Preface: Painlevé’s Problem

Let $K$ be a compact subset of the complex plane. Call $K$ removable for bounded analytic functions, or more concisely removable, if for each open superset $U$ of $K$ in the complex plane, each function that is bounded and analytic on $U \setminus K$ extends across $K$ to be analytic on the whole of $U$. In 1888 Paul Painlevé became the first to seriously investigate the nature of removable sets in his thesis [PAIN]. Because of this the removable subsets of the complex plane are often referred to as Painlevé null sets and the task of giving them a “geometric” characterization has come to be known as Painlevé’s Problem. In addition to being an academic, Painlevé was also a politician and statesman who served as War Minister and Prime Minister of France at various times in his life. For more on this interesting and multifaceted individual see Section 6 of Chapter 5 of [PAJ2].

The notion of “geometric” here is unavoidably vague and intuitive. On the one hand, a necessary but not sufficient condition for such a characterization is that it should make no reference to analytic functions. On the other hand, a sufficient but not necessary condition for such a characterization is that it be couched in terms of the cardinality of $K$ or the topological, metric, or rectifiability properties of $K$. At the very end of this book the following question will command our attention: Should a characterization involving totally arbitrary measures be counted as “geometric”?

The goal of this book is to present a complete proof of the recent affirmative resolution of a special case of Painlevé’s Problem known as Vitushkin’s Conjecture. This conjecture states that a compact set with finite linear Hausdorff measure is removable if and only if it intersects every rectifiable curve in a set of zero arclength measure. We note in passing that arclength measure here can be replaced by linear Hausdorff measure since the two have the same zero sets among subsets of rectifiable curves. More importantly, we note that the forward implication of Vitushkin’s Conjecture is equivalent to an earlier conjecture about a still more special case of Painlevé’s Problem known as Denjoy’s Conjecture. This conjecture states that a compact subset of a rectifiable curve with positive arclength measure is non-removable. So to prove Vitushkin’s Conjecture, we must also prove Denjoy’s Conjecture.

To understand this book a prospective reader should have a firm grasp of the first 14 chapters of Walter Rudin’s Real and Complex Analysis, 3rd Edition (hereafter referred to as [RUD]). Indeed, the author has somewhat eccentrically sought to make
this book, when used in conjunction with [RUD], entirely self-contained. Thus any
standard result of analysis which is needed but is not contained in [RUD] is proved
in this book (e.g., Besicovitch’s Covering Lemma), and conversely, any standard
result of analysis which is needed and is contained in [RUD] is always given a
citation from [RUD] (e.g., Lebesgue’s Dominated Convergence Theorem). Another
eccentricity of the book is a deliberate exclusion of figures but an equally delib-
erate inclusion of verbal descriptions precise enough to enable an attentive reader
to reconstruct the excluded figures. To a great extent the author wrote this book to
convince himself of the truth of Vitushkin’s Conjecture “beyond a reasonable doubt”
and so has elected to err on the side of too much detail rather than too little. Finally,
the author believes his notation is fairly standard or obvious but has nevertheless
spelled out the meaning of a number of symbols upon first use and appended a
symbol glossary and list to the back of the book for the reader’s convenience.

We now turn to detailing the contents of the book, chapter by chapter.

Chapter 1 introduces and then proves various standard elementary results about
the notions of removability and analytic capacity. The analytic capacity of a compact
subset \( K \) of the complex plane is a nonnegative number \( \gamma(K) \) which can be thought
of as a quantitative measure of removability/nonremovability since \( K \) is removable
if and only if \( \gamma(K) = 0 \). This result does not solve Painlevé’s Problem since \( \gamma(K) \)
is not a geometric quantity – its definition (see Section 1.2) involves suping over a
space of bounded analytic functions!

Chapter 2 introduces the notions of \( s \)-dimensional Hausdorff measure \( H^s \) and
Hausdorff dimension \( \dim H \) – these are not dealt with in [RUD] – and then relates
them to removability. It turns out that a result of Painlevé implies that a compact \( K \) is
removable whenever \( \dim H(K) < 1 \) and a result of Frostman implies that a compact
\( K \) is nonremovable whenever \( \dim H(K) > 1 \). So Painlevé’s Problem is reduced
to determining the removability of those compact \( K \) for which \( \dim H(K) = 1 \). At
the end of this chapter a natural conjecture presents itself which would finish off
Painlevé’s Problem if true. It is couched in terms of \( H^1 \) but is summarily slain by a
counterexample!

Chapter 3 proves a special case of Garabedian duality needed for our proof of
Denjoy’s Conjecture. Analytic capacity, whose definition involves suping over a
space of bounded analytic functions, is an \( L^\infty \) object. It has an \( L^2 \) analog and
Garabedian duality asserts that these two capacities, one \( L^\infty \) and the other \( L^2 \), are
related in a manner that makes it clear that they vanish for the same sets. The impor-
tance of Garabedian duality is that it thus allows us to use Hilbert space methods to
study an \( L^\infty \) problem – it is frequently easier to estimate an \( L^2 \) norm than it is to
estimate an \( L^\infty \) norm.

Chapter 4 introduces the notion of the Melnikov curvature of a measure and
the notion of a measure with linear growth. Garabedian duality is then used to
prove a result called Melnikov’s Lower Capacity Estimate. Given a compact set
supporting a nontrivial positive Borel measure with finite Melnikov curvature and
linear growth, this estimate gives a positive lower bound on the analytic capacity of
the set in terms of the Melnikov curvature, the linear growth bound, and the mass of
the measure. Of course this quantitative result trivially implies a qualitative one: a compact set which supports a nontrivial positive Borel measure with finite Melnikov curvature and linear growth is nonremovable. A Fourier transform argument due to Mark Melnikov and Joan Verdera is then given that shows that Lipschitz graphs support many such measures. After some preliminaries dealing with arclength and arclength measure, these two results combine to give a nice proof of Denjoy’s Conjecture. At the end of this chapter a natural conjecture presents itself which would finish off Painlevé’s Problem if true. It is couched in terms of rectifiable curves but meets the same fate as the earlier conjecture, i.e., it is summarily slain by a counterexample!

Chapter 5 is a grab bag of the measure theory needed to carry us forward. Amazingly, up to this point in the book it has sufficed to just know that $s$-dimensional Hausdorff measure is an outer measure defined on all subsets of the complex plane! Not so for what follows where we must know that it is an honest-to-god measure on a $\sigma$-algebra of subsets containing the Borel sets. The chapter has more in it than one would expect. The reason is that measures in [RUD] are typically obtained via the Riesz Representation Theorem and, in consequence, always put finite mass on any compact set. This is a property that $s$-dimensional Hausdorff measure on the complex plane has only when $s = 2$. So we cannot simply rely on [RUD] here for our measure theory.

Chapter 6 has a proof of Vitushkin’s Conjecture modulo two difficult results. The next two chapters, comprising roughly half the book, are taken up with proving these results.

Chapter 7 has a proof of the first difficult result, a $T(b)$ theorem due to Fedor Nazarov, Sergei Treil, and Alexander Volberg for measures that need not satisfy a doubling condition. The complexity of this proof precludes us from saying anything enlightening about it just now.

Chapter 8 has a proof of the second difficult result, a curvature theorem for arbitrary measures due to Guy David and Jean-Christophe Léger. The complexity of this proof precludes us from saying anything enlightening about it just now.

With the end of Chapter 8, the goal of this book, the presentation of a complete proof of Vitushkin’s Conjecture, has been achieved. But Vitushkin’s Conjecture, although a big part of Painlevé’s Problem, is not all of it. With the affirmative resolution of Vitushkin’s Conjecture, Painlevé’s Problem has been reduced to determining the removability of those compact sets $K$ for which $\dim_{H^s}(K) = 1$ but $H^1(K) = \infty$. A Postscript following Chapter 8 seeks to shed some light on these sets. This Postscript deals with two items: first, the extension of Vitushkin’s Conjecture to compact sets that are $\sigma$-finite for $H^1$, and second, a conjecture due to Melnikov which essentially says that the qualitative consequence of Melnikov’s Lower Capacity Estimate mentioned a few paragraphs ago is reversible. Both of these matters are resolved affirmatively with the aid of a quite recent and deep theorem, which we state but do not prove, due to Xavier Tolsa.

In writing this book the author has found three useful sources on Hausdorff measure and dimension: [ROG], [FALC], and [MAT3]. These items have been listed in order of increasing depth. For the purposes of this book [FALC] proved to be
ideal. The author was also helped by several excellent survey articles dealing with the status of Painlevé’s Problem and the various subproblems it has spawned. These are, in chronological order: [MARSH], [VER], [MAT5], [MAT6], [DAV2], [DAV3], [TOL4], and [PAJ3]. The author is also indebted to two books that are of a much more comprehensive scope than this one but deal with Painlevé’s Problem: [GAR2], from the pre-Melnikov-curvature era, and [PAJ2], from the post-Melnikov-curvature era. Finally, it should be noted that [MAT3], a very comprehensive and deep book on Hausdorff measure and rectifiability that appeared at the cusp between the two eras, has an excellent chapter devoted to the status of Painlevé’s Problem at that time. These sources also have superb and complete bibliographies. The bibliography of this book, being restricted solely to those articles and books that the author found necessary to cite, is spare by comparison.

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