{spark}(A) \geq 1 + 1/\mu \).

We have the following analog of the previous uniqueness theorems, and this time based on the mutual-coherence.

**Theorem 2.5.** (Uniqueness – Mutual-Coherence): If a system of linear equations \( Ax = b \) has a solution \( x \) obeying \( \|x\|_0 < \frac{1}{2} \left( 1 + 1/\mu(A) \right) \), this solution is necessarily the sparsest possible.

Compare Theorems 2.4 and 2.5. They are parallel in form, but with different assumptions. In general, Theorem 2.4, which uses \( \text{spark} \), is sharp and is far more powerful than Theorem 2.5, which uses the coherence and so only a lower bound on \( \text{spark} \). The coherence can never be smaller than \( 1/\sqrt{n} \), and therefore, the cardinality bound of Theorem 2.5 is never larger than \( \sqrt{n}/2 \). However, the \( \text{spark} \) can easily be as large as \( n \), and Theorem 2.4 then gives a bound as large as \( n/2 \).

\(^5\) For a general (possibly complex) matrix \( H \) of size \( n \times n \), Gershgorin’s disks are the \( n \) disks formed by the centers \( h(i, i) \) and radiuses \( \sum_{j \neq i} |h(i, j)| \). The theorem states that all eigenvalues of \( H \) must lie within the union of these disks.
In fact, for the special matrix $A = [\Psi, \Phi]$ we have also obtained such a rule. Interestingly, the lower bound for the general case becomes $(1 + 1/\mu(A))/2$ while the special two-ortho case gave a stronger (i.e., higher) lower bound. The general case bound is nearly a factor of 2 weaker than (2.18), because (2.18) uses the special structure $A = [\Psi, \Phi]$.

### 2.2.3 Uniqueness via the Babel Function

In the proof of Lemma 2.1 we considered minors of size $p \times p$ extracted from the Gram matrix of the normalized matrix $\tilde{A}$. The positive-definiteness of all such minors imply that every $p$ columns are linearly-independent. However, for simplification of the analysis we bounded all the off-diagonal entries of $G$ by a single value $\mu(A)$, and thus, lost robustness to possibly few extreme entries in this matrix.

Since we need to check whether the sum of the $p - 1$ off-diagonal entries in every row in these minors is less than 1 (for the Gershgorin property to hold true), we can define the following Babel function, following Tropp:

**Definition 2.4.** For a given matrix $\tilde{A}$ with normalized columns, we consider a subset $\Lambda$ of $p$ columns from $\tilde{A}$, and compute the sum of the absolute values of their inner product with a column outside this set. Maximizing over both the set $\Lambda$ and the outside column $j$ we obtain the Babel function:

$$
\mu_1(p) = \max_{\Lambda, |\Lambda|=p} \max_{j \notin \Lambda} \sum_{i \in \Lambda} |\tilde{a}_i^T \tilde{a}_j|.
$$

(2.23)

Clearly, for $p = 1$, we get that $\mu_1(1) = \mu(A)$. For $p = 2$ the above implies that we need to sweep through all possible triplets, considering two as those belonging to $\Lambda$, and the third as the external vector to compute inner products with. This definition implies that this function is monotonically non-decreasing, and the slower its growth the better the analysis it leads to, compared to the use of the cruder coherence measure.

It might seem that computing this function for large $p$ becomes exponential and thus prohibitive, but this is in fact not true. By computing $|G| = |\tilde{A}^T \tilde{A}|$, and sorting every row in descending order, we obtain the matrix $G_S$. The first entry in every row is 1, being the main diagonal entry, and thus should be disregarded. Computing the sum of the $p$ leading entries in every row (second and beyond), we get for every $j$ in the above definition the set of worst $\Lambda$’s, and among them we should choose the maximal one, thus

$$
\mu_1(p) = \max_{1 \leq j \leq m} \sum_{i=2}^{p+1} |G_S(j, i)|.
$$

(2.24)
We note that for every \( p \) we have that \( \mu_1(p) \leq p \cdot \mu(A) \), and the two become equal for Grassmannian matrices, where all the off-diagonal entries in the Gram matrix are of the same magnitude.

How can we use the Babel function for better assessing the uniqueness? It is clear that if \( \mu_1(p) < 1 \), we deduce that all \( p + 1 \) sets are linearly-independent. Thus, a lower-bound on the spark could be

\[
\text{spark}(A) \geq \min_{1 \leq p \leq n} \{ p | \mu_1(p - 1) \geq 1 \}.
\]

(2.25)

The uncertainty and uniqueness properties follow immediately.

### 2.2.4 Upper-Bounding the Spark

The spark is impossible to evaluate in general, as it is even harder than solving \((P_0)\). This is because its evaluation requires a sweep through all possible groups of columns from \( A \) with varying cardinalities, seeking for a linearly-dependent subset of columns. This is a combinatorial search of exponential complexity with \( m \).

While this difficulty explains the necessity to replace the spark with the mutual-coherence, the price paid in loss of tightness of the uniqueness result may be considered as too dear to be permitted. This motivates an alternative method for approximating the spark, and this time using an upper bound. Such a bound implies that we cannot guarantee uniqueness based on the obtained value, but it would give us a rough evaluation of the region of uniqueness.

In order to develop the upper-bound, we redefine the spark as the outcome of \( m \) optimization problems \((P_{0i})\) for \( i = 1, 2, \ldots, m \) of the form:

\[
(P_{0i}^i) : \quad \mathbf{x}_{opt}^i = \arg \min_{\mathbf{x}} \|\mathbf{x}\|_0 \text{ subject to } A\mathbf{x} = \mathbf{0} \text{ and } x_i = 1.
\]

(2.26)

Each of these problems assumes that the sparsest vector in the null-space of \( A \) uses the \( i \)-th entry. By solving this sequence of \( (P_0)\)-like problems, the sparsest result among \( \{x_{opt}^i\}_{i=1}^m \) gives the spark,

\[
\text{spark}(A) = \min_{1 \leq i \leq m} \|x_{opt}^i\|_0.
\]

(2.27)

Since the set of problems \((P_{0i}^i)\) is too complex, we define an alternative set of problems, replacing the \( \ell_0 \) with the \( \ell_1 \)-norm,

\[
(P_{1i}^i) : \quad \mathbf{z}_{opt}^i = \arg \min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ subject to } A\mathbf{x} = \mathbf{0} \text{ and } x_i = 1.
\]

(2.28)

As we have seen in Chapter 1, these problems have a linear programming structure, they are convex, and solvable in reasonable time. Furthermore, it is clear that for every \( i \), we have that \( \|x_{opt}^i\|_0 \leq \|z_{opt}^i\|_0 \), since \( \|x_{opt}^i\|_0 \) is the sparsest solution possible for this problem by definition, and thus we have the promised bound
2.3 Constructing Grassmannian Matrices

\[ \text{spark}(A) \leq \min_{1 \leq i \leq m} \|z_{opt}^i\|_0. \quad (2.29) \]

Numerical experiments show that this bound tends to be quite tight, and close to the true \textit{spark}.

2.3 Constructing Grassmannian Matrices

A Grassmannian (real) matrix \( A \) of size \( n \times m \) with \( m \geq n \) is a matrix with normalized columns such that its Gram matrix \( G = A^T A \) satisfies

\[ \forall k \neq j, |G_{k,j}| = \sqrt{\frac{m-n}{n(m-1)}}. \quad (2.30) \]

As we have stated above, this is the smallest possible mutual-coherence possible. Such matrices do not necessarily exist, and in particular, they are possible only if \( m < \min(n(n+1)/2, (m-n)(m-n+1)/2) \).

Grassmannian matrices are special in the sense that the angle between each and every pair of columns in it is the same, and it is also the smallest possible. Thus, construction of such matrices has a strong connection with packing of vectors/subspaces in the \( \mathbb{R}^n \)-space. While the case \( m = n \) leads trivially to unitary matrices that are easy to construct, constructing a general Grassmannian matrix is very hard. A numerical algorithm for this task was proposed by Tropp et al. and we bring it here as an illustration of such design procedures. We should add that while image processing finds little interest in such constructions, they find important application in channel coding, wireless communication, and more.

The key idea in the proposed algorithm is to iterate between projections onto the requirements such matrix should satisfy. Starting with an arbitrary matrix \( A \), these projections should refer to the following requirements:

1. The columns in \( A \) are \( \ell_2 \)-normalized: This can be forced by normalizing every column.
2. Property (2.30): We should compute \( G = A^T A \), detect the off-diagonal entries that are above some threshold, and decrease them. Similarly, since too small values are also not permitted in such Gram matrices, one might also add an increase to such off-diagonal entries.
3. The rank of \( G \) should not exceed \( n \): Since the above modifications cause \( G \) to become a full-rank one, an SVD operation and truncation of the singular-values beyond the first \( n \) ones brings the new Gram matrix to the proper rank.

This process can and should be repeated many times. There is no guarantee for convergence, or arrival at a Grassmannian matrix, but tests show that one can get closer to such matrices with this numerical scheme. A Matlab code that follows these guidelines is given in Figure 2.2.
D=randn(N,L); % initialization
D=D*diag(1./sqrt(diag(D'*D))); % normalize columns
G=D'*D; % compute the Gram matrix
mu=sqrt((L-N)/N/(L-1));
for k=1:1:Iter,
    % shrink the high inner products
    gg=sort(abs(G(:)));
    pos=find(abs(G(:))>gg(round(dd1*(L*L-L))) & abs(G(:))) <1;
    G(pos)=G(pos)*dd2;
    % reduce the rank back to N
    [U,S,V]=svd(G);
    S(N+1:end,1+N:end)=0;
    G=U*S*V;
    % Normalize the columns
    G=diag(1./sqrt(diag(G)))*G*diag(1./sqrt(diag(G)));
    % Show status
    gg=sort(abs(G(:)));
    pos=find(abs(G(:))>gg(round(dd1*(L*L-L))) & abs(G(:))) <1;
    disp([k,mu,mean(abs(G(pos))),max(abs(G(pos)))]);
end;
[U,S,V]=svd(G);
D=sqrt(S(1:N,1:N))*U(:,1:N)´;

Fig. 2.2 Matlab code for building a Grassmannian matrix.

Figure 2.3 shows the results for this procedure. For a matrix of size $50 \times 100$, the minimal possible coherence is $\sqrt{1/99} = 0.1005$. Figure 2.3 shows the initial Gram matrix and the one obtained after 10,000 iterations, using $dd1 = dd2 = 0.9$. Figure 2.4 shows the sorted off-diagonal entries in these two Gram matrices, and as can be seen, the iterative procedure succeeds very well in obtaining a matrix very close to a Grassmannian one. In fact, in this run, the maximal off-diagonal entry is 0.119. Figure 2.5 presents the mutual-coherence as it evolves through the iterations of the algorithm.

2.4 Summary

We have now given answers to the questions posed at the start of this chapter. We have seen that any sufficiently sparse solution is guaranteed to be unique among all possible solutions. Consequently, any sufficiently sparse solution is necessarily the global optimizer of $(P_0)$. These results show that searching for a sparse solution can lead to a well-posed question with interesting properties. We now turn to discuss practical methods for obtaining solutions for $(P_0)$.

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6 Courtesy of Joel Tropp.
Fig. 2.3 The initial Gram matrix (top-left) and the final one (top-right) in training a Grassmannian matrix of size $50 \times 100$. The bottom part of the figure shows the absolute of the final Gram matrix, showing that the off-diagonal elements tend to have the same value, as required.

Further Reading

Fig. 2.4 The sorted off-diagonal entries in the initial Gram matrix and the final one.

Fig. 2.5 The evolved *mutual-coherence* of the matrix as a function of the iteration.


