Chapter 2
Mathematical Foundations

In this chapter, we present certain mathematical and continuum mechanics principles that are relevant to constitutive modeling and boundary value analysis. One of the most important topics is the concept of tensors. Tensors are quantities that obey coordinate transformation rules. Examples of tensors include certain scalars (mass, area, and volume), vectors (displacements and forces) and certain quantities encountered in continuum mechanics (stresses and strains).

We will limit our discussions to rectangular Cartesian tensors, where tensors are defined with respect to a rectangular Cartesian (Euclidean) coordinate system. In a Cartesian coordinate system, the axes are orthogonal to each other. Tensors defined with respect to a curvilinear coordinate system require the consideration of the concepts of contravariant and covariant tensors. In a rectangular Cartesian system, there is no distinction between Cartesian, contravariant, and covariant tensors.

2.1 Transformation Rules

2.1.1 Scalars

Physical quantities such as the mass, volume, area, and length of a vector remain independent of the orientation of the coordinate axes. For example, denoting mass by \( m \)

\[
(m)_{x_1-x_2-x_3} = (m)_{x'_1-x'_2-x'_3}
\]  

(2.1)

where \( x_1-x_2-x_3 \) is the original coordinate system and \( x'_1-x'_2-x'_3 \) is the rotated coordinate system.
2.1.2 Vectors

Examples of vectors include position, displacement, force, velocity, and acceleration. Such vectors can be quantified using a certain coordinate system. The directions of the axes are defined in terms of what are known as basis vectors. Denoting the basis vectors in three dimensions in the $x_1 - x_2 - x_3$ system as $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ (Fig. 2.1), where

$$
\mathbf{e}_1 = (1, 0, 0); \quad \mathbf{e}_2 = (0, 1, 0); \quad \mathbf{e}_3 = (0, 0, 1)
$$

(2.2)

a vector such as a displacement vector $\mathbf{u}$ can be expressed as

$$
\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3
$$

(2.3)

where $u_1$, $u_2$, and $u_3$ are components of the vector $\mathbf{u}$.

The basis vectors must be independent of each other, i.e., $\alpha \mathbf{e}_1 + \beta \mathbf{e}_2 + \gamma \mathbf{e}_3 = 0$ only when $\alpha = 0$, $\beta = 0$ and $\gamma = 0$.

**Definition 2.1: Indicial notation (or tensor notation).** A tensor may be expressed by appending indices to the letter representing the tensor. Hence, a vector $\mathbf{x}$ may be expressed as $x_i$. Similarly, higher order tensors that we will define later may be expressed as $x_{ij}$ (second order), $x_{ijk}$ (fourth order), etc.

**Definition 2.2: Summation convention.** Unless explicitly stated, repeated indices appearing as subscripts on quantities on the same side of an equation imply summation. The number of terms in the summation equals the dimension of the space. The convention is used regardless of whether the quantity is a tensor or not.

For example, in a 3D Cartesian system,

$$
\alpha = x_i x_i = x_1 x_1 + x_2 x_2 + x_3 x_3 = x_1^2 + x_2^2 + x_3^2
$$

where $\alpha$ is a scalar, and $x_1$, $x_2$ and $x_3$ are the components of the vector $\mathbf{x}$. The subscript $i$ repeats on the right side of the equation. The subscript $i$ used to indicate summation is known as the “dummy” index since any arbitrary letter can be used for the index. For example

$$
\alpha = x_i x_j = x_k x_k
$$

![Fig. 2.1 Basis vectors for rectangular Cartesian coordinate system](image)
Definition 2.3: The Kronecker delta. The Kronecker delta \( \delta_{ij} \) is defined as
\[
\delta_{ij} = \begin{cases} 
1 & \text{for } i = j \\
0 & \text{for } i \neq j 
\end{cases}
\]

Hence, the Kronecker delta can be thought of as a \( 3 \times 3 \) identity matrix in three dimensions and a \( 2 \times 2 \) identity matrix in two dimensions.

Using the summation convention (Definition 2.2), (2.3) may be written as
\[
u = u_i e_i
\] (2.4a)

Note that the summation convention eliminates the need for using the customary summation sign as
\[
u = \sum_{i=1}^{3} u_i e_i
\] (2.4b)

Let us first consider coordinate transformation of a vector in two dimensions. The \( x_1 - x_2 \) system is rotated through an angle \( \theta \) in the counter clockwise direction to obtain the \( x'_1 - x'_2 \) system as shown in Fig. 2.2.

Let the components of a vector \( \bar{u} \) in the two systems be
\[
(u_1, u_2) \text{ and } (u'_1, u'_2)
\] (2.5)

Projecting the vector \( \bar{u} \) onto the \( x'_1 - x'_2 \) system results in
\[
\begin{align*}
   u'_1 &= u_1 \cos \theta + u_2 \sin \theta \\
   u'_2 &= -u_1 \sin \theta + u_2 \cos \theta
\end{align*}
\] (2.6a)

Placing the components \( (u_1, u_2) \) and \( (u'_1, u'_2) \) in vectors as
\[
\begin{align*}
u &= (u_1, u_2) \text{ and } \nu' &= (u'_1, u'_2)
\end{align*}
\]
Equation (2.6) can be expressed in a matrix-vector form as

$$\mathbf{u}' = \mathbf{a}\mathbf{u}$$  \hspace{1cm} (2.7a)

where

$$\mathbf{a} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$  \hspace{1cm} (2.7b)

In indicial notation (Definition 2.1), (2.7a) is expressed as

$$u_i' = a_{ik}u_k$$  \hspace{1cm} (2.7c)

Notice the use of summation convention in (2.7c).

By projecting the vector $\bar{\mathbf{u}}$ onto the $x_1 - x_2$ system

$$u_1 = u_1'\cos \theta - u_2'\sin \theta$$  \hspace{1cm} (2.8a)

$$u_2 = u_1'\sin \theta + u_2' \cos \theta$$  \hspace{1cm} (2.8b)

In a matrix-vector form

$$\mathbf{u} = \mathbf{b}\mathbf{u}'$$  \hspace{1cm} (2.9a)

where

$$\mathbf{b} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$  \hspace{1cm} (2.9b)

In indicial notation

$$u_i = b_{ik}u_k'$$  \hspace{1cm} (2.9c)

The matrices $\mathbf{a}$ and $\mathbf{b}$ are known as the transformation or rotation matrices.

The properties of the transformation matrices are

$$\mathbf{a} = \mathbf{b}^T \text{ (or } a_{ij} = b_{ji}) \quad \text{and} \quad \mathbf{b} = \mathbf{a}^T \text{ (or } b_{ij} = a_{ji})$$  \hspace{1cm} (2.10a)

$$\mathbf{a}^T\mathbf{a} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$  \hspace{1cm} (2.10b)

$$\mathbf{a}^{-1} = \mathbf{a}^T$$  \hspace{1cm} (2.10c)
where $I$ is the identity matrix. In indicial notation, (2.10b) is written as

$$a_{ki}a_{kj} = \delta_{ij} \quad (2.10d)$$

where $\delta_{ij}$ is the Kronecker delta (Definition 2.3). Due to property (2.10c)

$$aa^T = aa^{-1} = I$$

or

$$a_{ik}a_{jk} = \delta_{ij} \quad (2.10e)$$

Hence, in general:

$$a^Ta = aa^T = I \quad (2.10f)$$

$$(2.10g)$$

**Definition 2.4: Linear transformation.** The transformation of vectors due to the rotation of coordinate system (2.7c)

$$u'_i = a_{ik}u_k$$

is known as the linear transformation since there is a linear relationship between the original and rotated components of the vector.

**Definition 2.5: Bilinear transformation and bilinear function.** A bilinear transformation assigns a scalar $f$ to two vectors $u$ and $v$ as

$$f = S_{ij}u_iv_j$$

Note that $f$ is linear in the components of $u$ as well as in the components of $v$. Such functions are called the bilinear functions.

**Definition 2.6: Orthogonal matrices.** Matrices satisfying the property $a^Ta = I$ are known as orthogonal matrices. Hence, the transformation matrix in the transformation of a vector in a Cartesian coordinate system is orthogonal.

**Definition 2.7: Orthogonal transformation.** When $a$ appearing in the linear transformation $u'_i = a_{ik}u_k$ is orthogonal, the transformation is also called the orthogonal transformation.

Examination of the components of $a$ reveals that they are the direction cosines of the unit vectors along the axes. This may be formally shown as follows.
Let \( \mathbf{u} \) be a two-dimensional unit vector along the \( x_1 \)-axis. Then in the \( x_1 - x_2 \) system,

\[
\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

Defining \( \cos(x'_i, x_j) \) as the cosine of angle between \( x'_i \) and \( x_j \) axes, the components of \( \mathbf{u} \) in the \( x'_1 - x'_2 \) system are

\[
\mathbf{u}' = \begin{bmatrix} \cos(x'_1, x_1) \\ \cos(x'_2, x_1) \end{bmatrix}
\]

Now by the relation

\[
\mathbf{u}' = \mathbf{a} \mathbf{u}
\]

\[
\begin{bmatrix} \cos(x'_1, x_1) \\ \cos(x'_2, x_1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}
\]

(2.11a)

The above result will hold for other cases as well. Hence, in general

\[
a_{ij} = \cos(x'_i, x_j)
\]

(2.11b)

In two dimensions:

\[
\mathbf{a} = \begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) \end{bmatrix}
\]

(2.11c)

By expressing the cosines in (2.11c) in \( \theta \) shown in Fig. 2.2, we get (2.7b).

In three dimensions:

\[
\mathbf{a} = \begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) & \cos(x'_1, x_3) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) & \cos(x'_2, x_3) \\ \cos(x'_3, x_1) & \cos(x'_3, x_2) & \cos(x'_3, x_3) \end{bmatrix}
\]

(2.11d)

According to Definition 2.4, the transformation \( u'_i = a_{ik} u_k \) (2.7c), which relates the vector \( \mathbf{u} \) to \( \mathbf{u}' \) in a Cartesian coordinate system is a linear transformation. The function on the right-hand side of in (2.7c) is linear in the components of \( \mathbf{u} \).

Consider a projection of the vector \( \mathbf{u} \) onto an arbitrary direction \( \eta = (\eta_1, \eta_2) \)

\[
F_{\eta} = u_i \eta_i
\]

(2.12a)
Let us show that the numerical value of $F_\eta$ is independent of the orientation of the coordinate system:

$$F'_\eta = u'_i \eta'_i$$

$$= (a_{ik} u_k)(a_{i\ell} \eta_\ell) \text{ (using (2.7c))}$$

$$= \delta_{k\ell} u_k \eta_\ell \text{ (using (2.10g))}$$

$$= u_k \eta_k \text{ (using contraction operation, defined below)}$$

$$= F_\eta \quad \text{(2.12b)}$$

Note that when $\eta = u$, we have $F_\eta = u_i u_i = |u|$. Hence, the length of a vector is independent of the orientation of the coordinate system in which the components of $u$ are expressed.

While we will formally introduce the contraction operation for tensors later, the operation involved in the last step of (2.12b) may be easily understood by expanding the term $\delta_{k\ell} \eta_\ell$ and noting that a term in the summation is nonzero only when $\ell = k$ (from the definition of the Kronecker delta) as.

$$\delta_{k\ell} \eta_\ell = \delta_{k1} \eta_1 + \delta_{k2} \eta_2 + \delta_{k3} \eta_3$$

$$\delta_{k\ell} \eta_\ell = \delta_{11} \eta_1 + \delta_{12} \eta_2 + \delta_{13} \eta_3 = \eta_1 \quad \text{for } k = 1$$

$$\delta_{k\ell} \eta_\ell = \delta_{21} \eta_1 + \delta_{22} \eta_2 + \delta_{23} \eta_3 = \eta_2 \quad \text{for } k = 2$$

$$\delta_{k\ell} \eta_\ell = \delta_{31} \eta_1 + \delta_{32} \eta_2 + \delta_{33} \eta_3 = \eta_3 \quad \text{for } k = 3$$

Hence,

$$\delta_{k\ell} \eta_\ell = \eta_k \quad \text{(2.12c)}$$

That is, replace the index of $\eta$ (i.e., $\ell$) with one of the indices of $\delta$ that is different from the index of $\eta$ (i.e., $k$).

Equation (2.12b) proves that the component of the vector $u$ in any arbitrary direction $\eta$ is invariant with respect to the orientation of the coordinate system in which the components of $u$ and $\eta$ are expressed. Combining this with the fact that $F_\eta$ is linear in $\eta_i$, we say that a vector $u$ in an arbitrary direction $\eta$ is an invariant linear function of the components of the direction $\eta$ (Butkov 1968).

### 2.1.3 Stress

We discussed certain properties of a vector in the preceding section. We will see in the next section, after giving a formal definition of a tensor that vectors are first-order tensors. We will also see that stress is an example of a second-order tensor.
To facilitate the definition of tensors in general, we will first present some analysis of stress in this section.

First, consider a two-dimensional case. The stresses acting on a differential element is shown in Fig. 2.3a with respect to the $x_1 - x_2$ coordinate system. We are interested in finding the traction vector $t = (t_1, t_2)$ on an inclined plane shown in Fig. 2.3b. Traction is defined as force per unit area; hence, traction is the resultant stress at a point on a plane.

The inclined plane has a unit normal $n = (a_{n1}, a_{n2})$; hence

$$n = (a_{n1}, a_{n2}) = [\cos(n, e_1), \cos(n, e_2)] = (\cos \theta, \sin \theta)$$

(2.13)

Let the length of the inclined side be $\ell$. Now let us enforce equilibrium in $x_1$ – and $x_2$ – directions as

$$t_1 \ell = \sigma_{11} \ell \cos \theta + \sigma_{21} \ell \sin \theta \Rightarrow t_1 = \sigma_{11} a_{n1} + \sigma_{21} a_{n2}$$

(2.14a)

$$t_2 \ell = \sigma_{22} \ell \sin \theta + \sigma_{12} \ell \cos \theta \Rightarrow t_1 = \sigma_{12} a_{n1} + \sigma_{22} a_{n2}$$

(2.14b)

Hence,

$$t = \mathbf{\sigma}^T n; \quad \mathbf{\sigma}^T = \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

(2.14c)

or

$$t_i = \sigma_{ji} a_{nj}$$

(2.14d)

It is assumed that the reader is familiar with the result that shear stresses acting on the faces parallel to the $x_1$ – and $x_2$ – axes are equal to each other (i.e., $\sigma_{12} = \sigma_{21}$) – these
are called the complementary shear stresses. Utilizing this symmetry property, (2.14d) is written as

\[ t_i = \sigma_{ij}a_{nj} \quad \text{(2.15)} \]

The normal and shear stresses on the inclined plane may now be found by projecting \( t \) onto the normal \( n \) and shear directions.

But here we are interested in projecting onto an arbitrary direction (unit vector) \( m \) shown in Fig. 2.3c, where

\[ m = (m_1, m_2) = (\cos(m,e_1), \cos(m,e_2)) = (\cos\gamma', \sin\gamma') \quad \text{(2.16)} \]

We will call this \( \sigma_{nm} \) (stress component in direction \( m \) on a plane whose unit normal is \( n \)). Using (2.14d), the expression for \( \sigma_{nm} \) is

\[ \sigma_{nm} = m \cdot t = m^T \sigma n = a_{mi} (\sigma_{ji}a_{nj}) = a_{nj} \sigma_{ji}a_{mi} \quad \text{(2.17)} \]

According to Definition 2.5, \( \sigma_{nm} \) given by (2.17) is a \textit{bilinear function} in the components of the directions \( m \) and \( n \).

Let the stress components with respect to the \( x'_1 - x'_2 \) system be \( \sigma'_{ij} \). When \( n = e'_1 \) and \( m = e'_1 \), then \( \sigma_{nm} = \sigma'_{11} \). When \( n = e'_1 \) and \( m = e'_2 \), then \( \sigma_{nm} = \sigma'_{12} \). Hence, it may be seen that, upon letting \( n \) and \( m \) coincide with \( e'_1 \) and \( e'_2 \) in an alternating manner, (2.17) generates the components of \( \sigma'_{ij} \). Hence, (2.17) is a transformation rule that gives \( \sigma'_{ij} \) (i.e., components in the \( x'_1 - x'_2 \) system) in terms of \( \sigma_{ij} \) (i.e., components in the \( x_1 - x_2 \) system), i.e.,

\[ \sigma' = a^T \sigma a \quad \text{or} \quad \sigma'_{ij} = a_{ik} \sigma_{ij} a_{jl} = \sigma_{kl} a_{ik} a_{jl} \quad \text{(2.18a)} \]

It is further noted that the transformation matrix \( a \) is the same transformation matrix that rotated a vector as \( u' = au \) (2.7a).

Using the orthogonality property of \( a \) (2.10f), (2.18a) may be written (by pre- and post-multiplying (2.18a) with \( a^T \) and \( a \), respectively, and simplifying) as

\[ \sigma = a^T \sigma' a \quad \text{or} \quad \sigma_{ij} = a_{ki} \sigma'_{kl} a_{lj} = \sigma_{kl} a_{ki} a_{lj} \quad \text{(2.18b)} \]

As in the case with vectors (2.12b), there exist an invariant property for stresses. Let us examine this below. In a Cartesian coordinate system, the numerical value of \( \sigma_{nm} \) given by (2.17) is independent of the orientation of the coordinate system in which the components \( \sigma_{ij} \) appearing on the right-hand side of (2.17) are expressed. While this is evident on physical grounds, it may be theoretically proved as follows. First to avoid confusion of notations, let us introduce the following notations

\[ \mu = (\mu_1, \mu_2) = m = (a_{m1}, a_{m2}) \quad \text{(2.19a)} \]

\[ \eta = (\eta_1, \eta_2) = n = (a_{n1}, a_{n2}) \quad \text{(2.19b)} \]
Now consider the term
\begin{align*}
a_{mi} \sigma'_{ij} \eta'_{nj} &= \sigma'_{ij} \mu'_{i} \eta'_{j} \\
&= (\sigma_{kl} a_{ik} a_{jk})(a_{ip} \mu_{p})(a_{iq} \eta_{q}) \quad \text{(using (2.7c) and (2.18a))} \\
&= \sigma_{kl} (a_{ik} a_{ip})(a_{jk} \mu_{p})(\mu_{p} \eta_{q}) \quad \text{(rearranging)} \\
&= \sigma_{kl} \delta_{ip} \delta_{jq} \mu_{p} \eta_{q} \quad \text{(using the orthogonality property in (2.10g))} \\
&= \sigma_{kl} \mu_{k} \eta_{l} \quad \text{(using contraction operations (2.12c))}
\end{align*}

Hence, (2.17) becomes
\begin{equation}
\sigma_{mn} = \sigma'_{ij} \mu'_{i} \eta'_{j} = \sigma_{ij} \mu_{i} \eta_{j} \tag{2.21}
\end{equation}

which proves that \( \sigma_{mn} \) is independent (invariant) of the orientation of the coordinate system in which the components \( \sigma_{ij} \) appearing on the right-hand side of (2.17) are expressed.

It follows from (2.21) that stress with respect to two arbitrary directions \( \mu \) and \( \eta \) is an invariant bilinear function of the components of the two directions \( \mu \) and \( \eta \).

The relationships derived in this section in two dimensions are valid in three dimensions as well; the derivation is carried out by following the same approaches taken in this section. This is the subject of Problem 2.3.

### 2.2 Definition of a Cartesian Tensor

In Sect. 2.1, we examined the manner in which scalars, vectors, and stress transform when the coordinate system is rotated. In particular, we noted that (1) vectors and stress projected onto arbitrary directions remain invariant with regard to the orientation of the original coordinate system, and (2) the components of vectors and stress in a rotated coordinate system are obtained from the components in the original coordinate system through a coordinate transformation rule. Most scalars obey these rules (they remain unchanged when the coordinate system is rotated). These are the properties that define a tensor. Note, however, that the two properties listed above are equivalent (we will show this later).

One of the following two definitions may be used for a Cartesian tensor (Butkov 1968):

1. A tensor \( T_{ij \ldots n} \) (with \( m \) indices) yields a multi-linear, invariant function in arbitrary directions \( \eta_{i}, \mu_{i}, \ldots, \xi_{i} \) (\( m \) unit vectors) as
\begin{equation}
\alpha = T_{ij \ldots n} \eta_{i} \mu_{j} \ldots \xi_{n} \tag{2.22}
\end{equation}
The summations are done from 1 to $N$, where $N$ is the dimension of the space. $T_{ij...n}$ are called the component of the tensor, and are defined with respect to the same bases as those of $\eta, \mu, \ldots \xi$. The tensor $T_{ij...n}$ is known as the tensor of order (or rank) $m$.

2. A tensor $T_{ij...n}$ (with $m$ indices) is one that transforms under rotations of the coordinate system according to the equation

$$T'_{ij...n} = T_{pq...ra_{ip}a_{jq}...a_{nr}}$$

where $a$ is the transformation matrix that transforms a vector from the original coordinate system to a rotated coordinate system as $u' = au$. The tensor $T_{ij...n}$ is known as the tensor of order (or rank) $m$.

It may now be verified that vectors are first order tensors and stress is a second order tensor (by either the first or second definition). It is customary to consider scalars as tensors of zero order.

It is worth repeating the other examples mentioned earlier: velocity, acceleration, and force are vectors and hence are first order tensors. Strain is a second order tensor. In flow through porous media, one encounters a permeability tensor which is a second order tensor. The analogous quantity in heat flow is the heat conductivity tensor and in electric flow is the electric conductivity tensor.

It should also be noted that a quantity represented by a matrix is not always a tensor. A matrix is a mathematical concept and may or may not satisfy the requirements of a tensor as illustrated in Examples 2.1 and 2.2.

**Example 2.1.**

**Question:** Determine if the following matrix is a tensor or not.

$$A = \begin{bmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{bmatrix}$$

where $x_1$ and $x_2$ are components of a vector $\mathbf{x}$ in two dimensions.

**Answer:** To examine this using the invariance property (2.22), consider

$$\mathbf{x} = A_{ij} \mu_i \eta_j = x_1^2 \mu_1 \eta_1 + x_1x_2 \mu_1 \eta_2 + x_1x_2 \mu_2 \eta_1 + x_2^2 \mu_2 \eta_2$$

Since $\mu$ and $\eta$ are arbitrary, let

$$\mu = \eta = \frac{\mathbf{x}}{||\mathbf{x}||} = \frac{1}{||\mathbf{x}||} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
Then
\[
\begin{align*}
\alpha &= \frac{1}{|\mathbf{x}|} (x_1^4 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4) \\
&= \frac{(x_1^2 + x_2^2)^2}{|\mathbf{x}|} \\
&= |\mathbf{x}|^3
\end{align*}
\]

Since length of a vector is an invariant under rotation of coordinate systems, \(\alpha\) is an invariant as well. Hence, the matrix \(A\) is a tensor.

To carry out the investigation by the transformation property (2.23), let us write \(A\) as
\[
\begin{bmatrix}
x_1' \\
x_2'
\end{bmatrix}
= \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\begin{bmatrix}
x_1' \\
x_2'
\end{bmatrix}
= \mathbf{x}' \mathbf{x}'^T
= \mathbf{a} \mathbf{x} \mathbf{x}^T \mathbf{a}^T
= \mathbf{a} \mathbf{a}^T
\]

Hence, \(A\) is a tensor.

**Example 2.2.**

**Question:** Determine if the following matrix is a tensor or not.

\[
B = \begin{bmatrix}
x_1' x_2' \\
x_1 x_2 \\
x_1^2 \\
x_1 x_2
\end{bmatrix}
\]

where \(x_1\) and \(x_2\) are components of a vector \(\mathbf{x}\) in two dimensions.

**Answer:** To use the first definition of tensors (2.22), examine:
\[
\alpha = B_{ij} \mu_i \eta_j = x_1 x_2 \mu_1 \eta_1 + x_2^2 \mu_1 \eta_2 + x_1^2 \mu_2 \eta_1 - x_1 x_2 \mu_2 \eta_2
\]

Let
\[
\begin{pmatrix}
\mu \\
\eta
\end{pmatrix}
= \frac{1}{|\mathbf{x}|} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

Then
\[
\begin{align*}
\alpha &= C_{ij} \mu_i \eta_j \\
&= \frac{1}{|\mathbf{x}|} (x_1^3 x_2 + x_1 x_2^3 + x_1^3 x_2 - x_1 x_2^3) \\
&= \frac{1}{|\mathbf{x}|} [2x_1^3 x_2]
\end{align*}
\]
which is not an invariant under rotations of coordinate systems. For example, when the coordinate system is rotated such that
\[
x' = |x|(\cos 45^\circ, \sin 45^\circ) = \frac{|x|}{\sqrt{2}} (1, 1), \text{ then } x' = \frac{|x|^3}{2}
\]

When the coordinate system is rotated such that
\[
x' = |x|(\cos 30^\circ, \sin 30^\circ) = \frac{|x|}{2} (\sqrt{3}, 1), \text{ then } x' = \frac{3\sqrt{3}|x|^3}{8}
\]

**Example 2.3.**

**Question:** Prove that the two definitions of a tensor (2.22) and (2.23) are equivalent.

**Answer:** It was proved earlier (2.21) that, for stress, which is a second order tensor, the transformation property (2.22) leads to the invariant property (2.23). Extension of the proof to higher order tensors is conceptually straightforward.

To prove the inverse (that the invariance property leads to the transformation property), let us consider a second order tensor again. By the invariance property
\[
T_{ij}' \eta_i' \mu_j' = T_{kl} \eta_k \mu_l
\]

By the inverse transformation (2.9c) and using (2.10a)
\[
\eta_k = b_{kp} \eta_p' = a_{pk} \eta_p'
\]
\[
\mu_l = b_{lq} \mu_q' = a_{ql} \mu_q'
\]

Hence,
\[
T_{ij}' \eta_i' \mu_j' = T_{kl}(a_{pk} \eta_p')(a_{ql} \mu_q') = (T_{kl} a_{pk} a_{q\ell}) \eta_p' \mu_q'
\]

Let us consider two dimensions. Since the above relation must hold for any arbitrary directions \( \eta' \) and \( \mu' \), let us choose \( \eta' = (1, 0) \) and \( \mu' = (0, 1) \). Then
\[
T_{12}' = T_{k\ell} a_{1k} a_{2\ell}
\]

By repeating this process with appropriate choices for \( \eta' \) and \( \mu' \), it can be shown
\[
T_{ij}' = T_{k\ell} a_{ik} a_{j\ell}
\]

Hence, we have proven that the invariance property leads to the transformation property.
2.3 Operations with Tensors

2.3.1 Addition of Tensors

When a tensor is multiplied by a scalar constant, it remains a tensor with its order unchanged. Tensors of the same order may be added or subtracted to produce another tensor of the same order as

\[ \alpha T_{ij...n} \pm \beta S_{ij...n} = U_{ij...n} \]  

(2.24)

where \( \alpha \) and \( \beta \) are scalars.

2.3.2 Contraction Operation

In a contraction operation, two of the indices are set equal to each other. The summation convention applies over these repeated indices. For example, \( T_{ijkk} \) is a contraction of \( T_{ijk} \) over the last two indices, and expands as

\[ T_{ijkk} = T_{ij11} + T_{ij22} + T_{ij33} \]  

(2.25)

A contraction of a tensor of order \( m \) results in a tensor of order \( m-2 \). In the above example, a fourth order tensor becomes a second order tensor. To prove this, consider the contraction of a third order tensor \( A'_{ijk} \) and simplify using the transformation property as

\[
A'_{ijk} = A_{pqr} a_{ip} a_{jq} a_{kr} \\
A'_{ijk} = A_{pqr} a_{ip} a_{iq} a_{kr} = A_{pqr} \delta_{pq} a_{kr} = A_{ppr} a_{kr} \text{ (using (2.10d))}
\]

Hence, \( A_{ijk} \) transforms as a first order tensor. The proof for tensors of other orders is similarly obtained.

Contraction of a second order tensor \( A_{ij} \) results in its trace \( A_{ii} = A_{11} + A_{22} + A_{33} \).

Some examples of contraction operation are

- \( B_{ij} = T_{ijkk} \)
- \( B_{ijk} = T_{ijkl} a_{kl} \)
- \( B_{ij} = A_{ik} C_{kj} \)

2.3.3 Outer Product of Tensors

An outer product (also called the tensor product) of two tensors of arbitrary orders is multiplication of its components with distinct indices for each as
If the order of \( A \) and \( B \) are \( m \) and \( n \), respectively, then the order of \( C \) is \( m+n \). Some examples of outer products are

- \( C_{ijkl} = A_{ij}B_{kl} \)
- \( C_{ipqrs} = A_{ij}B_{pqrs} \)
- \( C_{ij} = u_{ij}v_{ij} \)

The last outer product \( C_{ij} = u_{ij}v_{ij} \) is the outer product between two vectors. This is also known as the dyad.

### 2.3.4 Inner Product of Tensors

The inner product of two tensors is formed by contraction of their outer products. For example, contraction of the outer products of \( A \) and \( B \) in (2.26) in index \( j \) and \( q \) yields

\[
C_{ij...kpq...r} = A_{ij...k}B_{pq...r}
\]

(2.27)

It is seen that the inner product of two tensors of order \( m \) and \( n \) yields a tensor of order \( m + n - 2 \).

Some examples of inner products are

- \( x = u_{ij}v_{ij} \) (inner product between two vectors)
- \( C_{ij} = A_{ik}B_{kj} \)
- \( C_{ij} = A_{ik}B_{ik} \)
- \( C_{ij} = A_{ik}B_{k\ell\mu\nu} \)

### 2.3.5 The Permutation Symbol

The permutation symbol (also known as the alternator symbol or the Levi-Civita symbol) \( \varepsilon_{ijk} \) is a third order tensor defined as

- \( \varepsilon_{ijk} = 1 \) if \( ijk \) is an even permutation of 123 (i.e., if they appear in a sequence as in the arrangement 12312)
- \( \varepsilon_{ijk} = -1 \) if \( ijk \) is an odd permutation of 123 (i.e., if they appear in a sequence as in the arrangement 32132)
- \( \varepsilon_{ijk} = 0 \) if \( ijk \) is not a permutation of 123 (i.e., if two or more of its indices have the same value).

For example, \( \varepsilon_{123} = 1 \) (even); \( \varepsilon_{213} = -1 \) (odd); \( \varepsilon_{231} = 1 \) (even); \( \varepsilon_{213} = -1 \) (odd); \( \varepsilon_{113} = 0 \) (repeated indices).
2.3.6 Cross Product of Two Vectors

Referring to Fig. 2.4, the cross product of two vectors \( \mathbf{x} \) and \( \mathbf{y} \) is defined as

\[
\mathbf{v} = \mathbf{x} \times \mathbf{y} = |\mathbf{x}||\mathbf{y}|(\sin\theta)\mathbf{e}
\]

(2.28)

Hence, the cross product of two vectors \( \mathbf{x} \) and \( \mathbf{y} \) is vector \( \mathbf{v} \), whose direction \( \mathbf{e} \) is normal to the plane in which \( \mathbf{x} \) and \( \mathbf{y} \) lie. The magnitude of the cross product is the area of the parallelogram formed by the vectors \( \mathbf{x} \) and \( \mathbf{y} \) as shown in Fig. 2.4.

The components of \( \mathbf{v} \) may be expressed in two different ways:

1. \[
    v_i = e_{ik\ell} x_k y_\ell
\]

(2.29a)

2. From the determinant of the following matrix

\[
\mathbf{v} = \begin{vmatrix}
    e_1 & e_2 & e_3 \\
    x_1 & x_2 & x_3 \\
    y_1 & y_2 & y_3
\end{vmatrix}
= (x_2y_3 - x_3y_2)e_1 + (x_3y_1 - x_1y_3)e_2 + (x_1y_2 - x_2y_1)e_3
\]

(2.29b)

It may be verified that both expressions are identical (Problem 2.7).

2.3.7 Symmetry of Tensors

A second order tensor \( T_{ij} \) is symmetric if

\[
T_{ij} = T_{ji} \quad \text{or} \quad \mathbf{T} = \mathbf{T}^T
\]

(2.30a)

Examples of such tensors are the stress and strain tensors.

A second order tensor \( T_{ij} \) is skew-symmetric or antisymmetric if

\[
T_{ij} = -T_{ji} \quad \text{or} \quad \mathbf{T} = -\mathbf{T}^T
\]

(2.30b)

\[\text{Fig. 2.4} \quad \text{Cross product of vectors}\]
Any second order tensor can be uniquely decomposed into a symmetric and anti-symmetric part as

\[ T_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}) \]  

(2.31a)

Denoting

\[ D_{ij} = \frac{1}{2}(T_{ij} + T_{ji}); \quad S_{ij} = \frac{1}{2}(T_{ij} - T_{ji}) \]  

(2.31b)

Equation (2.31a) is written as

\[ T_{ij} = D_{ij} + S_{ij} \]  

(2.31c)

Noting that

\[ D_{ji} = \frac{1}{2}(T_{ji} + T_{ij}) = D_{ij} \quad \text{and} \quad S_{ji} = \frac{1}{2}(T_{ji} - T_{ij}) = -S_{ij} \]

it is seen that \( D_{ij} \) is the symmetric part of \( T_{ij} \) and \( S_{ij} \) is the antisymmetric part of \( T_{ij} \).

Note that

\[ S_{ij} = 0 \quad \text{for} \quad i = j \]  

(2.32)

Hence, the diagonal elements of the antisymmetric part of second order tensors such as stress and strain are zero.

Extending these ideas to a tensor of any order, we define symmetric and antisymmetric tensors with respect to two of the indices of the tensor.

Some examples of such tensors are

- \( T_{ijk\ell} = T_{kij\ell} \) (symmetric in indices \( i \) and \( k \))
- \( T_{ijk\ell} = T_{kjitel} \) (antisymmetric in indices \( i \) and \( k \))
- \( T_{ijk\ell} = T_{kij\ell} \) (symmetric in indices \( i \) and \( k \), and in \( j \) and \( \ell \))
- \( e_{ijk} = e_{jik} \) (antisymmetric in indices \( i \) and \( j \))

**Example 2.4.**

**Question:** Given that \( A_{ij} \) and \( B_{ij} \) are symmetric and antisymmetric tensors, show that \( A_{ij}B_{ij} = 0 \).

**Answer:** To prove, let us expand \( A_{ij}B_{ij} \) as

\[ A_{ij}B_{ij} = -A_{ij}B_{ji} = -A_{ji}B_{ji} = -A_{pq}B_{pq} \]

\[ 2A_{ij}B_{ij} = 0 \]

\[ A_{ij}B_{ij} = 0 \]
2.3.8 Further Operations Involving the Kronecker and Levi-Civita Symbols

First, let us prove that the Kronecker delta \( \delta_{ij} \) is a tensor using the invariant property of tensors, i.e., consider

\[
\alpha = \delta_{ij} \eta_i \mu_j = \eta_i \mu_i
\]

Hence, \( \alpha \) equals the inner product between the two unit vectors \( \eta \) and \( \mu \). This is equal to the cosine of the angle between \( \eta \) and \( \mu \), which is independent of the orientation of the coordinate system. Hence, the Kronecker delta \( \delta_{ij} \) is a second order tensor.

Now let us use the transformation property

\[
\delta'_{ij} = \delta_{kl} a_{ik} a_{jl} = a_{il} a_{j\ell} = \delta_{ij} \text{ (using (2.10g))}
\]

Hence, the components of \( \delta_{ij} \) are independent of the orientation of the coordinate system. Such tensors (i.e., tensors whose components remain unchanged upon rotation of the coordinate system) are known as isotropic tensors.

Recall that, it was shown earlier \( \delta_{k\ell \eta \ell} = \eta_k \) ((2.12c)). Using this result, the following results are obtained:

- \( \delta_{ik} \delta_{kj} = \delta_{ij} \)
- \( \delta_{ij} \delta_{ij} = \delta_{ii} = 3 \)
- \( \delta_{ik} \delta_{k\ell} \delta_{\ell i} = \delta_{ii} = 3 \)
- \( T_{ijk\ell} \delta_{ij} = T_{ii\ell\ell} = B_{k\ell} \) (second order tensor)

It can be shown that

\[
\varepsilon_{ijk\ellrst} = \begin{vmatrix} \delta_{ir} & \delta_{is} & \delta_{it} \\ \delta_{jr} & \delta_{js} & \delta_{jt} \\ \delta_{kr} & \delta_{ks} & \delta_{kt} \end{vmatrix}
\]

To show this, consider the determinant

\[
\begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = 1
\]

The magnitude is 1 since the off diagonal terms of the matrix are all 0’s and the diagonal terms of the matrix are all 1’s.
We recall from the definition of determinants that the magnitude of the determinant of a matrix changes sign when two rows or columns are switched. For instance, switching columns,
\[
\begin{vmatrix}
\delta_{11} & \delta_{12} & \delta_{13} \\
\delta_{21} & \delta_{22} & \delta_{23} \\
\delta_{31} & \delta_{32} & \delta_{33}
\end{vmatrix}
= -
\begin{vmatrix}
\delta_{12} & \delta_{11} & \delta_{13} \\
\delta_{22} & \delta_{21} & \delta_{23} \\
\delta_{32} & \delta_{31} & \delta_{33}
\end{vmatrix}
\] (switching the first and second columns)

Based on the definition of the permutation symbol, this is achieved for any arbitrary column switches as
\[
\begin{vmatrix}
\delta_{1r} & \delta_{1s} & \delta_{1t} \\
\delta_{2r} & \delta_{2s} & \delta_{2t} \\
\delta_{3r} & \delta_{3s} & \delta_{3t}
\end{vmatrix}
= \varepsilon_{rst}
\begin{vmatrix}
\delta_{11} & \delta_{12} & \delta_{13} \\
\delta_{21} & \delta_{22} & \delta_{23} \\
\delta_{31} & \delta_{32} & \delta_{33}
\end{vmatrix}
= \varepsilon_{rst} \times 1 = \varepsilon_{rst}
\]

By the same logic, the sign change due to an arbitrary number of row switches is represented by
\[
\begin{vmatrix}
\delta_{lr} & \delta_{ls} & \delta_{lt} \\
\delta_{jr} & \delta_{js} & \delta_{jt} \\
\delta_{kr} & \delta_{ks} & \delta_{kt}
\end{vmatrix}
= \varepsilon_{ijk} \varepsilon_{rst}
\]
which is the required result.

Using this result, the following results may be obtained (Problem 2.9):

- \( \varepsilon_{ijk} \varepsilon_{ist} = \delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks} \)
- \( \varepsilon_{ijk} \varepsilon_{ijt} = 2 \delta_{kt} \)
- \( \varepsilon_{ijk} \varepsilon_{ijk} = 6 \)

### 2.4 Tensor Calculus

#### 2.4.1 Tensor Field

Denoting the position in space and time as \( x \) and \( t \), a tensor expressed as a function of \( x \) and \( t \) is known as the tensor field. In general
\[
T = T(x, t) \quad \text{or} \quad T_{ij...n} = T_{ij...n}(x_i, t) \tag{2.34}
\]

#### 2.4.2 The del Operator

The del operator (or the differential operator) is defined as
\[
\nabla = e_i \frac{\partial}{\partial x_i} \tag{2.35a}
\]
Hence, the *del* operator is a vector

\[
\nabla = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}
\]  

(2.35b)

When the *del* operator operates on a scalar \( \phi(\mathbf{x}, t) \), we get a vector as

\[
\text{grad} \phi = \nabla \phi = e_1 \frac{\partial \phi}{\partial x_1} + e_2 \frac{\partial \phi}{\partial x_2} + e_3 \frac{\partial \phi}{\partial x_3} \text{(vector)}
\]  

(2.36a)

This is the gradient (or “grad” for short) of \( \phi \). The components of the gradient vector are

\[
\frac{\partial \phi}{\partial x_i} \text{ or } \phi_{,i} \text{ (another common notation)}
\]  

(2.36b)

Consider a surface in space mathematically defined as

\[
\phi(\mathbf{x}, t) = c
\]  

(2.37)

where \( c \) is a constant. Figure 2.5a shows a schematic of the surface. Then \( \nabla \phi \) is a vector normal to the surface at \( \mathbf{x} \) for a fixed \( t \).

To prove this, consider two points A and B shown in Fig. 2.5b. As these points are near each other and lie on the surface, as you go from A to B:

\[
\begin{align*}
d\phi &= 0 \\
\phi_{,1} \, dx_1 + \phi_{,2} \, dx_2 + \phi_{,3} \, dx_3 &= 0 \\
\nabla \phi \cdot dx &= 0
\end{align*}
\]

Hence, \( \nabla \phi \) and \( dx \) are orthogonal, proving that \( \nabla \phi \) is normal to the surface.

The gradient of a general tensor is given by

\[
T_{ij...nk} = \frac{\partial}{\partial x_k} T_{ij...n}
\]
If the original tensor $T_{ij...n}$ is an $m$th order tensor, then the gradient defined above is an $(m + 1)$th order tensor. We saw this to be the case above in the case of a scalar (i.e., a scalar became a vector, (2.36a)). To prove this, let us consider the gradient of a second order tensor $T_{ij}$

$$T'_{ij,x_\ell} = \frac{\partial}{\partial x'_k} T'_{ij} = \frac{\partial}{\partial x'_k} (T_{pq}a_{ip}a_{jq}) \frac{\partial x'_j}{\partial x'_k}$$

(-using the transformation law, (2.23), and the chain rule)

From (2.9c) and (2.10a)

$$x_i = b_{ik}x'_k = a_{ki}x'_k \Rightarrow \frac{\partial x_i}{\partial x'_r} = a_{ki} \frac{\partial x'_k}{\partial x'_r} = a_{ki} \delta_{kr} = a_{ri}$$

Hence,

$$T'_{ij,x_\ell} = \frac{\partial}{\partial x'_k} (T_{pq}a_{ip}a_{jq}a_{kl}) = T_{pq, x_\ell}a_{ip}a_{jq}a_{kl}$$

which proves that $T'_{ij,x_\ell}$ transforms as a third order tensor. Hence, the derivative of a second order tensor is a third order tensor.

The other common operations involving the del operator are

The divergence operator:

$$\text{div} \mathbf{u} = \nabla \cdot \mathbf{u} = u_{i,i} \quad \text{(scalar), where } \mathbf{u} \text{ is a vector} \quad (2.38a)$$

The Laplace operator:

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \phi_{ii} = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} \quad \text{(scalar)} \quad (2.38b)$$

The Curl operator:

$$\mathbf{v} = \text{Curl} \ \mathbf{u} = \nabla \times \mathbf{u}; \quad \text{or} \quad v_i = \epsilon_{ijk} \partial_j u_k = \epsilon_{ijk} u_{k,j} \quad (2.38c)$$

### 2.4.3 Stokes’ Theorem

Stokes’ theorem relates a line integral of a vector $\mathbf{F}$ around a closed curve to a surface integral over a two-sided surface as shown in Fig. 2.6. The theorem states

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = \int_S \mathbf{n} \cdot (\nabla \times \mathbf{F}) \, dS \quad (2.39a)$$
2.4.4 Divergence Theorem of Gauss

The divergence theorem of Gauss relates a surface integral to a volume integral (Fig. 2.7). Considering first a vector \( \mathbf{u} \), the Gauss theorem in indicial notation is

\[
\int_{S} \mathbf{u} \cdot \mathbf{n} \, dS = \int_{\Omega} \text{div} \mathbf{u} \, dV \tag{2.40a}
\]

and in symbolic notation is

\[
\int_{S} \mathbf{u} \cdot \mathbf{n} \, dS = \int_{\Omega} \text{div} \mathbf{u} \, dV \tag{2.40b}
\]

\( \mathbf{u} \cdot \mathbf{n} \) is the flux of the vector field \( \mathbf{u} \) through the surface \( S \).

The theorem is generalized to a tensor as

\[
\int_{S} T_{ij...k} \mathbf{n} \, dS = \int_{\Omega} T_{ij...k} \, dV \tag{2.40c}
\]

2.4.5 Differentiation of a Tensor with Respect to another Tensor

Let us define a tensor \( T(\mathbf{A}, \mathbf{B}, \ldots) \) of arbitrary order that is a function of other tensors \( \mathbf{A}, \mathbf{B}, \) etc., of arbitrary orders. The partial derivative of \( T(\mathbf{A}, \mathbf{B}, \ldots) \) with respect to one of its argument tensors (say, \( \mathbf{B} \)) is
If the orders of tensors $T$ and $B$ are $m$ and $n$, respectively, the order of the derivative $G$ is $m + n$.

Some examples are

- $G_{ij} = \frac{\partial \phi}{\partial T_{ij}}$ (derivative of a scalar with respect to a second order tensor)
- $G_{ijkl} = \frac{\partial A_{ij}}{\partial T_{kl}}$ (derivative of a second order tensor with respect to another second order tensor)

Let us consider the derivative of a tensor by itself as (for example, for a second order tensor)

$$C_{ijkl} = \frac{\partial A_{ij}}{\partial A_{kl}}$$

Noting that $C_{ijkl} = 0$ except when the indices coincide as $i = k$ and $j = \ell$, it follows

$$C_{ijkl} = \frac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik}\delta_{j\ell}$$

(2.42)

To prove that $G_{ij} = \frac{\partial \phi}{\partial T_{ij}}$ is a second order tensor, let us consider

$$G'_{ij} = \frac{\partial \phi'(T')}{\partial T'_{ij}}$$

$$= \frac{\partial \phi'(T')}{\partial T_{kl}} \frac{\partial T_{kl}}{\partial T_{ij}}$$
From (2.18b):

\[ T_{k\ell} = T'_{pq}a_{pk}a_{q\ell} \]

\[ \frac{\partial T_{k\ell}}{\partial T'_{ij}} = \delta_{pi}\delta_{qj}a_{pk}a_{q\ell} \]

\[ = a_{i\ell}a_{j\ell} \]

Assuming \( \phi' = \phi \),

\[ G'_{ij} = \frac{\partial \phi}{\partial T_{k\ell}}a_{ik}a_{j\ell} \]

\[ = G_{k\ell}a_{ik}a_{j\ell} \]

Hence, \( G_{ij} \) is a second order tensor ((2.23)).

The usual chain rule applies to tensor derivatives as well. For example, let \( \phi \) be a function of a scalar \( \alpha \) as \( \phi(\alpha) \), where \( \alpha = (\sigma_{k\ell}\sigma_{k\ell})^{1/2} \), then

\[ C_{ij} = \frac{\partial \phi}{\partial \sigma_{ij}} = \frac{\partial \phi}{\partial \alpha} \frac{\partial \alpha}{\partial \sigma_{ij}} \]  \hspace{1cm} (2.43)

Now differentiate \( \alpha \) as

\[ \alpha^2 = \sigma_{k\ell}\sigma_{k\ell} \]

\[ 2\alpha \frac{\partial \alpha}{\partial \sigma_{ij}} = 2\delta_{ki}\delta_{lj}\sigma_{k\ell} \]

\[ \frac{\partial \alpha}{\partial \sigma_{ij}} = \sigma_{ij} \]

\[ \frac{\partial \alpha}{\partial \sigma_{ij}} = \alpha \]

Then

\[ C_{ij} = \frac{\partial \phi}{\partial \sigma_{ij}} = \frac{\partial \phi}{\partial \alpha} \frac{\sigma_{ij}}{\alpha} \]

A more comprehensive application of this is addressed in Problem 2.12.

### 2.5 Invariants of Stresses and Strains

In most problems that the finite element method is used to solve (i.e., two- and three-dimensional problems), more than one component of the stress tensor varies with loading. Furthermore, the variation can be quite arbitrary. For instance, in a plane strain problem, all three of the nonzero components of the stress tensor
(σ_{xx}, σ_{yy}, σ_{xy}) can vary with the applied load, and the interrelationship among the three components also can be quite arbitrary. Thus, the three stress components (σ_{xx}, σ_{yy}, σ_{xy}) and the corresponding three strain components (ε_{xx}, ε_{yy}, ε_{xy}) can vary in a complex manner. In a truly three-dimensional problem, all six stress components (σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, σ_{xz}, σ_{yz}) and the corresponding six strain components (ε_{xx}, ε_{yy}, ε_{zz}, ε_{xy}, ε_{xz}, ε_{yz}) can vary with the interrelationships being very complex. Hence, there is a need to seek simpler and/or more rational methods of relating the stresses and strains.

For these reasons, it is of interest to seek other measures of stresses and strains. Examples of such measures are

- Principal stresses and principal strains
- Volumetric and deviatoric components of the stress and strain tensors
- Invariants of stress and strain tensors

In the following section, we will define these quantities and present geometrical meaning of them.

### 2.5.1 Principal Stresses and Strains

The principal planes are those planes where the shear stresses are zero. Considering a Cartesian coordinate system, there are three perpendicular planes where this occurs. The normal stresses on the principal planes are called the principal stresses (also principal values). The principal strains are similarly defined.

It is assumed that the reader is familiar with the Mohr-circle method of finding the principal stresses, principal strains and their directions. Here we describe a general method of finding the principal values and directions that is useful in three-dimensional analyses.

It was shown earlier (2.14c) that the traction \( t \) on a plane having a unit normal \( n \) is related to the stress tensor at that point as

\[
\mathbf{t} = \sigma \mathbf{n}
\]  

(2.44)

If the shear stress on this plane is to be zero, then the direction of \( t \) must coincide with the direction of \( n \), i.e.,

\[
\mathbf{t} = \lambda \mathbf{n}
\]  

(2.45)

where \( \lambda \) is a scalar. Combining (2.44) and (2.45)

\[
\sigma \mathbf{n} = \lambda \mathbf{n}
\]  

(2.46a)

which the reader can recognize as the standard eigenvalue problem. \( \lambda \) is the eigenvalue of \( \sigma \) and \( n \) is the corresponding eigenvector. Since the size of \( \sigma \) is
3×3, there are three eigenvalues (not necessarily distinct) and eigenvectors, which are, respectively, the principal stresses and their directions. Equation (2.46a) may be written as

\[(\mathbf{\sigma} - \lambda \mathbf{I})\mathbf{n} = 0 \]  
\[(2.46b)\]

where \(\mathbf{I}\) is identity matrix. For (2.46b) to be true for a nontrivial \(\mathbf{n}\), the determinant of the matrix must be zero, i.e.,

\[|\mathbf{\sigma} - \lambda \mathbf{I}| = 0 \]  
\[(2.46c)\]

Expanding (2.46c), one obtains a cubic equation in \(\lambda\), with three solutions, which are the principal stresses associated with \(\mathbf{\sigma}\). We illustrate this for a specific problem in Example 2.5.

**Example 2.5.**

**Question:** Find the principal stresses and principal directions for the following stress tensor:

\[
\mathbf{\sigma} = \begin{bmatrix}
50 & 20 & 0 \\
20 & 60 & 0 \\
0 & 0 & 40
\end{bmatrix}
\]

**Answer:** From (2.46c),

\[
\begin{vmatrix}
50 - \lambda & 20 & 0 \\
20 & 60 - \lambda & 0 \\
0 & 0 & 40 - \lambda
\end{vmatrix} = 0
\]

\[(50 - \lambda)(60 - \lambda)(40 - \lambda) - 20 \times 20 \times (40 - \lambda) = 0\]

\[(40 - \lambda)[(60 - \lambda)(50 - \lambda) - 400] = 0\]

\[\lambda_1 = 40, \quad \lambda_2 = 75.61, \quad \lambda_3 = 34.39\]

The principal stress tensor is

\[
\mathbf{\sigma} = \begin{bmatrix}
40.00 & 0 & 0 \\
0 & 75.61 & 0 \\
0 & 0 & 34.39
\end{bmatrix}
\]

In finding the principal directions, it must be remembered that the three equations in (2.46b) are dependent, and there are two independent equations. A common
procedure is to assume one of the nonzero components of the vector \( \mathbf{n} \) to be 1.0 and find the remaining two.

In this specific problem, it is seen that direction 3 is already a principal direction (notice that the off diagonal terms on the third column and row are zero). This is why one of the principal values coincided with the diagonal term of 40.0 of the stress tensor. The analysis will ascertain this as well.

Let us evaluate the principal directions. First, consider the three equations (2.46b) for \( \lambda = 40 \):

\[
(50 - 40)n_1 + 20n_2 + 0 \times n_3 = 0 \Rightarrow 10n_1 + 20n_2 = 0
\]
\[
20n_1 + (60 - 40)n_2 + 0 \times n_3 = 0 \Rightarrow 20n_1 + 20n_2 = 0
\]
\[
0 \times n_1 + 0 \times n_2 + (40 - 40)n_3 = 0
\]

It is clear that \( n_3 \) cannot be solved from the first two equations. The third equation suggests that \( n_3 \) can assume any value. Let us assume \( n_3 = 1.0 \). From the first two equations, we get: \( n_1 = 0 \) and \( n_2 = 0 \). Thus, \( \mathbf{n}_1 = \{0, 0, 1\} \), which is the direction along the \( x_3 \)-axis. As expected, the principal direction coincides with the \( x_3 \)-axis.

Now, let us consider the three equations for \( \lambda = 75.61 \):

\[
(50 - 75.61)n_1 + 20n_2 + 0 \times n_3 = 0 \Rightarrow -25.61n_1 + 20n_2 = 0
\]
\[
20n_1 + (60 - 75.61)n_2 + 0 \times n_3 = 0 \Rightarrow 20n_1 - 15.61n_2 = 0
\]
\[
0 \times n_1 + 0 \times n_2 + (40 - 75.61)n_3 = 0 \Rightarrow -35.61n_3 = 0
\]

This time, the third equation leads to \( n_3 = 0 \). It is seen that the first two equations are identical. With \( n_1 = 1 \), these equations yield \( n_2 = 1.28 \). Thus, the eigenvector is \( \mathbf{n}_2 = \{1, 1.28, 0\} \). When a unit vector is desired, \( \mathbf{n}_2 = \{0.616, 0.788, 0\} \).

By following the same procedure, it can be shown that the eigenvector corresponding to \( \lambda = 34.39 \) is \( \mathbf{n}_3 = \{0.788, -0.616, 0\} \).

Also note that vectors \( -\mathbf{n}_1 \), \( -\mathbf{n}_2 \) and \( -\mathbf{n}_3 \) will also satisfy (2.46b), and therefore are legitimate eigenvectors. Noting that each principal plane has two sides, the meaning of \( \mathbf{n}_i \) and \( -\mathbf{n}_i \) for \( i = 1–3 \) are easily understood.

### 2.5.2 Spherical and Deviatoric Components

Any second order tensor can be split up into spherical and deviatoric components. Let us first consider the stress tensor \( \sigma_{ij} \). A coordinate invariant \( I \) is defined as

\[
I = \sigma_{kk} = \sigma_{kk} \delta_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} \quad (2.47a)
\]
$I$ is, therefore, the trace of the stress tensor (sum of the diagonals). $I$ is related to the mean normal pressure $p$ as

$$I = 3p \quad (2.47b)$$

The spherical (also called the hydrostatic) part of $\sigma_{ij}$ is

$$\frac{1}{3}I\delta_{ij} \quad (2.47c)$$

The deviatoric component of the stress tensor is obtained by subtracting the spherical part from the stress tensor as

$$s_{ij} = \sigma_{ij} - \frac{1}{3}I\delta_{ij} \quad (2.48)$$

Denoting the strain tensor by $e_{ij}$, the corresponding quantities for the strain tensor are similarly defined as follows:

$$e_v = e_{kk} = e_{11} + e_{22} + e_{33} \quad (2.49)$$

$$e_{ij} = e_{ij} - \frac{1}{3}e_v\delta_{ij} \quad (2.50)$$

where $(1/3)e_v\delta_{ij}$ is the spherical (also called as the volumetric) part of $e_{ij}$, and $e_{ij}$ is the deviatoric part of $e_{ij}$. $e_v$ is the volumetric strain.

### 2.5.3 Invariants of Stress and Strain Tensors

Invariants are scalar quantities which remain unchanged upon rotation of the coordinate system. Many physical scalars, which are 0th order tensors, are invariants. For example, mass of a body is invariant. For a vector, the linear function defined as part of the first definition of a tensor ((2.22) and (2.12a)) is an invariant. When the arbitrary vector $\eta$ in (2.12a) is along the direction of the vector itself, the linear invariant function is the length of the vector; hence, the length of a vector is an invariant, which is easy to understand from physical grounds.

For a second order tensor, such as the stress and strain tensors, three scalar invariants can be defined. These invariants can be defined in a number of different ways. For example, when you expand the determinant in (2.46c), the following cubic equation in $\lambda$, known as the characteristic equation, is obtained.

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$
where \( I_1 = \sigma_{kk} \) (trace of \( \sigma_{ij} \))

\[
I_1 = \frac{1}{2} (\sigma_{ii} - \sigma_{ij})
\]

\[
I_2 = \frac{1}{2} (\sigma_{ij} \sigma_{jj} - \sigma_{ij} \sigma_{ij})
\]

\[
I_3 = \det(\mathbf{\sigma})
\]

\( I_1, I_2, \) and \( I_3 \) are invariants.

Here, we will define the invariants in a different way. Our first invariant is \( I \) and is taken to be the same as \( I_1 \) defined above. That is,

\[
I = \sigma_{kk} = \sigma_{ki} \delta_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} \quad (2.51a)
\]

Note that \( I \) was defined earlier in (2.47a) as part of the definition of the spherical part of the stress tensor. The second (\( J \)) and the third (\( S \)) invariants are defined on the basis of the deviatoric stress tensor \( s_{ij} \) (2.48) as follows:

\[
J = \left( \frac{1}{2} s_{ij} s_{ij} \right)^{1/2} \quad (2.51b)
\]

\[
S = \left( \frac{1}{3} s_{ik} s_{kl} s_{li} \right)^{1/3} \quad (2.51c)
\]

In addition to these, we define a fourth invariant, known as the “Lode” angle, by combining \( J \) and \( S \) as follows:

\[
-\frac{\pi}{6} \leq \alpha = \frac{1}{3} \sin^{-1} \left[ \frac{3 \sqrt{3}}{2} \left( \frac{S}{J} \right)^3 \right] \leq \frac{\pi}{6} \quad (2.52)
\]

The invariants \( I, J \) and \( \alpha \) have geometrical meaning as we will see later, and have special significance for constitutive modeling. These are the reasons for these particular choices of invariants for the discussion here.

The first invariant of the strain tensor is the volumetric strain \( \varepsilon_v \), defined in (2.49), which is repeated below for convenience:

\[
\varepsilon_v = \varepsilon_{kk} = \varepsilon_{ki} \delta_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \quad (2.53a)
\]

The second and third invariants of the strain tensor (i.e., the counterparts of \( J \) and \( S \)) are defined from the deviatoric strain tensor \( e_{ij} \) (2.50) as follows:

\[
J_e = \left( \frac{1}{2} e_{ij} e_{ij} \right)^{1/2} \quad (2.53b)
\]

\[
S_e = \left( \frac{1}{3} e_{ik} e_{kl} e_{li} \right)^{1/3} \quad (2.53c)
\]
Example 2.6.

**Question:** Determine the volumetric and deviatoric components and the invariants \((I, J, S, \alpha)\) of the following stress tensor:

\[
\mathbf{\sigma} = \begin{bmatrix} 50 & 20 & 0 \\ 20 & 60 & 0 \\ 0 & 0 & 40 \end{bmatrix}
\]

**Answer:**

\[
I = 50 + 60 + 40 = 150
\]

\[
s = \begin{bmatrix} 50 & 20 & 0 \\ 20 & 60 & 0 \\ 0 & 0 & 40 \end{bmatrix} - \frac{150}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 20 & 0 \\ 20 & 10 & 0 \\ 0 & 0 & -10 \end{bmatrix}
\]

\[
J = \frac{1}{\sqrt{2}} \left( s_{11}^2 + s_{22}^2 + \ldots \right)^{1/2} = \frac{1}{\sqrt{2}} \left( 0 + 400 + 0 + 400 + 100 + 0 + 0 + 0 + 100 \right)^{1/2} = 22.36
\]

\[
S = \frac{1}{\sqrt{3}} \begin{bmatrix} s_{11}s_{11}s_{11} + s_{11}s_{12}s_{21} + s_{11}s_{13}s_{31} + s_{12}s_{21}s_{11} + s_{12}s_{22}s_{21} + s_{12}s_{23}s_{31} + \ldots, \text{ etc} \end{bmatrix} \quad \text{for } i = 1, k = 1, \ell = 1 : 3
\]

\[
= 15.87
\]

\[
\alpha = \frac{1}{3} \sin^{-1} \left[ \frac{3\sqrt{3}}{2} \left( \frac{15.87}{22.36} \right)^3 \right] = 22.75^\circ \leq \left( \frac{\pi}{6} = 30^\circ \right)
\]

2.5.4 **Analysis of Stress Invariants in the Principal Stress Space**

Let us now consider the principal stress space (where the stress tensor only has the diagonal terms), simplify the equations presented in the preceding section, and examine the geometrical meaning of the deviatoric tensor and the stress invariants.

In the principal stress space, the deviatoric stress tensor is diagonal as well. This property allows us to treat the stress tensor as a vector and represent the state of stress by a point in the principal stress space, as shown in Fig. 2.8. The stress and the deviatoric stress tensors are expressed in vectors as:

\[
\mathbf{\sigma}^* = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}; \quad \mathbf{s}^* = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 - I/3 \\ \sigma_2 - I/3 \\ \sigma_3 - I/3 \end{bmatrix} = \begin{bmatrix} 2\sigma_1 - \sigma_2 - \sigma_3 \\ 2\sigma_2 - \sigma_3 - \sigma_1 \\ 2\sigma_3 - \sigma_1 - \sigma_2 \end{bmatrix}
\]  

(2.54a)
\[ s_i^* = \sigma_i - p; \quad p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \quad (2.54b) \]

\[ J = \frac{1}{\sqrt{2}}(s_1^2 + s_2^2 + s_3^2)^{1/2} \quad \text{and} \quad S = \frac{1}{\sqrt{3}}(s_1^3 + s_2^3 + s_3^3)^{1/3} \quad (2.54c) \]

\( p \) is the mean normal pressure. The Kronecker delta is expressed in a vector as:

\[ \delta^* = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (2.55) \]

Let us provide geometrical interpretation of these quantities. The space diagonal is the line in the \((\sigma_1, \sigma_2, \sigma_3)\) space that is inclined at equal angle \( \gamma \) to \( \sigma_1-\), \( \sigma_2-\), and \( \sigma_3-\) axes.

The direction cosine of the space diagonal with each of the axes is \( \cos \gamma \). From the property

\[ \cos^2 \gamma + \cos^2 \gamma + \cos^2 \gamma = 1 \Rightarrow \cos \gamma = \frac{1}{\sqrt{3}} \quad (2.56) \]

A unit vector along the space diagonal is \( \delta^*/\sqrt{3} \). Then, referring to Fig. 2.8a,

\[ |OA| = \frac{1}{\sqrt{3}} \sigma^* \cdot \delta^* = \frac{I}{\sqrt{3}} \quad (2.57a) \]

\[ \overrightarrow{AB} = \sigma^* - \overrightarrow{OA} = \sigma^* - \frac{I}{\sqrt{3}} \frac{\delta^*}{\sqrt{3}} = s^* \quad (2.57b) \]
Thus, the projection of the stress vector onto the space diagonal is proportional to $I$. The projection of the stress vector onto a plane perpendicular to the space diagonal is the deviatoric stress tensor. The magnitude of the vector $AB$ is proportional to the second invariant $J$ as

$$J = \frac{\overline{AB}}{\sqrt{2}} \quad (2.58)$$

Now we define two planes as follows: (a) *meridional plane*: A plane that contains the space diagonal and (b) *octahedral plane*: A plane normal to the space diagonal (Fig. 2.8b). The octahedral plane that passes through the origin is known as the $\pi$-plane. On the $\pi$-plane, $p = I/3 = 0$.

Let the projections of the $s_1$, $s_2$, and $s_3$ axes onto the $\pi$-plane be $\tilde{s}_1$, $\tilde{s}_2$, $\tilde{s}_3$-axes (Fig. 2.8a, b). The angle between the $s_1$- and $\tilde{s}_1$- axes is $90 - \gamma$ as shown in Fig. 2.8a; hence the direction cosine of $\tilde{s}_1$- axis with the $s_1$- axis is $\cos(90 - \gamma) = \sin \gamma$. The $\tilde{s}_1$- axis is inclined at equal angles to $s_2$- and $s_3$- axes and hence have equal direction cosines with $s_2$- and $s_3$- axes; let this be $x$. Then

$$\sin^2 \gamma + 2x^2 = 1 \Rightarrow \frac{2}{3} + 2x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{6}}$$

Noting that $\tilde{s}_1$-axis is in the negative direction of $s_2$- and $s_3$-axes, the unit normal along the $\tilde{s}_1$-axis is

$$m = \frac{1}{\sqrt{6}} \{2, -1, -1\} \quad (2.59)$$

Now equating the projection of vector $\overline{AB}$ onto the $\tilde{s}_1$–axis, and using the property $s_1 + s_2 + s_3 = 0 \Rightarrow s_2 + s_3 = -s_1$, and (2.58),

$$s^* \cdot m = (OB') \cos \theta$$

$$\frac{1}{\sqrt{6}} (2s_1 - s_2 - s_3) = J\sqrt{2} \cos \theta$$

$$\frac{3}{\sqrt{6}} s_1 = J\sqrt{2} \cos \theta$$

$$s_1 = \frac{2}{\sqrt{3}} J \cos \theta$$

Now from (2.48)

$$\sigma_1 = \frac{2}{\sqrt{3}} J \cos \theta + p$$
Similarly, by finding the projection of $\overrightarrow{AB}$ onto $\bar{\sigma}_1$- and $\bar{\sigma}_2$-axes, it can be shown (Problem 2.18)

\[
\begin{align*}
\sigma_1 &= r_0 \cos \theta + p \\
\sigma_2 &= -r_0 \cos(\theta - 60^\circ) + p \\
\sigma_3 &= -r_0 \cos(\theta + 60^\circ) + p 
\end{align*}
\]  
(2.60a)  
(2.60b)  
(2.60c)

where

\[
r_0 = \frac{2}{\sqrt{3}} J
\]  
(2.60d)

We will discuss more about the angle $\theta$ and $\alpha$ in the next section. Let us now present some other commonly used expressions. In the principle stress space, the octahedral normal stress $\sigma_{\text{oct}}$ is given by (with the aid of using (2.14c))

\[
\sigma_{\text{oct}} = n^T \sigma n = \frac{1}{3} \delta^{*T} \sigma \delta = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) = p
\]  
(2.61)

where $n = (1/\sqrt{3}) \delta^{*T}$ is the unit normal to the octahedral plane, which is identical to the unit vector along the space diagonal. The octahedral shear stress $\tau_{\text{oct}}$ is

\[
\tau_{\text{oct}} = \sqrt{|\sigma n|^2 - \sigma_{\text{oct}}^2}
\]  
(2.62)

which, after some manipulations, yields the following in the principal stress space

\[
\tau_{\text{oct}} = \frac{1}{3} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 \right]^{1/2}
\]  
(2.63)

In the general stress space,

\[
\tau_{\text{oct}} = \frac{1}{3} \left[ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2 + 6\sigma_{12}^2 + 6\sigma_{23}^2 + 6\sigma_{13}^2 \right]^{1/2}
\]  
(2.64)

It can be further shown

\[
\tau_{\text{oct}} = J \sqrt{\frac{2}{3}}
\]  
(2.65)
2.5.5 Analysis of Invariants in the Equibiaxial (Triaxial) Stress Space

This is a special case of the principal stress space where two of the principal stresses are equal to each other. Taking 2 and 3 to be the directions in which the stresses are equal to each other, an equibiaxial (also known as “triaxial” in disciplines that deal with soil, rock, and concrete) state of stress is then formally defined as: \( \sigma_1 \neq \sigma_2 = \sigma_3 \). The two variables that are used in constitutive theories are the mean normal pressure,

\[
p = \frac{1}{3}(\sigma_1 + 2\sigma_3) \quad (2.66a)
\]

and the deviatoric stress,

\[
q = \sigma_1 - \sigma_3 \quad (2.66b)
\]

An isotropic compression is one where \( \sigma_1 = \sigma_2 = \sigma_3 \). An equibiaxial compression loading is one where \( \sigma_1 > \sigma_2 = \sigma_3 \) and equibiaxial extension loading is one where \( \sigma_1 < \sigma_2 = \sigma_3 \). It then follows that

\[
q > 0 \quad \text{for equibiaxial compression} \quad (2.67a)
\]

\[
q < 0 \quad \text{for equibiaxial extension} \quad (2.67b)
\]

\[
q = 0 \quad \text{for isotropic (or hydrostatic) compression loading} \quad (2.67c)
\]

For equibiaxial state of stress:

\[
\sigma^* = \begin{bmatrix} \sigma_1 \\ \sigma_3 \\ \sigma_3 \end{bmatrix};
\]

\[
s^* = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 - 1/3 \\ \sigma_3 - 1/3 \\ \sigma_3 - 1/3 \end{bmatrix} = \begin{bmatrix} 2\sigma_1 - \sigma_3 - \sigma_3 \\ 3 \sigma_3 - \sigma_3 - \sigma_1 \\ 3 \sigma_3 - \sigma_1 - \sigma_3 \end{bmatrix} = \left[ \begin{array}{c} \frac{2}{3}q \\ -\frac{1}{3}q \\ -\frac{1}{3}q \end{array} \right] \quad (2.68a)
\]

\[
J = \frac{1}{\sqrt{2}}(s_1^2 + s_2^2 + s_3^2)^{1/2} \quad \text{and} \quad S = \frac{1}{\sqrt{3}}(s_1^3 + s_2^3 + s_3^3)^{1/3}
\]

\[
= \frac{q}{\sqrt{3}} \quad \text{and} \quad S = \frac{2^{1/3}q}{3} \quad (2.68b)
\]
Now from (2.52),

$$
\alpha = \frac{1}{3} \sin^{-1} \left[ \frac{3\sqrt{3}}{2} \left( \pm \frac{6\sqrt{3}}{27} \right) \right] 
$$

(2.69a)

$$
= \pm \frac{\pi}{6} \text{ with } + \text{ for equibiaxial compression and } \\
- \text{ for equibiaxial extension} 
$$

(2.69b)

Referring to Fig. 2.8b, the “Lode” angle \( \alpha \) is an angular measure in this plane, with \( \alpha = \pi/6 \) on the positive \( \sigma_1-\), \( \sigma_2-\), and \( \sigma_3-\) axes, and \( \alpha = -(\pi/6) \) on the negative \( \sigma_1-\), \( \sigma_2-\), and \( \sigma_1-\) axes. \( \alpha \) takes on a value between \( \pi/6 \) and \( -(\pi/6) \) at other points on the octahedral plane. \( J \) is a radial measure of the magnitude of stress on the \( \pi \)-plane.

Instead of using \( (\sigma_1, \sigma_2, \sigma_3) \), an alternate way to represent the state of stress is using the invariants \( (I, J, \alpha) \).

There are two advantages of this representation:

1. The stress is split up into spherical and deviatoric parts. This allows the constitutive models to be developed on the basis of the volumetric and deviatoric stress–strain relationships of the material.
2. In three-dimensional analyses, the stress tensor has six independent stress components. However, for isotropic materials, it suffices to develop the stress–strain behavior of the material in terms of the three invariants \( (I, J, \alpha) \).

### Problems

**Problem 2.1** Determine the transformation matrix \( a \) for the original and rotated coordinate systems shown in Fig. 2.9a and b. Verify that the transformation matrices are orthogonal.

![Fig. 2.9 Rotated coordinate systems](image-url)
Problem 2.2 Determine the components of the vector $\mathbf{u} = (1, -1, 1)$ in rotated coordinate systems shown in Fig. 2.9a and b using the transformation rule in (2.7a) and the transformation matrices computed in Problem 2.1. Verify your answers by directly projecting $\mathbf{u}$ onto the rotated system.

Problem 2.3 An inclined plane ABC is defined in three-dimensional space in Fig. 2.10a. The vectors $\mathbf{n}$, $\mathbf{m}$, and $\mathbf{t}$ denote, respectively, the unit vector normal to the plane, an arbitrary unit vector and traction on the plane (refer to Fig. 2.3b and c for their counter parts in two dimensions).

Derive the following equations:

1. $t_i = \sigma_{ik}a_{nk}$
2. $\sigma_{\mu\nu} = \sigma'_{ij}m_i^\prime \eta'_j = \sigma_{ij}\eta_i\eta_j$
3. $\sigma'_{ij} = \sigma_{kl}a_{ik}a_{j\ell}$

where $\mathbf{a}$ is the transformation matrix, and $\mathbf{\mu} = \mathbf{m}$ and $\mathbf{\eta} = \mathbf{n}$ (two arbitrary directions).

Hint: (area OAC) = (area ABC)(\mathbf{n.e}_2) = (area ABC)$a_{n2}$. Hence, $(1/2)dx_1dx_3 = ds_n a_{n2}$, where $ds_n = (area ABC)$.

Problem 2.4 The stress tensor in the $x_1$–$x_2$–$x_3$ system is

$$\mathbf{\sigma} = \begin{bmatrix} 10 & 5 & -6 \\ 5 & 20 & 15 \\ -6 & 15 & 8 \end{bmatrix}$$

Determine the components in the rotated systems shown in Fig. 2.9a and b.

Problem 2.5 Determine if the vector outer product

$$B_{ij} = u_i v_j$$

is a tensor or not. $u_i$ and $v_i$ are components of vectors $\mathbf{u}$ and $\mathbf{v}$, respectively.
Problem 2.6 Assuming that \( x_1 \) and \( x_2 \) transform as components of a vector, determine which of the following matrices are tensors (Butkov 1968)

\[
C = \begin{bmatrix}
   x_2^2 & x_1x_2 \\
   x_1x_2 & x_1^2
\end{bmatrix}; \quad D = \begin{bmatrix}
   -x_1x_2 & x_2^2 \\
   -x_2^2 & x_1x_2
\end{bmatrix}
\]

Problem 2.7 Verify that the expressions given by (2.29a) and (2.29b) are identical.

Problem 2.8 Prove that \( T_{ij}u_iu_j = D_{ij}u_iu_j \), where \( D_{ij} \) is the symmetric part of \( T_{ij} \)

Problem 2.9 Using the result of (2.33), show

- \( \varepsilon_{ijk} \varepsilon_{ist} = \delta_{jt} \delta_{kt} - \delta_{jt} \delta_{ks} \)
- \( \varepsilon_{ijk} \varepsilon_{ijt} = 2 \delta_{kt} \)
- \( \varepsilon_{ijk} \varepsilon_{ijk} = 6 \)

Problem 2.10 Prove that \( (P_{ijk} + P_{jki} + P_{jik})x_i x_j x_k = 3P_{ijk}x_i x_j x_k \) (from Mase 1970)

Problem 2.11 Prove that the following quantities derived from a stress tensor \( \sigma_{ij} \) are invariants under rotations of the coordinate system:

(a) \( I = \sigma_{ii} \)
(b) \( J = \left( \frac{1}{2} s_{ij} s_{ij} \right)^{1/2} \)
(c) \( \varepsilon_{ijk} \varepsilon_{kjp} T_{ip} \) (from Mase 1970)

where \( s_{ij} = \sigma_{ij} - (1/3)I \delta_{ij} \).

Problem 2.12 An yield surface is given by the following function

\[
\phi(I, J, \alpha, \zeta) = 0
\]

where

\[
I = \sigma_{kl} \delta_{k\ell} \]

\[
s_{ij} = \sigma_{ij} - \frac{1}{3} I \delta_{ij}; \quad J = \left( \frac{1}{2} s_{kl} s_{k\ell} \right)^{1/2}
\]

\[
S = \left( \frac{1}{3} s_{ij} s_{jk} s_{ki} \right)^{1/3}
\]

\[
-\frac{\pi}{6} \leq \alpha = \frac{1}{3} \sin^{-1} \left[ \frac{3\sqrt{3}}{2} \left( \frac{S}{J} \right)^3 \right] \leq \frac{\pi}{6}
\]
Show that the gradient of the function (normal to the surface) in the stress space is given by

$$\frac{\partial \phi}{\partial \sigma_{ij}} = \frac{\partial \phi}{\partial \hat{I}} \delta_{ij} + \frac{\partial \phi}{\partial \hat{J}} \frac{s_{ij}}{2\hat{J}} + \frac{\partial \phi}{\partial \hat{K}} \frac{\sqrt{3}}{2\hat{J}} \cos 3\alpha \left[ \frac{s_{ij} - s_{jj}}{\hat{J}^2} - \frac{2}{3} \delta_{ij} - \frac{3}{2} \left( \frac{S}{\hat{J}} \right)^3 S_{ij} \right]$$

**Problem 2.13** For the function $\lambda = A_{ij} x_i x_j$ where $A_{ij}$ is a constant, show that

- $\frac{\partial \lambda}{\partial x_i} = (A_{ij} + A_{ji}) x_j$
- $\frac{\partial^2 \lambda}{\partial x_i \partial x_j} = A_{ij} + A_{ji}$

Simplify these derivatives for the case $A_{ij} = A_{ij}$ (from Mase 1970).

**Problem 2.14** Show that the eigenvalues of the tensor

$$\sigma = \begin{bmatrix} 30 & -10 & 0 \\ -10 & 30 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

are 10, 20, and 40. Also show that the eigen-vectors corresponding to these eigenvalues are

- $n_1 = \{0, 0, \pm1\}$, $n_2 = \{\pm1/\sqrt{2}, \pm1/\sqrt{2}, 0\}$ and
- $n_3 = \{\mp1/\sqrt{2}, \pm1/\sqrt{2}, 0\}$ respectively

**Problem 2.15** Show that the principal stresses are real.

**Problem 2.16** Show that for a symmetric second order tensor, when $\lambda_i \neq \lambda_j$, the eigen-vectors are orthogonal to each other.

**Problem 2.17** Form a matrix $T$ by placing the eigenvectors in the columns of $T$ as:

$$T = \{n_1 \quad n_2 \quad n_3\}$$

Then show:

$$T^T \sigma T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

**Remark:** It is clear from the above equation, that $T$ is the transformation matrix that transforms the stress tensor into the principal stress tensor.
Problem 2.18 By following the procedure presented in Sect. 2.5.4, show

\[ \sigma_2 = -r_0 \cos(\theta - 60^\circ) + p \]
\[ \sigma_3 = -r_0 \cos(\theta + 60^\circ) + p \]

where

\[ r_0 = \frac{2}{\sqrt{3}} J \]
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Solids and Porous Media
Anandarajah, A.
2010, XV, 653 p., Hardcover