Chapter 2
Removable Sets and Hausdorff Measure

2.1 Hausdorff Measure and Dimension

At a fuzzy intuitive level, removable sets have small “size” and nonremovable sets big “size.” A precise notion of “size” applicable to arbitrary subsets of \( \mathbb{C} \) and appropriate to our problem is given by Hausdorff measure (and Hausdorff dimension). So in this section we will simply introduce Hausdorff measure as a gauge of the smallness of a set and as a necessary preliminary for another such gauge, Hausdorff dimension. Surprisingly, the assertions 2.1 through 2.4 below are enough to get us through to the end of Chapter 4. It is only after, in Section 5.1, that we shall need to take up the fact that Hausdorff measure is indeed a positive measure defined on a \( \sigma \)-algebra containing the Borel subsets of \( \mathbb{C} \).

Given an arbitrary subset \( E \) of \( \mathbb{C} \) and \( \delta > 0 \), a \( \delta \)-cover of \( E \) is simply a countable collection of subsets \( \{ U_n \} \) of \( \mathbb{C} \) such that \( E \subseteq \bigcup_n U_n \) and \( 0 < |U_n| < \delta \) for each \( n \). For any \( s \geq 0 \), define

\[
\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_n |U_n|^s : \{U_n\} \text{ is a } \delta \text{-cover of } E \right\}.
\]

Clearly \( \mathcal{H}^s_\delta(E) \) increases as \( \delta \) decreases and so converges to a limit in \([0, \infty]\) as \( \delta \downarrow 0 \). This limit is called the \( s \)-dimensional Hausdorff measure (or \( s \)-dimensional Hausdorff–Besicovitch measure) of \( E \) and denoted \( \mathcal{H}^s(E) \). Thus

\[
\mathcal{H}^s(E) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E).
\]

Since the diameters of a set, its convex hull, its closure, and its closed convex hull are the same, \( \mathcal{H}^s(E) \) may be computed by restricting ones attention to \( \delta \)-covers of \( E \) by convex, closed, or closed convex sets. Similarly, since any nonempty set \( U \) is contained in the open set \( \{ z : \text{dist}(z, U) < \varepsilon \} \) whose diameter is \( |U| + 2\varepsilon \) and since \( \{ z : \text{dist}(z, U) < \varepsilon \} \) is convex whenever \( U \) is convex, \( \mathcal{H}^s(E) \) may be computed by restricting ones attention to \( \delta \)-covers of \( E \) by open or open convex sets. Lastly, when
Proposition 2.1 For any $s \geq 0$, $\mathcal{H}^s$ is a set function defined on all subsets of $\mathbb{C}$ and taking values in $[0, \infty]$ such that

$$\mathcal{H}^s(\emptyset) = 0,$$

$$\mathcal{H}^s(E) \leq \mathcal{H}^s(F) \text{ whenever } E \subseteq F \subseteq \mathbb{C},$$

and

$$\mathcal{H}^s \left( \bigcup_n E_n \right) \leq \sum_n \mathcal{H}^s(E_n) \text{ whenever } \{E_n\} \text{ is a countable collection of subsets of } \mathbb{C}.$$

Proposition 2.2 Let $E$ be a subset of $\mathbb{C}$ and let $f$ be a mapping of $E$ into $\mathbb{C}$ such that there exists a constant $c \geq 0$ for which $|f(z) - f(w)| \leq c|z - w|$ whenever $z, w \in E$. Then $\mathcal{H}^s(f(E)) \leq c^s\mathcal{H}^s(E)$.

Corollary 2.3 For $\alpha$ and $\beta$ complex numbers and $E$ a subset of $\mathbb{C}$, $\mathcal{H}^s(\alpha E + \beta) = |\alpha|^s\mathcal{H}^s(E)$.

If $t > s \geq 0$, $0 < \delta < 1$, and $\{U_n\}$ is a $\delta$-cover of $E$, then $\sum_n |U_n|^t \leq \sum_n |U_n|^s$. By infing over all $\delta$-covers and then letting $\delta \downarrow 0$, we see that $\mathcal{H}^t(E) \leq \mathcal{H}^s(E)$. Thus $\mathcal{H}^s(E)$ decreases as $s$ increases. But more can be said in this situation: $\sum_n |U_n|^t \leq \delta^{t-s} \sum_n |U_n|^s$, and so by infing over all $\delta$-covers,

$$\mathcal{H}^t_\delta(E) \leq \delta^{t-s}\mathcal{H}^s_\delta(E).$$

Letting $\delta \downarrow 0$, we see that if $\mathcal{H}^t(E) < \infty$, then $\mathcal{H}^t(E) = 0$ for all $t > s$, and that if $\mathcal{H}^t(E) > 0$, then $\mathcal{H}^s(E) = \infty$ for all $s < t$. In consequence, there exists a unique nonnegative number called the Hausdorff dimension (or Hausdorff–Besicovitch dimension) of $E$ and denoted $\dim_{\mathcal{H}}(E)$ such that

$$\mathcal{H}^s(E) = \begin{cases} \infty & \text{when } s < \dim_{\mathcal{H}}(E) \\ 0 & \text{when } s > \dim_{\mathcal{H}}(E). \end{cases}$$

What happens at $s = \dim_{\mathcal{H}}(E)$? In this case, $\mathcal{H}^s(E)$ can be 0, $\infty$, or anything in between. When $\mathcal{H}^s(E) = \infty$, it can even be the case that $E$ is non-$\sigma$-finite for $\mathcal{H}^s$, i.e., $E$ cannot be expressed as a countable union of sets of finite $\mathcal{H}^s$-measure. An example of this is the Joyce–Mörters set at the end of Chapter 4 (see Proposition 4.34).
Almost all of the following is fairly immediate from the definition of Hausdorff dimension and Proposition 2.1.

**Proposition 2.4** The Hausdorff dimension \( \dim_{\mathcal{H}} \) is a function defined on all subsets of \( \mathbb{C} \) and taking values in \([0, 2]\) such that

\[
\dim_{\mathcal{H}}(\emptyset) = 0,
\]

\[
\dim_{\mathcal{H}}(E) \leq \dim_{\mathcal{H}}(F) \text{ whenever } E \subseteq F \subseteq \mathbb{C},
\]

and

\[
\dim_{\mathcal{H}} \left( \bigcup_n E_n \right) = \sup_n \dim_{\mathcal{H}}(E_n) \text{ whenever } \{E_n\} \text{ is a countable collection of subsets of } \mathbb{C}.
\]

The nonimmediate item is that \( \dim_{\mathcal{H}}(E) \leq 2 \) always. Because of this \( \mathcal{H}^s(\emptyset) = 0 \) for any subset of \( \mathbb{C} \) whenever \( s > 2 \). Thus we will have no interest in \( \mathcal{H}^s \) for \( s > 2 \).

**Proof** To show that \( \dim_{\mathcal{H}}(E) \leq 2 \) always, it suffices to show that \( \dim_{\mathcal{H}}(Q) \leq 2 \) for any closed square \( Q \) in \( \mathbb{C} \) whose sides have length 1. Given \( \delta > 0 \), let \( n \) be a positive integer such that \( \sqrt{2}/n < \delta \). Cut up \( Q \) into \( n^2 \) squares whose sides have length \( 1/n \) in the obvious way. Then \( \mathcal{H}_{\delta}^2(Q) \leq n^2(\sqrt{2}/n)^2 = 2 \). Upon letting \( \delta \downarrow 0 \), we get \( \mathcal{H}^2(Q) \leq 2 \). Hence, \( \dim_{\mathcal{H}}(Q) \leq 2 \). \( \square \)

We close this section with a few examples to give the reader a feel for what Hausdorff measure and dimension are.

When \( s = 0 \), \( \mathcal{H}^0 \) is easily seen to be counting measure, i.e., \( \mathcal{H}^0(E) \) is just the cardinality of \( E \).

When \( s = 1 \), \( \mathcal{H}^s \) of a linear subset of \( \mathbb{C} \) is just the linear Lebesgue measure of the subset. To see this recall that for \( E \) a subset of a line \( L \), we have by definition that

\[
L^1(E) = \inf \left\{ \sum_n |I_n| : \{I_n\} \text{ is a countable cover of } E \text{ by intervals in } L \right\}.
\]

Also recall that in computing \( \mathcal{H}^1_{\delta}(E) \) we need only consider \( \delta \)-covers of \( E \) by convex sets in the plane. Now an interval in \( L \) is a convex set in the plane and the intersection of a convex set in the plane with \( L \) gives us an interval in \( L \) with smaller diameter than the convex set. Hence

\[
\mathcal{H}^1_{\delta}(E) = \inf \left\{ \sum_n |I_n| : \{I_n\} \text{ is a } \delta\text{-cover of } E \text{ by intervals in } L \right\}.
\]
Since any interval can be written as a finite union of intervals whose diameters are all less than \( \delta \) and which sum to the diameter of the original interval, one clearly then has \( \mathcal{H}^1(E) = L^1(E) \). Letting \( \delta \downarrow 0 \) yields \( \mathcal{H}^1(E) = L^1(E) \).

More generally, \( \mathcal{H}^1 \) of a subset of a rectifiable arc in \( \mathbb{C} \) is just the subset’s “arclength measure.” This and more will be shown later (Sections 4.5 and 5.2) when we consider rectifiable curves and define arclength measure on them. Thus one-dimensional Hausdorff measure, \( \mathcal{H}^1 \), may be thought of as a “generalized length” defined for any subset of \( \mathbb{C} \) and will be referred to more concisely as linear Hausdorff measure in what follows.

When \( s = 2 \), \( \mathcal{H}^s \) of any subset of \( \mathbb{C} \) is just a multiple of the area, i.e., the planar Lebesgue measure, of the subset. More precisely, \( \mathcal{H}^2(E) = (4/\pi)L^2(E) \) for any subset \( E \) of \( \mathbb{C} \). This will be shown later (Section 5.3) after we prove Vitali’s Covering Lemma and the Isodiametric Inequality. Clearly then \( \dim_{\mathcal{H}}(E) = 2 \), whenever \( L^2(E) > 0 \); in particular, whenever \( E \) has nonempty interior.

To get an example of a set of “fractional” Hausdorff dimension, consider the standard middle-thirds Cantor set \( K \). Recall that \( K \) is the countable intersection of closed sets \( C_n \) where \( C_0 = [0, 1], C_1 = [0, 1/3] \cup [2/3, 1], C_3 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1], \) etc. Note that each \( C_n \) is the union of \( 2^n \) closed intervals of length \( 3^{-n} \) and that each \( C_{n+1} \) is obtained from \( C_n \) by removing the open middle-third from each of these constituent intervals of \( C_n \).

Considering the obvious covering of \( K \) by the constituent intervals of \( C_n \), we see that for any \( \delta > 3^{-n} \), \( \mathcal{H}_\delta^s(K) \leq 2^n (3^{-n})^s = (2 \times 3^{-s})^n \). We would like to let \( \delta \downarrow 0 \), i.e., \( n \to \infty \), and have this upper estimate for \( \mathcal{H}_\delta^s(K) \) remain bounded. The smallest value of \( s \) for which this happens occurs when \( 2 \times 3^{-s} = 1 \), i.e., when \( s = \ln 2/\ln 3 \). So fixing \( s \) at this value and letting \( \delta \downarrow 0 \), we have that \( \mathcal{H}^s(K) \leq 1 \).

To show that \( \mathcal{H}^s(K) \geq 1/2 \) and so that \( \dim_{\mathcal{H}}(K) = s = \ln 2/\ln 3 \), it suffices to prove that

\[
\sum_{I \in \mathcal{C}} |I|^s \geq \frac{1}{2}
\]

for any finite covering \( \mathcal{C} \) of \( K \) by open intervals. If this inequality fails, then among all coverings for which it fails, there is one with a minimal number of intervals. Denoting this minimal failing covering by \( \mathcal{C} \), we will obtain a contradiction by constructing a failing covering \( \mathcal{C}' \) with fewer intervals.

For any interval \( I \) of length \( 1/3 \) or more, \( |I|^s \geq 3^{-s} = 1/2 \). Thus each interval of \( \mathcal{C} \) must have length less than \( 1/3 \) and so cannot intersect both \( K_l = K \cap [0, 1/3] \) and \( K_r = K \cap [2/3, 1] \). Let \( \mathcal{C}_l \) and \( \mathcal{C}_r \) consist of those intervals of \( \mathcal{C} \) intersecting \( K_l \) and \( K_r \) respectively. Then \( \mathcal{C}_l \) and \( \mathcal{C}_r \) are nonempty, have no common intervals, and have cardinality strictly less than that of \( \mathcal{C} \). Moreover, \( \mathcal{C}_l \) and \( \mathcal{C}_r \) are coverings of \( K_l \) and \( K_r \) respectively. If

\[
\sum_{I \in \mathcal{C}_l} |I|^s \leq \sum_{I \in \mathcal{C}_r} |I|^s,
\]
then, since the dilation $x \mapsto 3x$ maps $K_l$ onto $K$, $C' = \{3I : I \in C_l\}$ is the desired failing covering of $K$ with fewer intervals. It fails because $C$ fails and

$$\sum_{I \in C} |I|^s \geq \sum_{I \in C_l} |I|^s + \sum_{I \in C_r} |I|^s \geq 2 \sum_{I \in C_l} |I|^s = \sum_{I \in C_l} 3^s|I|^s = \sum_{I' \in C'} |I'|^s.$$  

If $C_r$ has the smaller associated sum, then proceed similarly using the dilation $x \mapsto 3x - 2$ of $K_r$ onto $K$ instead. In either case, we have produced the desired $C'$ and so are done.

The elegant analysis above is taken from 8.2.22 of [KK]. A more refined analysis, as in the proof of Theorem 1.14 of [FALC], which gives more attention to the lengths of the constituent intervals and the distances between them, leads to the conclusion that for this $K$ and $s$, $\mathcal{H}^s(K) = 1$ exactly.

Note that the upper estimate on the Hausdorff measure of $K$ was easy to come by, whereas the lower estimate was difficult to establish. This is no accident but typical of most sets and in the nature of things. According to the definition of Hausdorff measure, to establish an upper estimate one need only come up with a single economical $\delta$-covering, whereas to establish a lower estimate one needs to compute an infimum over all possible $\delta$-coverings.

The analysis just given for $K$ can be used to estimate the Hausdorff measure and obtain the Hausdorff dimension of other Cantor sets. The energetic reader may take the following two paragraphs as exercises to be worked out.

Given $\alpha$ strictly between 0 and 1/2, let $K_\alpha$ be the linear Cantor set gotten by the same procedure used to generate the standard middle-thirds set only now split each constituent interval from a generation, each “parent,” into two constituent intervals of the next generation, two “children,” each with length $\alpha$ times that of the parent interval (so the standard middle-thirds Cantor set just considered is, appropriately enough, $K_{1/3}$). Then $(1 - 2\alpha)^s \leq \mathcal{H}^s(K_\alpha) \leq 1$ for $s = \ln 2 / \ln(1/\alpha)$ and so $\dim_H(K_\alpha) = \ln 2 / \ln(1/\alpha)$. This shows that for any $s$ strictly between 0 and 1, there exist linear Cantor sets of Hausdorff dimension $s$. (The more careful analysis in the proof of Theorem 1.14 of [FALC] can easily be adapted to show that for these $K_\alpha$ and $s$, $\mathcal{H}^s(K_\alpha) = 1$ exactly.)

Given $\alpha$ strictly between 0 and 1/2, let $C_0$ be the closed unit square $[0, 1] \times [0, 1]$. Let $C_1$ be the union of the four closed squares contained in $C_0$ each of which contains a corner of $C_0$ and has edges of length $\alpha$. Let $C_2$ be the union of the sixteen closed squares contained in $C_1$ each of which contains a corner of a constituent square of $C_1$ and has edges of length $\alpha^2$. Continue in this fashion to generate a sequence of nested closed sets $\{C_n\}$, each $C_n$ consisting of the disjoint union of $4^n$ closed squares with edges of length $\alpha^n$. Set $K_\alpha = \bigcap_n C_n$. Then $(1 - 2\alpha)^s \leq \mathcal{H}^s(K_\alpha) \leq \sqrt{2}^s$ for $s = \ln 4 / \ln(1/\alpha)$ and so $\dim_H(K_\alpha) = \ln 4 / \ln(1/\alpha)$. This shows that for any $s$ strictly between 0 and 2, there exist planar Cantor sets of Hausdorff dimension $s$. (A more careful analysis which can be found in [McM] shows that $\mathcal{H}^1(K_{1/4}) = \sqrt{2}$ exactly. We shall return to this set $K_{1/4}$ and examine it more carefully in the last section of this chapter.)
\[ \gamma(N_{\Phi_1}) \text{is the number of sets } \gamma \text{ by bounded open sets such that } \exists \gamma, \eta > 0 \text{ so that } |U_{\gamma_1} - U_{\gamma_2}| < \varepsilon/2 \pi. \]

\text{Proof (Loose)} The definition of } H_{\infty}(K) \text{ provides us with a finite cover } \{U_n\} \text{ of } K \text{ by bounded open sets such that } \sum |U_n| < H_{\infty}(K) + \varepsilon/2\pi. \text{ Without loss of generality, each } U_n \text{ intersects } K. \text{ Replace each } U_n \text{ by an open disk of radius } |U_n| \text{ centered anywhere on } U_n. \text{ Let the cycle } \Gamma \text{ be the counterclockwise boundary of the union of the disks so produced. The reader may “easily” verify that } \Gamma \text{ works.} \]

\section*{2.2 Painlevé’s Theorem}

The main concern of this section, Painlevé’s Theorem from [PAIN], will enable us to dispose of sets of Hausdorff dimension less than one – they will all be shown removable! Painlevé’s Theorem will follow easily from a more general quantitative result which bounds } \gamma(K) \text{ by } H_{\infty}(K). \text{ (Note that } H_{\infty}(K) \text{ is just } H_1(K) \text{ with } \delta = \infty! \text{ It is usually referred to as the linear Hausdorff content of } K. \text{) For this we need a lemma which constructs cycles surrounding } K \text{ with length just a little more than } 2\pi H_{\infty}(K). \text{ We give two proofs of this lemma: one, short but loose, sufficient for those who are willing to take certain topological assertions as obvious, and another, more rigorous but longer, addressed to those who are finicky about such things.}

\begin{lemma}
\text{Let } K \text{ be a compact subset of } \mathbb{C} \text{ and } \varepsilon > 0. \text{ Then there exists a cycle } \Gamma \text{ in } \mathbb{C} \setminus K \text{ of length at most } 2\pi H_{\infty}(K) + \varepsilon \text{ with winding number one about each point of } K.
\end{lemma}

\text{Proof (Loose)} The definition of } H_{\infty}(K) \text{ provides us with a finite cover } \{U_n\} \text{ of } K \text{ by bounded open sets such that } \sum |U_n| < H_{\infty}(K) + \varepsilon/2\pi. \text{ Without loss of generality, each } U_n \text{ intersects } K. \text{ Replace each } U_n \text{ by an open disk of radius } |U_n| \text{ centered anywhere on } U_n. \text{ Let the cycle } \Gamma \text{ be the counterclockwise boundary of the union of the disks so produced. The reader may “easily” verify that } \Gamma \text{ works.} \]

\text{The next proof is a slight modification of the proof of [RUD, 13.5].}

\text{Proof (More Rigorous)} The definition of } H_{\infty}(K) \text{ provides us with a finite cover } \{U_n\} \text{ of } K \text{ by bounded open sets such that } \sum |U_n| < H_{\infty}(K) + \varepsilon/4. \text{ Without loss of generality, each } U_n \text{ intersects } K. \text{ Choose } \eta > 0 \text{ so that } 4 \sum |U_n| + 8N\eta < 4H_{\infty}(K) + \varepsilon \text{ where } N \text{ is the number of sets } U_n \text{ in our finite cover. Cover the plane with a grid of closed squares whose interiors are nonintersecting and whose edges have length } \eta. \text{ Let } R_n \text{ be the smallest closed rectangle whose interior contains } U_n \text{ and whose sides lie in the lines formed by our grid. Clearly the perimeter of } R_n \text{ is at most } 4(|U_n| + 2\eta). \text{ Let } Q_1, \ldots, Q_M \text{ be those squares of our grid contained in } R_1 \cup \cdots \cup R_N. \text{ Each square } Q_m \text{ has four boundary intervals. Direct these four intervals so that } Q_m \text{’s boundary is given a counterclockwise orientation. Let } \Sigma \text{ be the collection of the } 4M \text{ directed intervals that result.}

\text{Then } \Sigma \text{ is a balanced collection of directed intervals, which means that for every point } p \text{ of the plane, the number of intervals of } \Sigma \text{ with } p \text{ as initial point is equal to the number of intervals of } \Sigma \text{ with } p \text{ as terminal point. Remove those intervals from } \Sigma \text{ whose oppositely oriented intervals are also in } \Sigma \text{ and denote the collection of remaining intervals of } \Sigma \text{ by } \Phi. \text{ The collection } \Phi \text{ is still balanced and each interval of } \Phi \text{ is contained in the boundary of some } R_n. \]

\text{Construct a cycle } \Gamma \text{ from } \Phi \text{ as follows. Pick any } \gamma_1 \text{ from } \Phi. \text{ Since } \Phi \text{ is balanced, there is an interval } \gamma_2 \text{ in } \Phi \setminus \{\gamma_1\} \text{ whose initial point is the terminal point of } \gamma_1. \text{ Since } \Phi \text{ is balanced, there is an interval } \gamma_3 \text{ in } \Phi \setminus \{\gamma_1, \gamma_2\} \text{ whose initial point is the terminal point of } \gamma_2. \text{ Continue in this manner until you reach an interval } \gamma_k \text{ whose endpoint is the initial point of } \gamma_1. \text{ This must occur since } \Phi \text{ is finite and balanced. Then the intervals } \gamma_1, \gamma_2, \ldots, \gamma_k \text{ form a closed path. The collection } \Phi \setminus \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \text{ is still
balanced so, if it is nonempty, we may repeat our procedure again. Continuing in
this manner until \( \Phi \) is empty, we see that the intervals of \( \Phi \) form a chain consisting
of closed paths, i.e., the intervals of \( \Phi \) form a cycle, \( \Gamma \).

If an edge of some \( Q_m \) intersects \( K \), then it must be common to two adjacent
\( Q_m \)'s since \( K \subseteq \operatorname{int} \bigcup_{m=1}^M Q_m \). Hence \( \Sigma \) contains the two oppositely oriented
intervals determined by the edge and so these intervals are not in \( \Phi \). In consequence,
\( \Gamma \) is a cycle in \( \mathbb{C} \setminus K \).

Clearly the length of \( \Gamma \) is at most the sum of the perimeters of the rectangles \( R_n \)
and so less than \( 4\mathcal{H}_\infty^1(K) + \varepsilon \).

For any \( z \in K \) but not on the boundary of any \( Q_m \), the winding number of \( \Gamma \)
about \( z \), being the sum of the winding numbers of the boundaries of the \( Q_m \)'s about
\( z \), is 1. This and the constancy of the winding number of a cycle on each component
of the complement of its range \([RUD, 10.10]\) make the last assertion of the lemma
clear.

Note that the more rigorous proof has produced a cycle with a better estimate on
its length: the \( 2\pi \) has been reduced to 4. We will say more about this after proving
the main result of this section which now follows quite easily . . .

**Theorem 2.6** \( \gamma(K) \leq \mathcal{H}_\infty^1(K) \).

**Proof** Let \( f \in H^\infty(C^* \setminus K) \) be such that \( \|f\|_\infty \leq 1 \). Given \( \varepsilon > 0 \), obtain a cycle
\( \Gamma \) as in the previous lemma. Then, by the integral representation for \( f'(\infty) \) in the
discussion just after Proposition 1.1,

\[
|f'(\infty)| = \left| \frac{1}{2\pi i} \int_\Gamma f(\zeta) \, d\zeta \right| \leq \frac{2\pi \mathcal{H}_\infty^1(K) + \varepsilon}{2\pi}.
\]

Letting \( \varepsilon \downarrow 0 \) and then suping over the \( f \)'s in question, we see that \( \gamma(K) \leq \mathcal{H}_\infty^1(K) \). \( \square \)

Of course what has been shown is that if the cycle \( \Gamma \) of the lemma has length
at most \( c\mathcal{H}_\infty^1(K) + \varepsilon \), then \( \gamma(K) \leq \kappa \mathcal{H}_\infty^1(K) \) where \( \kappa = c/2\pi \). The loose proof
led to \( c = 2\pi \) and so \( \kappa = 1 \); while the more rigorous proof led to \( c = 4 \) and so
\( \kappa = 2/\pi = 0.6366 \cdots \). If in the loose proof one uses Jung’s Theorem (recall the
paragraph after the proof of Proposition 1.18) instead of simply enclosing each \( U_n \)
in a ball of radius \( |U_n| \), then one has \( c = 2\pi/\sqrt{3} \) and so \( \kappa = 1/\sqrt{3} = 0.5773 \cdots \). It
turns out that it is possible to take \( c = \pi \) and so \( \kappa = 1/2 = 0.5 \). To obtain this one
must do three things however: first, extend ones complex integration theory from
chains of segments and circular arcs to chains of rectifiable arcs; second, show that
the boundary of a bounded open convex set is rectifiable; and third, show a (sharp)
“peri-diametric inequality” to the effect that the perimeter of a bounded open convex
set is at most \( \pi \) times its diameter. We dispense with these complications and content
ourselves with the cruder estimate enunciated in the theorem above since from the
point of view of removability it suffices simply to have \( \kappa < \infty \! \).

However, before passing to Painlevé’s Theorem, let us take the estimate
\((\star) \ \gamma(K) \leq \mathcal{H}_\infty^1(K)/2 \) as established and play with it to see a few things. By
using the trivial covering of $K$ by itself we have (⋆⋆) $\mathcal{H}_{\infty}^1(K) \leq |K|$ always. Thus (⋆⋆⋆) $\gamma(K) \leq |K|/2$ always, which is a sharper form of the first estimate of Proposition 1.18 and beats out the improvement of it via Jung’s Theorem noted immediately after its proof. Next, invoking Corollary 1.13 we see that for $K$ a closed ball of radius $r$,

$$r = \gamma(K) \leq \frac{\mathcal{H}_{\infty}^1(K)}{2} \leq \frac{|K|}{2} = r,$$

and so (⋆), (⋆⋆), and (⋆⋆⋆) are all as sharp as possible.

The theorem which gives this section its title is more in the nature of a corollary to the main result above since one trivially has $\mathcal{H}_{\infty}^1(K) \leq \mathcal{H}^1(K)$, but we label it a theorem due to its importance.

**Theorem 2.7 (Painlevé)** If $\mathcal{H}^1(K) = 0$, then $K$ is removable.

Since any set of Hausdorff dimension less than 1 has linear Hausdorff measure 0, we immediately obtain the following.

**Corollary 2.8** If $\dim_{\mathcal{H}}(K) < 1$, then $K$ is removable.

### 2.3 Frostman’s Lemma

Painlevé’s Theorem (2.7) has disposed of sets with dimension less than 1. The main concern of this section, a potential-theoretic lemma from the thesis of Otto Frostman, will enable us to dispose of sets with dimension greater than one – they will all be shown nonremovable! We prove only a simple version of this lemma sufficient for our purposes. Throughout this book we shall denote the radius of a closed ball $B$ by $\operatorname{rad}(B)$. (Note that $\mathcal{H}_{\infty}^s(K)$ in the next lemma is just $\mathcal{H}_{\delta}^s(K)$ with $\delta = \infty$! It is usually referred to as the $s$-dimensional Hausdorff content of $K$.)

**Lemma 2.9 (Frostman)** Given $1 < s \leq 2$, set $M = 9 \times 4^s$. Then for any compact subset $K$ of $\mathbb{C}$, there exists a regular positive Borel measure $\mu$ on $K$ with mass at least $\mathcal{H}_{\infty}^s(K)$ such that $\mu(B) \leq M \operatorname{rad}(B)^s$ for all closed balls $B$ in $\mathbb{C}$.

**Proof** Call a square half-closed when it is obtained from its closure by removing the top and right closed edges. Note that any half-closed square can be decomposed, in the obvious way, into four pairwise disjoint, half-closed squares whose diameters are half that of the original. We call these four squares the children of the original square. In what follows we shall also use the terms parent, grandparent, and ancestor in a self-evident manner given the way the term children has just been defined.

Choose any half-closed square $Q_0$ containing $K$ in its interior and set $G_0 = \{Q_0\}$. Let $G_1$ consist of the four children of the single square of $G_0$. Let $G_2$ consist of the 16 children of the 4 squares of $G_1$. Continuing in this manner, we obtain the generations $G_0, G_1, G_2, \ldots$ of the dyadic grid $G = \bigcup_{n \geq 0} G_n$.

Fixing $n$ for the moment, let $\nu_n$ be the regular positive Borel measure on $Q_0$ whose restriction to each $Q \in G_n$ is equal to $\kappa$ times the restriction of $L^2$ to $Q$. 


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where $\kappa = |Q|^\delta / L^2(Q)$ if $Q$ intersects $K$ and 0 otherwise. Clearly $v_n(Q) \leq |Q|^\delta$ for every $Q \in \mathcal{G}_n$ and $v_n(Q) = |Q|^\delta$ for every $Q \in \mathcal{G}_n$ which intersects $K$.

Let $v_{n-1}$ be the regular positive Borel measure on $Q_0$ whose restriction to each $Q \in \mathcal{G}_{n-1}$ is equal to $\kappa$ times the restriction of $v_n$ to $Q$ where $\kappa = |Q|^\delta / v_n(Q)$ if $|Q|^\delta < v_n(Q)$ and 1 otherwise. Note that the measure $v_{n-1}$ just defined has been gotten by uniformly reducing $v_n$ or by leaving it undisturbed on each square of $\mathcal{G}_{n-1}$. In consequence, $v_{n-1}(Q) \leq |Q|^\delta$ for every $Q \in \mathcal{G}_{n-1} \cup \mathcal{G}_n$. Furthermore, given any square from $\mathcal{G}_n$ which intersects $K$, we have $v_{n-1}(Q) = |Q|^\delta$ for $Q$ the square or its parent.

Let $v_{n-2}$ be the regular positive Borel measure on $Q_0$ whose restriction to each $Q \in \mathcal{G}_{n-2}$ is equal to $\kappa$ times the restriction of $v_{n-1}$ to $Q$ where $\kappa = |Q|^\delta / v_{n-1}(Q)$ if $|Q|^\delta < v_{n-1}(Q)$ and 1 otherwise. Note that the measure $v_{n-2}$ just defined has been gotten by uniformly reducing $v_{n-1}$ or by leaving it undisturbed on each square of $\mathcal{G}_{n-2}$. In consequence, $v_{n-2}(Q) \leq |Q|^\delta$ for every $Q \in \mathcal{G}_{n-2} \cup \mathcal{G}_{n-1} \cup \mathcal{G}_n$. Furthermore, given any square from $\mathcal{G}_n$ which intersects $K$, we have $v_{n-2}(Q) = |Q|^\delta$ for $Q$ the square, its parent, or its grandparent.

Continuing in this manner, we obtain measures $v_n$, $v_{n-1}$, $v_{n-2}$, ..., $v_1$, and $v_0$. Denote the measure $v_0$ by $\mu_n$. Then $(\star) \mu_n(Q) \leq |Q|^\delta$ for every $Q$ in $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_n$. Furthermore, given any square from $\mathcal{G}_n$ which intersects $K$, we have $\mu_n(Q) = |Q|^\delta$ for $Q$ the square or one of its ancestors. In consequence, each point of $K$ is contained in a unique largest square $Q_z$ in $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_n$ for which $\mu_n(Q) = |Q|^\delta$. The collection $\{Q_z : z \in K\}$ is actually finite and so can be written $\{Q_1, Q_2, \ldots, Q_M\}$. Since these squares are pairwise disjoint and $K$ is contained in their union, we have on the one hand

$$(\star \star) \mu_n(\text{cl } Q_0) \geq \sum_{m=1}^{M} \mu_n(Q_m) = \sum_{m=1}^{M} |Q_m|^\delta \geq \mathcal{H}_\infty^\delta(K),$$

while on the other hand we obviously have $(\star \star \star) \mu_n(\text{cl } Q_0) \leq |Q_0|^\delta$.

Unfix $n$ now and realize that, by $(\star \star \star)$, we have produced a sequence $\{\mu_n\}$ of uniformly bounded linear functionals on the Banach space of continuous functions on $\text{cl } Q_0$. Moreover, this space is separable (consider the polynomials in $z$ and $\bar{z}$ with rational coefficients and invoke Stone-Weierstrass). We may thus apply [RUD, 11.29] and [RUD, 6.19] to obtain a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ which converges weakly to some regular complex Borel measure $\mu$ on $\text{cl } Q_0$.

Given a compact subset $C$ of $\text{cl } Q_0$, consider the functions $f_N(z) = \max[1 - N \text{ dist}(z, C), 0]$. By the weak convergence and positiveness of the measures $\mu_{n_k}$, $\int f_N \, d\mu \geq 0$ for all $N \geq 1$. Letting $N \to \infty$ and invoking Lebesgue’s Dominated Convergence Theorem [RUD, 1.34], we conclude that $\mu(C) \geq 0$. If $C$ is also disjoint from $K$, then the same argument shows that $\mu(C) = 0$ since each $\mu_n$ is supported on the closed $|Q_0|/2^n$-neighborhood of $K$. The inner regularity of $\mu$ now implies that $\mu$ is a positive measure supported on $K$. Of course $(\star \star)$ and the weak convergence imply that $\mu$ has mass at least $\mathcal{H}_\infty^\delta(K)$. So it only remains to verify the growth condition on $\mu$. 

2.3 Frostman’s Lemma
We first show that for any \( n \), \( \mu_n(B) \leq M \text{rad}(B)^s \) for all closed balls \( B \) such that \( |B \cap Q_0| > |Q_0|/2^{n+1} \). Let \( m \) be the integer between 0 and \( n \) such that

\[
\frac{|Q_0|}{2^{m+1}} < |B \cap Q_0| \leq \frac{|Q_0|}{2^m}.
\]

For this \( m \), \( B \cap Q_0 \) is contained in the union of at most nine squares from \( \mathcal{G}_m \). Thus by (⋆) we have

\[
\mu_n(B) = \mu_n(B \cap Q_0) \leq 9 \left\{ \frac{|Q_0|}{2^m} \right\}^s \leq 9 |B \cap Q_0|^s \leq M \text{rad}(B)^s.
\]

Now consider any closed ball \( B \). Without loss of generality, \( B \) intersects \( \text{cl} \, Q_0 \).

Given \( N \geq 1 \), set \( g_N(z) = \max\{1 - N \text{dist}(z, B), 0\} \) and let \( B_N \) be the closed ball concentric with \( B \) and of radius \( \text{rad}(B) + 1/N \). Then

\[
\mu(B) \leq \int g_N \, d\mu = \lim_{k \to \infty} \int g_N \, d\mu_{n_k} \leq \lim \sup_{k \to \infty} \mu_{n_k}(B_N) \leq M \text{rad}(B_N)^s
\]

\[
= M(\text{rad}(B) + 1/N)^s.
\]

Letting \( N \to \infty \), we are done. \( \Box \)

The second ingredient needed for the main result of this section is an estimate on the Newtonian potential of a regular positive Borel measure satisfying a growth condition as in Frostman’s Lemma.

**Lemma 2.10** Let \( 0 < M < \infty \) and \( 1 < s \leq 2 \). Suppose \( \mu \) is a nontrivial regular positive Borel measure on a compact subset \( K \) of \( \mathbb{C} \) such that \( \mu(B) \leq M \text{rad}(B)^s \) for all closed balls \( B \) in \( \mathbb{C} \). Then for any \( z \in \mathbb{C} \setminus K \),

\[
\int \frac{1}{|\xi - z|} \, d\mu(\xi) \leq \frac{s}{s - 1} M^{1/s} \mu(K)^{1-1/s}.
\]

**Proof** By Fubini’s Theorem [RUD, 8.8],

\[
\int \frac{1}{|\xi - z|} \, d\mu(\xi) = \int \left\{ \int_0^\infty \frac{1}{t^2} \, d\mu(\xi) \right\} \, dt
\]

\[
= \int \left\{ \int_0^\infty \frac{1}{t^2} \chi_{[|\xi - z| \leq t]} \, dt \right\} \, d\mu(\xi)
\]

\[
= \int_0^\infty \left\{ \int_0^\infty \frac{1}{t^2} \chi_{[|\xi - z| \leq t]} \, d\mu(\xi) \right\} \, dt
\]

\[
= \int_0^\infty \frac{\mu(B(z; t))}{t^2} \, dt.
\]
We now wish to plug in our growth estimate on $\mu(B(z; t))$, but only for $t$ not too large. Obviously past some point it is better to estimate $\mu(B(z; t))$ simply by $\mu(K)$ rather than by $Mt^s$. This change-over point is the $T$ such that $MT^s = \mu(K)$, i.e., $T = [\mu(K)/M]^{1/s}$. This leads to

$$\int \frac{1}{|\xi - \zeta|} d\mu(\xi) \leq \int_0^T \frac{Mt^s}{t^2} dt + \int_T^\infty \frac{\mu(K)}{t^2} dt = \frac{MT^{s-1}}{s-1} + \frac{\mu(K)}{T}.$$ 

Substituting in our choice for $T$ and doing a bit of algebra, we obtain what we want. \hfill \square

The main result of this section now follows quite easily.

**Theorem 2.11** For $1 < s \leq 2$, $\gamma(K) \geq \frac{s-1}{4s} \left\{ \frac{\mathcal{H}_s(K)}{9} \right\}^{1/s}$.

*Proof* Without loss of generality, $\mathcal{H}_s(K) > 0$. (As an aside, we note that $\mathcal{H}_s(K) \leq |K|^s < \infty$ always.) Get a measure $\mu$ as in Frostman’s Lemma (2.9). Clearly $\mu$ is nontrivial since $\mu(K) \geq \mathcal{H}_s(K)$. Consider the function $f$ equal to $\hat{\mu}$, the Cauchy transform of $\mu$. Thus

$$f(z) = \int \frac{1}{\xi - z} d\mu(\xi).$$

Clearly $f$ is defined and analytic on $\mathbb{C} \setminus K$ since $\mu$ is supported on $K$. By the last lemma,

$$\|f\|_\infty \leq \frac{s}{s-1} \left\{ 9 \times 4^s \right\}^{1/s} \mu(K)^{1-1/s} \leq \frac{4s}{s-1} \left\{ \frac{9}{\mathcal{H}_s(K)} \right\}^{1/s} \mu(K).$$

But $f(\infty) = \lim_{z \to \infty} f(z) = 0$, so $|f'(\infty)| = \lim_{z \to \infty} |zf(z)| = \mu(K)$. Thus

$$\gamma(K) \geq \frac{|f'(\infty)|}{\|f\|_\infty} \geq \frac{s-1}{4s} \left\{ \frac{\mathcal{H}_s(K)}{9} \right\}^{1/s} \mu(K).$$

\hfill \square

**Corollary 2.12** If $\dim_{\mathcal{H}}(K) > 1$, then $K$ is nonremovable.

*Proof* Let $s$ be any number strictly between 1 and $\dim_{\mathcal{H}}(K)$. Then $\mathcal{H}^s(K) = \infty$, so there exists a $\delta > 0$ such that $\mathcal{H}^s_\delta(K) > 0$. Then for any countable cover of $K$ by bounded sets $\{U_n\}$ we must have $\sum_n |U_n|^s \geq \mathcal{H}_s^s(K)$ if $\{U_n\}$ is a $\delta$-cover and $\sum_n |U_n|^s \geq \delta^s$ otherwise. Thus $\mathcal{H}_s^s(K) \geq \min\{\mathcal{H}^s_\delta(K), \delta^s\} > 0$. By the theorem we are done. \hfill \square

Note how the theorem is quantitative: given $\mathcal{H}_s^\infty(K) > 0$, it allows us to conclude $\gamma(K) > 0$ and even to estimate how much bigger than zero $\gamma(K)$ is in terms of $\mathcal{H}_s^\infty(K)$. The corollary however is merely qualitative: although we see
that $\gamma(K) > 0$ when $\dim\mathcal{H}(K) > 1$, we do not obtain an effective estimate of how much bigger than zero $\gamma(K)$ is in terms of $\dim\mathcal{H}(K)$. This is to be expected since under a similitude $\gamma(K)$ scales linearly (Proposition 1.9) while $\dim\mathcal{H}(K)$ clearly remains invariant. In the proof this shows up as a lack of effective control over the quantity $\min\{\mathcal{H}^s_\delta(K), \delta^s\}$; although positive it could be arbitrarily small since $\delta$ could be arbitrarily small!

2.4 Conjecture and Refutation: The Planar Cantor Quarter Set

The results of the last two sections, which settle the question of removability for all compact sets except those of dimension one, are all consistent with the conjecture that Painlevé’s Theorem (2.7) is reversible. If true, one would have that a compact subset $K$ of $\mathbb{C}$ is removable if and only if $\mathcal{H}^1(K) = 0$. This beautiful conjecture, which would end our quest in a quite tidy manner, is false however! The first example of a removable compact set with positive, and even finite, linear Hausdorff measure was due to Anatoli Vitushkin (see [VIT1], or Section 3 of Chapter IV of [GAR2]). His example is quite complicated, slaying other conjectures than just the one of interest to us here. Later John Garnett (see [GAR1], or Section 2 of Chapter IV of [GAR2]) realized that a planar Cantor quarter set is a much simpler example of a set with positive finite linear Hausdorff measure but zero analytic capacity. Since then other proofs that this set works have been published (see [JON1], [MAT1], and/or [MUR]). We give Garnett’s original proof below, it being the most elementary.

So consider the planar Cantor quarter set, i.e., the set $K_{1/4}$ of the last paragraph in Section 2.1. Recall that $K_{1/4} = \bigcap_n C_n$ where $C_0$ is the closed unit square $[0, 1] \times [0, 1]$, $C_1$ is the union of the four closed squares contained in $C_0$ each of which contains a corner of $C_0$ and has edges of length $1/4$, $C_2$ is the union of the 16 closed squares contained in $C_1$ each of which contains a corner of a constituent square of $C_1$ and has edges of length $1/16$, etc. As noted in Section 2.1, the coverings of $K_{1/4}$ by the $4^n$ closed constituent squares of $C_n$ with edges of length $1/4^n$ lead easily to the upper estimate $\mathcal{H}^1(K_{1/4}) \leq \sqrt{2} = 1.4142 \ldots$, while the more difficult self-similarity argument given there leads to the lower estimate $\mathcal{H}^1(K_{1/4}) \geq 1/2 = 0.5$. As an easy and nice application of Proposition 2.2, we now substantially improve this lower estimate. Note that the orthogonal projection of the plane onto the line $y = x/2$ takes each of the sets $C_n$, and so too $K_{1/4}$, onto the closed segment with endpoints $(0, 0)$ and $(6/5, 3/5)$. Since orthogonal projection onto any fixed line decreases distances, it can only decrease linear Hausdorff measure by Proposition 2.2. Recall that in Section 2.1 we also showed the coincidence of $\mathcal{H}^1$ and $\mathcal{L}^1$ for linear sets. Hence we obtain our substantial improvement: $\mathcal{H}^1(K_{1/4}) \geq 3\sqrt{5}/5 = 1.3416 \ldots$. Of course, as noted already in Section 2.1, there is a more careful and difficult analysis in [McM] which shows that $\mathcal{H}^1(K_{1/4}) = \sqrt{2}$ exactly.

The rest of this section is devoted to the task of showing that $\gamma(K_{1/4}) = 0$. A number of lemmas are required. The first is a version of Cauchy’s Integral Theorem
for functions analytic at $\infty$. We use $n(\Gamma; z)$ to denote the winding number of a cycle $\Gamma$ in $\mathbb{C}$ about a point $z$ in $\mathbb{C} \setminus \Gamma$.

**Lemma 2.13** Let $K$ be a compact subset of $\mathbb{C}$ and let $f$ be an analytic function on $\mathbb{C}^* \setminus K$. Suppose that $\Gamma$ is a cycle in $\mathbb{C} \setminus K$ with winding number one about every point of $K$ and that $z$ is a point off $K \cup \Gamma$.

Then

$$f(\infty) - \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} f(z) & \text{if } n(\Gamma; z) = 0 \\ 0 & \text{if } n(\Gamma; z) = 1 \end{cases}.$$  

**Proof** Consider any $R$ large enough so that $K \cup \Gamma \cup \{z\}$ is encircled by $C_R$, the counterclockwise circle about the origin of radius $R$. Cauchy’s Integral Theorem [RUD, 10.35] applied to the cycle $C_R - \Gamma$ implies that

$$\frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} f(z) & \text{if } n(\Gamma; z) = 0 \\ 0 & \text{if } n(\Gamma; z) = 1 \end{cases}.$$  

Thus, to finish we need only show that the first integral in the above equation converges to $f(\infty)$ as $R \to \infty$. Since $n(C_R; z) = 1$, we have

$$\left| \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta - f(\infty) \right| = \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta) - f(\infty)}{\zeta - z} d\zeta \right| \leq \frac{1}{2\pi} \cdot \sup\{|f(\zeta) - f(\infty)| : \zeta \in C_R\} \cdot 2\pi R \cdot \frac{R}{R - |z|} \cdot \sup\{|f(\zeta) - f(\infty)| : \zeta \in C_R\}.$$  

Since the last quantity converges to 0 as $R \to \infty$, we are done. (An aside: Being the only term involving $R$ in the penultimate displayed equation, the integral over $C_R$ does not really depend on $R$. Thus we have shown not merely that this integral converges to $f(\infty)$ as $R \to \infty$ but that it is actually equal to it for all large $R$!)  

When we proved Painlevé’s Theorem (2.7) we essentially verified (d) of Proposition 1.3 for sets of linear Hausdorff measure 0. As an exercise the energetic reader may reprove Painlevé with the help of this last lemma by verifying (c) of Proposition 1.3 for sets of linear Hausdorff measure 0!

For convenience’s sake, denote $K_{1/4}$ simply by $K$ for the remainder of this section. Let $G_0$ denote the collection consisting of the single square which is $C_0$, let $G_1$ denote the collection consisting of the four squares whose union comprise $C_1$, let $G_2$ denote the collection consisting of the 16 squares whose union comprise $C_2$, etc. Next, let $\mathcal{G}$ be the collection of all squares appearing in one of these generations, so $\mathcal{G} = \bigcup_n G_n$. Continuing, for every square $Q$ from $\mathcal{G}$, set $K_Q = K \cap Q$, $G(Q) = \{R \in \mathcal{G} : R \subseteq Q\}$, and $G_n(Q) = \{R \in G_n : R \subseteq Q\}$. Note that each $K_Q$
is geometrically similar to $K$ with similarity ratio $l(Q)$, the common length of the edges of $Q$. Finally, given a square $Q$ from $\mathcal{G}$, a square $R$ from $\mathcal{G}(Q)$, and a function $f$ from $H^\infty(\mathbb{C}^* \setminus K_Q)$ which vanishes at $\infty$, define a function $f_{Q,R}$ on $\mathbb{C}^* \setminus K_R$ by

$$f_{Q,R}(z) = \frac{-1}{2\pi i} \int_{\Gamma_{Q,R}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where $\Gamma_{Q,R}$ is any cycle in $\mathbb{C} \setminus (K_Q \cup \{z\})$ with winding number 1 about every point of $K_R$ and 0 about every point of $(K_Q \setminus K_R) \cup \{z\}$.

**Lemma 2.14** With the conventions and definitions as above, we have

(a) Each $f_{Q,R}$ is a well-defined element of $H^\infty(\mathbb{C}^* \setminus K_R)$ which vanishes at $\infty$.

(b) $\|f_{Q,R}\|_\infty \leq M\|f\|_\infty$ where $M = 1 + 6/\pi$.

(c) Fix $n \geq m$, where $m$ is the integer such that $G_m$ contains $Q$. Then on the set $\mathbb{C} \setminus K_Q$ one has

$$f = \sum_{R \in G_n(Q)} f_{Q,R}.$$

(d) Let $\varphi$ be a similitude of the complex plane onto itself involving no rotation, so $\varphi(z) = \alpha z + \beta$ where $\alpha > 0$ and $\beta$ is a complex number. Suppose there are squares $Q^*$ and $R^*$ from $\mathcal{G}$ such that $\varphi(Q^*) = Q$ and $\varphi(R^*) = R$. Then

$$f_{Q,R} \circ \varphi = (f \circ \varphi)_{Q^*,R^*}.$$

(e) For any square $S$ from $\mathcal{G}(R)$, $(f_{Q,R})_{R,S} = f_{Q,S}$.

**Proof** (a) The functions $f_{Q,R}$ are well defined because of Cauchy’s Integral Theorem [RUD, 10.35]. Analyticity and vanishing at infinity are clear. Boundedness follows from the next item.

(b) Let $\Gamma$ denote the counterclockwise boundary of the square concentric with $R$ and with 3 times the diameter. Then by Cauchy’s Integral Theorem [RUD, 10.35] applied to the cycle $\Gamma - \Gamma_{Q,R}$, for $z$ close to but not on $K_R$ we have

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta + f_{Q,R}(z).$$

Suppose $z$, not on $K_R$, is within a distance $\varepsilon > 0$ of $K_R$. Then

$$|f_{Q,R}(z)| \leq |f(z)| + \frac{1}{2\pi} \cdot \frac{\|f\|_\infty}{l(R) - \varepsilon} \cdot 12l(R) \leq \left\{ 1 + \frac{6}{\pi} \cdot \frac{l(R)}{l(R) - \varepsilon} \right\} \|f\|_\infty.$$

By the Maximum Modulus Principle [RUD, 12.1 modified to take into account regions containing $\infty$].
\[ \| f_{Q,R} \|_\infty \leq \left\{ 1 + \frac{6}{\pi} \cdot \frac{l(R)}{l(R) - \varepsilon} \right\} \| f \|_\infty. \]

Letting \( \varepsilon \downarrow 0 \), we have our estimate.

(c) Apply the last lemma to the cycle which is the sum of the cycles \( \Gamma_{Q,R} \) for \( R \in G_n(Q) \).

(d) Let \( z^* \) be such that \( \varphi(z^*) = z \). Simply use the change of variables \( \zeta = \varphi(\xi) \) in the equation defining \( f_{Q,R}(z) = (f_{Q,R} \circ \varphi)(z^*) \) and then notice that we are done since the cycle \( \varphi^{-1}(\Gamma_{Q,R}) \) will serve as a cycle \( \Gamma_{Q^*,R^*} \) as in the definition of \( (f \circ \varphi)_{Q^*,R^*}(z^*) \).

(e) What follows is a pain but (e) is absolutely crucial to our success! Fixing \( z \in \mathbb{C} \setminus K_S \), we must show that \( (f_{Q,R})_{R,S}(z) = f_{Q,S}(z) \).

Choose a cycle \( \Gamma_{R,S} \) in \( \mathbb{C} \setminus (K_Q \cup \{z\}) \) with winding number 1 about every point of \( K_S \) and 0 about every point of \( (K_Q \setminus K_S) \cup \{z\} \). Then \( \Gamma_{R,S} \) is a cycle in \( \mathbb{C} \setminus (K_R \cup \{z\}) \) with winding number 1 about every point of \( K_S \) and 0 about every point of \( (K_R \setminus K_S) \cup \{z\} \) and so

\[ \star \ (f_{Q,R})_{R,S}(z) = \frac{-1}{2\pi i} \int_{\Gamma_{R,S}} \frac{f_{Q,R}(\zeta)}{\zeta - z} \, d\zeta. \]

Let \( U \) be the open set of points in the plane where the winding number of \( \Gamma_{R,S} \) is 1. Choose a cycle \( \Gamma_{Q,S} \) in \( U \setminus K_S \) with winding number 1 about every point of \( K_S \) and 0 about every point of \( \mathbb{C} \setminus U \). Then \( \Gamma_{Q,S} \) is a cycle in \( \mathbb{C} \setminus (K_Q \cup \{z\}) \) with winding number 1 about every point of \( K_S \) and 0 about every point of \( (K_Q \setminus K_S) \cup \{z\} \) and so

\[ \star \star \ f_{Q,S}(z) = \frac{-1}{2\pi i} \int_{\Gamma_{Q,S}} \frac{f(\xi)}{\xi - z} \, d\xi. \]

Let \( V \) be the open set of points in the plane where the winding number of \( \Gamma_{R,S} \) is 0. Choose a cycle \( \Gamma \) in \( \{V \setminus (K_Q \setminus K_R)\} \setminus (K_R \setminus K_S) \) with winding number 1 about every point of \( K_R \setminus K_S \) and zero about every point of \( \mathbb{C} \setminus \{V \setminus (K_Q \setminus K_R)\} \). Then \( \Gamma_{Q,R} = \Gamma_{Q,S} + \Gamma \) is a cycle in \( \mathbb{C} \setminus (K_Q \cup \Gamma_{R,S}) \) with winding number 1 about every point of \( K_R \) and 0 about every point of \( (K_Q \setminus K_R) \cup \Gamma_{R,S} \) and so for every \( \zeta \) in \( \Gamma_{R,S} \),

\[ \star \star \star \ f_{Q,R}(\zeta) = \frac{-1}{2\pi i} \int_{\Gamma_{Q,R}} \frac{f(\xi)}{\xi - \zeta} \, d\xi. \]

Lastly, note that by simple algebra,

\[ \frac{1}{2\pi i} \int_{\Gamma_{R,S}} \frac{1}{(\zeta - \xi)(\zeta - z)} \, d\zeta = \frac{1}{\xi - z} \left\{ \frac{1}{2\pi i} \int_{\Gamma_{R,S}} \frac{1}{\zeta - \xi} \, d\zeta - \frac{1}{2\pi i} \int_{\Gamma_{R,S}} \frac{1}{\zeta - z} \, d\zeta \right\}. \]

The two integrals on the right-hand side of this equation are winding numbers. Taking into account the way we have constructed our cycles, we thus obtain
\[
(*) \quad \frac{1}{2\pi i} \int_{\Gamma_{R,S}} \frac{1}{(\xi - z)(\xi - z)} \, d\xi = \begin{cases} \\
\frac{1}{\xi - z} & \text{when } \xi \in \Gamma_{Q,S} \\
0 & \text{when } \xi \in \Gamma.
\end{cases}
\]

The desired equality is now just a computation: substitute (**) into (*), use Fubini’s Theorem [RUD, 8.8] on the double integral obtained, simplify by means of (***), and then recognize the result as what we want via (**).

Denote the southwest corner and the center of any square \( S \) from \( \mathcal{G} \) by \( z_S \) and \( c_S \), respectively.

**Lemma 2.15** For \( Q \in \mathcal{G}_m \), the following assertions hold:

(a) The sequence \( A_n(Q) = \sum_{R \in \mathcal{G}_n(Q)} \frac{l(R)^2}{|c_R - z_Q|^2} \), defined for \( n \geq m \), is bounded by 3.

(b) The sequence \( B_n(Q) = \sum_{R \in \mathcal{G}_n(Q)} \frac{l(R) \cdot \text{Re}(c_R - z_Q)}{|c_R - z_Q|^2} \), defined for \( n \geq m \), is unbounded.

**Proof** (a) Let \( Q_{sw} \) denote the southwest square of \( \mathcal{G}_{m+1}(Q) \), so \( Q_{sw} \) is the unique square in \( \mathcal{G}_{m+1}(Q) \) which contains \( z_Q \). Then for \( n > m \),

\[
A_n(Q) = \sum_{R \in \mathcal{G}_n(Q_{sw})} \frac{l(R)^2}{|c_R - z_Q|^2} + \sum_{R \in \mathcal{G}_n(Q) \setminus \mathcal{G}_n(Q_{sw})} \frac{l(R)^2}{|c_R - z_Q|^2}.
\]

Let \( R^* \) denote the result of applying the similitude with center \( z_Q \) and similarity ratio 4 to a square \( R \). Then \( l(R^*)^2/|c_R - z_Q|^2 = (R^*)^2/|c_R - z_Q|^2 \) and as \( R \)'s range over all of \( \mathcal{G}_n(Q_{sw}) \), the \( R^* \)'s range over all of \( \mathcal{G}_n(Q) \). Thus the first sum in the above is just \( A_{n-1}(Q) \).

The number of \( R \)'s involved in the second sum in the above is \((3/4)4^{n-m}\) and for each of these \( R \)'s, \( |c_R - z_Q| \geq (1/2)l(Q) = 4^{-m}/2 \), so the second sum in the above is at most \((3/4)4^{n-m} \cdot (4^{-m})^2/(4^{-m}/2)^2 = (3 \times 4^m)4^{-n}\).

Hence \( A_n(Q) \leq A_{n-1}(Q) + (3 \times 4^m)4^{-n} \) for all \( n > m \), which implies that the sequence \( A_n(Q), n \geq m \), is bounded and bounded by 3.

(b) For \( n > m \), split \( B_n(Q) \) up into two sums just as we splitted \( A_n(Q) \). Handle the resulting first sum in the same manner to conclude that it is \( B_{n-1}(Q) \).

With the resulting second sum throw away the terms, all positive, arising from \( R \)s lying above \( Q_{sw} \), i.e., \( R \)s lying in \( Q_{sw} \) so to speak. For each of the \( R \)s remaining, i.e., \( R \)s in \( Q_{ne} \) or \( Q_{se} \) so to speak, we have \( \text{Re}(c_R - z_Q) \geq (1/2)l(Q) = 4^{-m}/2 \) and \( |c_R - z_Q| \leq \sqrt{2} l(Q) = \sqrt{2} \cdot 4^{-m} \), so the resulting second sum is at least \((1/2)4^{n-m} \cdot (4^{-m})^2/(\sqrt{2} \cdot 4^{-m})^2 = 1/8\).

Hence \( B_n(Q) \geq B_{n-1}(Q) + 1/8 \) for all \( n > m \), which implies that the sequence \( B_n(Q), n \geq m \), is unbounded. \(\square\)
Given $Q$, $R$, and $f$ as in Lemma 2.14, define $g_{Q,R}(z) = f'_{Q,R}(\infty)/(c_R - z)$. (Warning: An ambiguous expression like “$f'_{Q,R}(\infty)$” is to be interpreted as “$(f_{Q,R})'(\infty)$” and not as “$(f')_{Q,R}(\infty)$”!) For $n \geq m$ where $m$ is the integer such that $G_m$ contains $Q$, define

$$g_{Q,n}(z) = \sum_{R \in G_n(Q)} g_{Q,R}(z) = \sum_{R \in G_n(Q)} \frac{f'_{Q,R}(\infty)}{c_R - z}.$$ 

**Lemma 2.16** For $Q \in G_m$, the sequence $g_{Q,n}(z_Q)$, defined for $n \geq m$, is bounded.

**Proof** Set $h_{Q,R} = f_{Q,R} + g_{Q,R}$ and let $B = B(c_R; l(R))$. Using (b) of Lemma 2.14 and Proposition 1.9, one sees that for any $z \in B \setminus R$,

$$|(c_R - z)^2 f_{Q,R}(z)| \leq l(R)^2 \|f_{Q,R}\|_\infty \leq l(R)^2 M \|f\|_\infty$$

and

$$|(c_R - z)^2 g_{Q,R}(z)| = |(c_R - z) f'_{Q,R}(\infty)| \leq l(R) \gamma(K) \|f_{Q,R}\|_\infty$$

$$\leq l(R)^2 \gamma(K)^2 M \|f\|_\infty.$$ 

So, setting $M_1 = \{1 + \gamma(K)\}M$ on $B \setminus R$, one has $|(c_R - z)^2 h_{Q,R}(z)| \leq l(R)^2 M_1 \|f\|_\infty$.

By the way $g_{Q,R}$ has been defined, $h_{Q,R} \in H^\infty(C^* \setminus R)$, $h_{Q,R}(\infty) = 0$, and $h'_{Q,R}(\infty) = 0$. Hence the function $(c_R - z)^2 h_{Q,R}(z)$, besides being analytic on $C \setminus R$, is also still analytic at $\infty$. Thus, by the Maximum Modulus Principle [RUD, 12.1 modified to take into account regions containing $\infty$], the supremum norm of this function on $C^* \setminus R$ is the same as its supremum norm on $B \setminus R$. We conclude that for any $z \in C^* \setminus R$,

$$(\star) \ |h_{Q,R}(z)| \leq \frac{l(R)^2 M_1 \|f\|_\infty}{|c_R - z|^2}.$$ 

For $z$ any point strictly southwest of $z_Q$, note that $(\star\star) \ |c_R - z| \geq |c_R - z_Q|$ for all $R \in G(Q)$. Hence by (c) of Lemma 2.14, $(\star)$, $(\star\star)$, and (a) of Lemma 2.15, we have

$$|g_{Q,n}(z)| \leq \sum_{R \in G_n(Q)} |h_{Q,R}(z)| + |f(z)| \leq \sum_{R \in G_n(Q)} \frac{l(R)^2 M_1 \|f\|_\infty}{|c_R - z_Q|^2} + \|f\|_\infty$$

$$\leq M_2 \|f\|_\infty$$

where $M_2 = 3M_1 + 1$. Finally, letting $z \to z_Q$, we are done. \qed

**Lemma 2.17** Suppose $Q \in G_m$ and $f \in H^\infty(C^* \setminus K_Q)$ with $f(\infty) = 0$. If $f'(\infty) \neq 0$, then for some $R \in G(Q)$, we must have
\[ f'_{Q,R}(\infty) \neq \frac{l(R)}{l(Q)} f'(\infty). \]

**Proof** We proceed by contradiction. Assuming the nonexistence of \( R \) as in the lemma’s conclusion, we have

\[ g_{Q,n}(z_Q) = \sum_{R \in \mathcal{G}_n(Q)} \frac{l(R)}{l(Q)} \frac{f'(\infty)}{c_R - z_Q} \]

and so

\[
\text{Re} \left\{ \frac{l(Q)}{f'(\infty)} g_{Q,n}(z_Q) \right\} = \text{Re} \left\{ \sum_{R \in \mathcal{G}_n(Q)} \frac{l(R)}{c_R - z_Q} \right\} = \sum_{R \in \mathcal{G}_n(Q)} \frac{l(R) \cdot \text{Re}(c_R - z_Q)}{|c_R - z_Q|^2} = B_n(Q).
\]

We are now in a pickle: on the one hand, the leftmost item in the above must be bounded by the last lemma, while on the other hand, the rightmost item in the above must be unbounded by (b) of the next to last lemma.

**Lemma 2.18** For \( M < \infty \) and \( a > 0 \), there exists a \( \delta > 0 \) such that for any \( Q \in \mathcal{G} \) and any \( f \in H^\infty(\mathbb{C}^* \setminus K_Q) \), if \( \|f\|_\infty \leq M \), \( f(\infty) = 0 \), and \( |f'(\infty)|/l(Q) \geq a \), then there exists an \( R \in \mathcal{G}(Q) \) such that

\[ |f'_{Q,R}(\infty)| \geq (1 + \delta) \frac{|f'(\infty)|}{l(Q)}. \]

**Proof** Given \( M < \infty \) and \( a > 0 \), we first find a \( \delta > 0 \) that works only for the largest square in \( \mathcal{G} \), i.e., the square \( Q_0 = C_0 \) for which \( l(Q_0) = 1 \), \( K_{Q_0} = K \), \( \mathcal{G}(Q_0) = \mathcal{G} \), and \( \mathcal{G}_n(Q_0) = \mathcal{G}_n \). If the lemma fails for this square \( Q_0 \), then there exist sequences \( \delta_k \downarrow 0 \) and \( \{f_k\} \) from \( H^\infty(\mathbb{C}^* \setminus K) \) such that \( \|f_k\|_\infty \leq M \), \( f_k(\infty) = 0 \), and \( |f'_k(\infty)| \geq a \), and yet for all \( k \) and all \( R \in \mathcal{G} \)

\[ \frac{|f'_{Q_0,R}(\infty)|}{l(R)} \leq (1 + \delta_k)|f'_k(\infty)|. \]

Since the functions \( \{f_k\} \) form a normal family [RUD, 14.6], a subsequence of them converges to a function \( f \in H^\infty(\mathbb{C}^* \setminus K) \) satisfying \( \|f\|_\infty \leq M \), \( f(\infty) = 0 \), \( |f'(\infty)| \geq a \), and \( |f'_{Q_0,R}(\infty)| \leq l(R)|f'(\infty)| \) for all \( R \in \mathcal{G} \). Now there are \( 4^n \) squares \( R \in \mathcal{G}_n = \mathcal{G}_n(Q_0) \) and for each of these squares we have \( l(R) = 4^{-n} \). Hence by (c) of Lemma 2.14, we have for every \( n \) that
\[|f'(\infty)| = \sum_{R \in \mathcal{G}_n(Q_0)} |f'_{Q_0,R}(\infty)| \leq \sum_{R \in \mathcal{G}_n(Q_0)} 4^{-n}|f'(\infty)| = |f'(\infty)|.\]

This forces \(f'_{Q_0,R}(\infty) = 4^{-n}f'(\infty) = (l(R)/l(Q_0))f'(\infty)\) for all \(R \in \bigcup_n \mathcal{G}_n(Q_0) = \mathcal{G}(Q_0)\), which, being a contradiction to the last lemma, shows that there exists a \(\delta > 0\) that works for \(Q_0\).

Now we show that the \(\delta > 0\) just gotten for \(Q_0\) actually works for any \(Q \in \mathcal{G}\). To this end, let \(f \in H^\infty(\mathbb{C}^* \setminus K_Q)\) be such that \(\|f\|_{\infty} \leq M\), \(f(\infty) = 0\), and \(|f'(\infty)|/l(Q) \geq a\). For the similitude \(\varphi(z) = l(Q)z + z_Q\), the square \(Q^*\) for which \(\varphi(Q^*) = Q\) is just \(Q_0\). Thus \(f \circ \varphi \in H^\infty(\mathbb{C}^* \setminus K)\) is such that \(\|f \circ \varphi\|_{\infty} \leq M\), \((f \circ \varphi)(\infty) = 0\), and \(|(f \circ \varphi)'(\infty)| = |f'(\infty)|/l(Q) \geq a\). So, by the last paragraph, for some square \(R^* \in \mathcal{G}\) we must have

\[
\frac{|(f \circ \varphi)'_{Q_0,R^*}(\infty)|}{l(R^*)} \geq (1 + \delta)|(f \circ \varphi)'(\infty)|.
\]

By (d) of Lemma 2.14, \((f \circ \varphi)_{Q_0,R^*} = f_{Q,R} \circ \varphi\) for \(R = \varphi(R^*)\). We also have \((f_{Q,R} \circ \varphi)'(\infty) = f_{Q,R}(\infty)/l(Q), l(R^*) = l(R)/l(Q)\), and \((f \circ \varphi)'(\infty) = f'(\infty)/l(Q)\). Applying all these equalities to the inequality above and doing a bit of algebra, we obtain our conclusion. \(\square\)

**Theorem 2.19** The planar Cantor quarter set has positive finite linear Hausdorff measure, yet is removable.

**Proof** We only need to suppose that \(\gamma(K) > 0\) and deduce a contradiction. So let \(f \in H^\infty(\mathbb{C}^* \setminus K)\) be such that \(\|f\|_{\infty} \leq 1\), \(f(\infty) = 0\), and \(|f'(\infty)| > 0\). Let \(\delta > 0\) be gotten from the last lemma with \(M = 1 + 6/\pi\) [as in (b) of Lemma 2.14] and \(a = |f'(\infty)|\). Since the last lemma applies to \(Q_0\) and \(f\), we get \(Q_1 \in \mathcal{G}(Q_0)\) such that

\[
\frac{|f'_{Q_0,Q_1}(\infty)|}{l(Q_1)} \geq (1 + \delta)|f'(\infty)|.
\]

Since the last lemma applies to \(Q_1\) and \(f_{Q_0,Q_1}\) [to see this use (a) and (b) of Lemma 2.14 here], we get \(Q_2 \in \mathcal{G}(Q_1)\) such that

\[
\frac{|(f_{Q_0,Q_1})'_{Q_1,Q_2}(\infty)|}{l(Q_2)} \geq (1 + \delta)\frac{|f'_{Q_0,Q_1}(\infty)|}{l(Q_1)}.
\]

Combining the last two inequalities and using (e) of Lemma 2.14, we have

\[
\frac{|f'_{Q_0,Q_2}(\infty)|}{l(Q_2)} \geq (1 + \delta)^2|f'(\infty)|.
\]
Continuing in this manner, we obtain a sequence of squares \( \{Q_n\} \) from \( \mathcal{G} \) such that
\[
\frac{|f'_{Q_0, Q_n}(\infty)|}{l(Q_n)} \geq (1 + \delta)^n |f'(\infty)|.
\]
However, by Proposition 1.9 and (b) of Lemma 2.14,
\[
|f'_{Q_0, Q_n}(\infty)| \leq \gamma(K_{Q_n}) \|f_{Q_0, Q_n}\|_{\infty} \leq l(Q_n) \gamma(K) M.
\]
From the last two inequalities we immediately obtain \((1 + \delta)^n |f'(\infty)| \leq M \gamma(K)\) for all integers \(n\). This is patently absurd given that \(\delta > 0\) and \(|f'(\infty)| > 0!\) \(\square\)