Chapter 2
On Semi-Holonomic Cosserat Media

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Dedicated to the memory of the Cosserat brothers on the centenary of the publication of their magnum opus [1]

Abstract The notions of semi-holonomic and quasi-holonomic Cosserat media are introduced and their differences outlined. Contrary to the classical holonomic and non-holonomic counterparts, the definition of semi- and quasi-holonomic media is not kinematic but constitutive. Possible applications include granular media embedded in a rigid matrix and colloidal suspensions in an ideal incompressible fluid.

2.1 Cosserat Bodies

In Continuum Mechanics, a material body $\mathcal{B}$ is defined as a 3-dimensional differentiable manifold that can be covered with a single coordinate chart. A configuration $\kappa$ is defined as an embedding of $\mathcal{B}$ into the 3-dimensional Euclidean space $\mathbb{E}^3$:

$$\kappa : \mathcal{B} \longrightarrow \mathbb{E}^3.$$  \hspace{1cm} (2.1)

In terms of coordinate charts $X^I$ ($I = 1, 2, 3$) and $x^i$ ($i = 1, 2, 3$) in the body and in space, respectively, the configuration $\kappa$ is given by three smooth functions:

$$X^I \mapsto x^i = \kappa^i(X^I).$$  \hspace{1cm} (2.2)

To convey the presence of extra kinematic degrees of freedom, however, these definitions need to be expanded so that the differential geometry can properly reflect the existence of the microstructure and its possible deformability. We recall that,
given an \( m \)-dimensional differentiable manifold \( \mathcal{M} \), its principal frame bundle \( F\mathcal{M} \) is obtained by adjoining at each point \( x \in \mathcal{M} \) the collection \( F_x\mathcal{M} \) of all the possible bases of its tangent space \( T_x\mathcal{M} \). The set thus obtained has a canonical structure of a differentiable manifold of dimension \( m + m^2 \). It is endowed with the natural projection map:

\[
\pi_M : F\mathcal{M} \longrightarrow \mathcal{M}
\]  

(2.3)

that assigns to each point \( p \in F\mathcal{M} \) the point \( \pi(p) \in \mathcal{M} \) to which it is attached. If \( x^i \) (\( i = 1, \ldots, m \)) is a coordinate chart on \( U \subset \mathcal{M} \) with natural basis \( e_i = \partial/\partial x^i \),

we can construct an associated chart in \( F\mathcal{M} \) by assigning to each point \( p \in \pi_M^{-1}(U) \) the numbers \( \{x^i, p^j_i\} \), where \( p^j_i \) is the \( i \)th component of the \( j \)th vector of the frame \( p \) in the natural basis \( \{e_1, \ldots, e_m\} \). Expressed in terms of coordinates, the natural projection is given by:

\[
x^i, p^j_i \mapsto x^i.
\]  

(2.4)

We define a Cosserat body as the principal frame bundle \( F\mathcal{B} \) of an ordinary material body \( \mathcal{B} \). The physical intent is that, while the underlying body \( \mathcal{B} \) represents the macro-medium, each fibre \( F_x\mathcal{B} \) represents the micro-particle or grain at \( x \in \mathcal{B} \).

Concomitantly with the enlargement of the scope of material bodies, we need to envisage a more general definition of the notion of configuration. To this end, we consider fibre-preserving maps:

\[
K : F\mathcal{B} \longrightarrow F\mathbb{E}^3
\]  

(2.5)

such that \( K \) is a principal fibre-bundle morphism between \( F\mathcal{B} \) and its image. By fibre preservation, we mean the commutativity of the diagram:

\[
\begin{array}{ccc}
F\mathcal{B} & \xrightarrow{K} & F\mathbb{E}^3 \\
\pi_B & & \pi_E \\
\mathcal{B} & \xrightarrow{\kappa} & \mathbb{E}^3
\end{array}
\]  

(2.6)

where \( \kappa \) is a well-defined map between the base manifolds. Thus, a Cosserat deformation \( K \) automatically implies the existence of an ordinary deformation \( \kappa \), representing the deformation of the macro-structure. By fibre-bundle morphism we imply that, fibre by fibre, each of the restrictions \( K|_X \) (\( X \in \mathcal{B} \)) commutes with the multiplicative right-action of the general linear group \( \text{GL}(3; \mathbb{R}) \). In terms of coordinates, this means that there exists an \( X \)-dependent matrix \( K^i_j \) such that any Cosserat configuration is completely defined by twelve smooth functions:

\[
x^i = \kappa^i(X^j),
\]  

(2.7)

and

\[
K^i_j = K^i_j(X^j).
\]  

(2.8)
The physical meaning of these assumptions is that each grain can undergo only homogeneous deformations, as represented by the local matrix $K^i_I$. In other words, each grain behaves as a pseudo-rigid body. A more detailed treatment can be found in [2–4].

Remark 2.1. As already pointed out, the original formulation by the Cosserat brothers considered the case in which $K^i_I$ is orthogonal. In the terminology of [5], this case corresponds to the micropolar continuum. The more general case in which $K^i_I$ is an arbitrary non-singular matrix corresponds to the micromorphic continuum of [5]. We use the terminology “Cosserat body” in this more general sense.

We can see that in a Cosserat body there exist two, in principle independent, mechanisms for dragging vectors by means of a deformation: The first mechanism is the ordinary dragging of vectors by means of the deformation gradient of the macro-medium, represented by the matrix with entries $F^j_i = x^j_i$. The second mechanism is the one associated with the deformation of the micro-particle or grain, and is represented by the matrix with entries $K^i_I$. Note that in a second-grade body these two mechanisms are identified with each other, thus suggesting that different kinds of Cosserat media may be obtained by either kinematic restrictions of this kind or by constitutive restrictions. In fact, the Cosserat brothers themselves already advanced these possibilities and introduced the outmoded terminology of “trièdre caché” (hidden triad) and “W caché” (hidden strain-energy function) to refer, respectively, to these kinematic or constitutive restrictions. We will follow in their steps.

2.2 Various Jets

Given two smooth manifolds, $\mathcal{M}$ and $\mathcal{N}$, of dimensions $m$ and $n$, respectively, we say that two maps $f, g : \mathcal{M} \to \mathcal{N}$ have the same $k$-jet at a point $X \in \mathcal{M}$ if:

(i) $f(X) = g(X)$;
(ii) in a coordinate chart in $\mathcal{M}$ containing $X$ and a coordinate chart in $\mathcal{N}$ containing the image $f(X)$, all the partial derivatives of $f$ and $g$ up to and including the order $k$ are respectively equal.

Although the above definition is formulated in terms of charts, it is not difficult to show by direct computation that the property of having the same derivatives up to and including order $k$ is, in fact, independent of the coordinate systems used in either manifold. Notice that, in order for this to work, it is imperative to equate all the lower-order derivatives. If, for example, we were to equate just the second derivatives, without regard to the first, the equality of the second derivatives would not be preserved under arbitrary coordinate transformations.

The property of having the same $k$-jet at a point is, clearly, an equivalence relation. The corresponding equivalence classes are called $k$-jets at $X$. Any function in a given $k$-jet is then called a representative of the $k$-jet. The $k$-jet at $X$ of which a given function $f : \mathcal{M} \to \mathcal{N}$ is a representative is denoted by $j^k_X f$. The collection of all $k$-jets at $X \in \mathcal{M}$ is denoted by $J^k_X(\mathcal{M}, \mathcal{N})$. The point $X$ is called the source of $j^k_X f$ and the image point $f(X)$ is called its target.
Let a smooth map \( f : \mathcal{M} \rightarrow \mathcal{N} \) be given in terms of coordinates \( X^I \) \((I = 1, \ldots, m)\) and \( x^i \) \((i = 1, \ldots, n)\) in \( \mathcal{M} \) and \( \mathcal{N} \), respectively, by the functions:

\[
x^i = x^i(X^1, \ldots, X^m), \quad i = 1, \ldots, n.
\]  

(2.9)

The jet \( j^2_X f \), for example, is then given by the following coordinate expressions:

\[
x^i(X^1, \ldots, X^m), \quad \left[ \frac{\partial x^i}{\partial X^I} \right]_X, \quad \left[ \frac{\partial^2 x^i}{\partial X^J \partial X^I} \right]_X,
\]

(2.10)
a total of \( n + mn + m^2 n \) numbers.

We are particularly interested in the case of 1-jets. Let us evaluate, accordingly, the coordinate expression of \( j^1_X K \), where \( K \) is a Cosserat configuration, as defined in coordinates by (2.7) and (2.8). Notice that the dimension of both the source and the target manifolds in this case is 12. Following the definition, we conclude that the \( j^1_X K \) consists of the following elements:

\[
x^i, \quad K^i_I, \quad \left[ \frac{\partial x^i}{\partial X^I} \right]_X, \quad \left[ \frac{\partial K^i_J}{\partial X^J} \right]_X,
\]

(2.11)

which we can abbreviate as:

\[
x^i, \quad K^i_I, \quad F^i_I = x^i_I, \quad K^i_{I,J}.
\]

(2.12)

If no further restrictions are imposed on \( K \), we speak of the components (2.12) as the representatives of a non-holonomic 1-jet at \( X \in \mathcal{B} \). It is possible, however, to demand in an intrinsic manner, independent of the coordinates, that the functions \( K \) under consideration satisfy the following compatibility requirement in a neighborhood of \( X \):

\[
K^i_I \equiv x^i_I.
\]

(2.13)

In this case, the collection of 1-jets obtained is smaller. Not only the second and third entries in (2.12) are the same, but also, by virtue of the identical satisfaction of (2.13) in a neighborhood of \( X \), we must have:

\[
K^i_{I,J} = x^i_{I,J} = K^i_{J,I}.
\]

(2.14)

In other words, the last element of the jet is symmetric with respect to its lower indices. We will indicate the coordinate expression of these holonomic jets as follows:

\[
x^i, \quad F^i_I, \quad K^i_{I,J} = K^i_{J,I}.
\]

(2.15)

Finally, there exists a third type of jet, somewhat intermediate between the two extremes just presented. It is obtained when the potential representatives \( K \) are restricted to satisfy the condition:

\[
K^i_I(X) = x^i_I(X).
\]

(2.16)
In other words, we demand the satisfaction of (2.13) not identically in a neighborhood of \( X \), but just at the point \( X \) itself. The 1-jets thus obtained are known as \textit{semi-holonomic jets}. The coordinate expression of a semi-holonomic jet is:

\[
x^i, \quad K^i_I, \quad K^i_{I,J}.
\]  
(2.17)

Notice that the last entry is no longer necessarily symmetric.

\textit{Remark 2.2.} Given an actual arbitrary configuration \( K \), it will give rise automatically to point-wise non-holonomic jets. If the configuration is restricted so that Condition (2.13) is satisfied over the whole base manifold \( \mathcal{B} \), it will give rise to everywhere holonomic jets. In this sense, it is possible to speak of non-holonomic or holonomic configurations, respectively. On the other hand, it is not possible to define semi-holonomic configurations. Indeed, if Condition (2.16) were to be imposed at each point, we would immediately revert to Condition (2.13), thus obtaining a holonomic configuration.

\section*{2.3 Semi-Holonomic Cosserat Media}

The last section ended in a definitely pessimistic note. Indeed, if semi-holonomic configurations cannot be properly defined, there seems to be no point in attempting a definition of semi-holonomic media. This kinematic impasse, however, can perhaps be resolved by means of a constitutive statement. We could say, for example, that a non-holonomic Cosserat medium is semi-holonomic if its constitutive equation involves only the semi-holonomic part of the 1-jet of the configuration. Physically, this would correspond to a response that is in some sense oblivious of the presence of the macro-medium. In this section, we look into this and other possibilities with some care.

Since we are contemplating a particular case of non-holonomic Cosserat media, it will be useful to record the law governing the change of constitutive law of such a medium under a change of reference configuration. For specificity, we will limit ourselves to a single scalar constitutive law, such as the free-energy density per unit mass \( \psi \). Let the constitutive law with respect to a reference configuration \( K_0 \) be given in a coordinate system \( X^I \) by the expression:

\[
\psi = \psi_0(K^i_I, F^i_I, K^i_{I,J}; X^I),
\]  
(2.18)

and let the counterpart for a reference configuration \( K_1 \) with coordinates \( Y^A \) be given by:

\[
\psi = \psi_1(K^i_A, F^i_A, K^i_{A,B}; Y^A),
\]  
(2.19)

with an obvious notational scheme. The deformation from \( K_0 \) to \( K_1 \) is given by twelve quantities denoted as:

\[
Y^A = Y^A(X^I), \quad K^A_I = K^A_I(X^J).
\]  
(2.20)
By the law of composition of jets (or derivatives), we obtain the following relation between the constitutive expressions:

$$
\psi_1(K^A_i, F^A_i, K^A_{i,A,B}; Y^A(X^J)) = \psi_0(K^A_i K^A_i, F^A_i F^A_i, K^A_{i,A,B} F^B_j + K^A_{i,A,i} F^A_{j,J}; X^J),
$$

(2.21)

where $F^A_i = Y^A_{i,A}$.

The point of bringing this transformation to bear is the proof of the following:

**Proposition 2.1.** If the constitutive law (2.18), in the reference configuration $K_0$, is independent of the second argument ($F^A_i$), so is the expression of the same constitutive law in any other reference configuration $K_1$ independent of the second argument ($F^A_i$).

**Proof.** The proof is an immediate consequence of the transformation law (2.21). $\square$

As a direct corollary of this proposition, we can propose the following definition.

**Definition 2.1.** A non-holonomic Cosserat medium is said to be *semi-holonomic* at $X$ if its constitutive law at $X$ is independent of the deformation gradient of the macro-medium.

From the mathematical standpoint, it is necessary to note that this definition does not imply the existence of a *canonical* projection of a non-holonomic jet onto a semi-holonomic part. In fact, such a canonical projection does not exist. What the definition implies is that, once a non-canonical choice is effected in one particular reference configuration, this choice can be convected to all other configurations by means of the correct application of the transformation equation (2.21). In particular, this convection involves the gradient of the change of reference configuration ($F^A_i$). Another way to state the choice of a particular “projection” is to say that a particular parallelism (whose physical meaning may, for example, be related to the existence of some particular stress-free configuration) must be chosen as part and parcel of the constitutive law of a semi-holonomic Cosserat medium.

From the physical point of view, a semi-holonomic Cosserat medium may be said to consist of an incoherent matrix upon which a coherent micro-medium has been installed. The interaction between the grains may “remember” the existence of a particular configuration of the macro-medium whereby the constitutive law takes a particularly simple form. It is interesting to remark that, since it plays no other role, the macro-medium of a semi-holonomic medium may be, in a possible application, assumed to be rigid.

The converse of the above statement is not true: a non-holonomic Cosserat medium with a rigid matrix is not automatically semi-holonomic. Indeed, by a direct application of the principle of frame indifference, the constitutive law (2.18) can be reduced to the form:

$$
\psi = \psi(R^T K, U, R^T \nabla K; X),
$$

(2.22)
where the polar decomposition $F = RU$ has been exploited and where block letters stand for the collections of homonymous indexed quantities used in previous formulas. Using now the polar decomposition:

$$K = R'U',$$  \hspace{1cm} (2.23)

we may write (2.22) as:

$$\psi = \psi(R'U', U, rR'T\nabla K; X),$$  \hspace{1cm} (2.24)

where:

$$R = R^T R'$$  \hspace{1cm} (2.25)

is the (referential) relative rotation of the grain with respect to the macro-medium. If the macro-medium is rigid, we must have necessarily $U = I$. But for a semi-holonomic body the constitutive law must be independent of both components $U$ and $R$ of the polar decomposition of $F$. It follows, therefore, that rigidity alone does not imply semi-holonomy. If, on the other hand, the constitutive law of a rigid-matrix Cosserat medium is independent of the rotation $R$, we may choose $R = R'$ (or, equivalently, $R = I$), thereby leading to the following reduced equation of a semi-holonomic Cosserat body:

$$\psi = \psi(U', R'T\nabla K; X).$$  \hspace{1cm} (2.26)

In the physical interpretation, we may say that the grains are attached to the rigid macro-medium by means of ideal frictionless pins, so that there is no energetic cost to produce a relative rotation between them. In the admittedly imperfect pictorial representation of Fig. 2.1, the grains in the reference configuration are depicted as squares pin-jointed at their centers to the rigid matrix and connected to their neighbors by means of springs (represented by broken lines) designed to detect differential stretches and rotations between contiguous grains. The grains themselves behave as pseudo-rigid bodies, so that their deformed versions are represented by parallelograms.

The reduced form (2.26) of the constitutive law of a semi-holonomic Cosserat material applies whether or not the matrix is rigid, since in either case the response is independent of both $U$ and $R$. 

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**Fig. 2.1** A rigid-matrix semi-holonomic Cosserat medium
2.4 Quasi-Holonomic Cosserat Media

As defined, a semi-holonomic Cosserat medium may not necessarily have any material symmetries. We want to contrast the above definition with the following one that, by demanding the maximum possible symmetry of the macro-medium, appears to carry the same physical meaning.

**Definition 2.2.** A non-holonomic Cosserat medium is said to be quasi-holonomic at $X$ if, for some (local) reference configuration, its symmetry group $\mathcal{H}$ at $X$ contains the subgroup given by:

$$\mathcal{G} = \{ \{I, G, 0\} \mid G \in \text{GL}(3; \mathbb{R}) \},$$

where $I$ is the unit of $\text{GL}(3; \mathbb{R})$.

The reason to suspect that this definition might be equivalent to the previous one is that, due to the assumed arbitrariness of $G$, it seems to imply that the deformation of the macro-medium plays no role in the constitutive response. A direct application of the definition of a non-holonomic symmetry, however, leads to the conclusion that a quasi-holonomic medium must have a constitutive law of the form:

$$\psi = \psi(K_1^i, K_{i,j}^i, F_{-j}^j; X^I),$$

in the special reference configuration used in the definition.  

Physically, this means that the price to pay for this large symmetry group is, surprisingly, the reappearance of the deformation gradient of the macro-medium in the last argument of the constitutive law so as to permit the interaction between the grains to take into account their relative spatial locations (rather than those pulled back to some putative, perhaps unstressed, reference configuration).

The purpose of the following simple example is to shed light on the subtle difference between semi-holonomic and quasi-holonomic media, as conceived in Definitions 2.1 and 2.2, respectively. To this end, we consider the successive application of two deformations, the first of which can be regarded as a change of reference configuration so as to bring the notation in line with that of the previous section. The (Cartesian) coordinate systems $X^I$, $Y^A$, $x^i$ are assumed to coincide with each other. The first deformation is a uniaxial contraction along the $X_1$-axis, namely:

$$Y_1 = 0.8X_1, \quad Y_2 = X_2, \quad Y_3 = X_3, \quad K_{i}^A = \delta_{i}^A.$$  

(2.29)

The second deformation is a micro-rotation about the $Y_3$ axis that increases linearly with $Y_1$. Specifically:

$$Y^1 = 0.8X^1, \quad Y^2 = X^2, \quad Y^3 = X^3, \quad K_{i}^A = \delta_{i}^A. \quad (2.29)$$

In any other reference configuration, the symmetry group will contain a conjugate of the group $\mathcal{G}$ and the form of the constitutive law will be, accordingly, somewhat more involved.
\[ x^1 = Y^1, \quad x^2 = Y^2, \quad x^3 = Y^3, \]
\[
\{ K_A^i \} = \begin{bmatrix}
\cos \left( \frac{\pi}{3} Y^1 \right) & -\sin \left( \frac{\pi}{3} Y^1 \right) & 0 \\
\sin \left( \frac{\pi}{3} Y^1 \right) & \cos \left( \frac{\pi}{3} Y^1 \right) & 0 \\
0 & 0 & 1
\end{bmatrix}.
\] (2.30)

The effect of each of the two deformations on a unit-width strip in the \( X^1, X^2 \) and \( Y^1, Y^2 \) planes, respectively, is shown in Figs. 2.2 and 2.3, while Fig. 2.4 shows the composition. Notice that, at the moment of composition, it is the already contracted strip that encounters the values of the rotation field already in place (as dictated by the second deformation), thus resulting in a maximum value for the rotation of the grain in the deformed strip of 48° rather than 60°, which was the value at the right-hand end of the strip as far as the second deformation alone was concerned. If the Cosserat body is semi-holonomic, the gradient of the rotation would be obtained by dividing 48° by the original unit width. On the other hand, if the Cosserat body is quasi-holonomic, it is the width measured in the final deformed configuration that matters in the calculation of the gradient. Since this width is of 0.8, we verify that the rotation gradient in the composite deformation turns out to be identical to the gradient in the second deformation. In other words, the pre-application of the first deformation (in this case a contraction of the macro-medium) is irrelevant for a quasi-holonomic medium.
Among various possible physical applications of both semi-holonomic and quasi-holonomic Cosserat media, beyond those with a rigid matrix, we mention the modeling of aggregates [7], such as colloidal suspensions [6], when the underlying continuum upon which the interacting particles dwell is, say, an ideal incompressible fluid. The choice of model depends on the physical nature of the interactions between the dispersed particles.

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References

Mechanics of Generalized Continua
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