Chapter 2
Kinematics of Mechanisms

2.1 Preamble

Robot kinematics is the study of the motion (kinematics) of robotic mechanisms. In a kinematic analysis, the position, velocity, and acceleration of all the links are calculated with respect to a fixed reference coordinate system, without considering the forces or moments. The relationship between motion and the associated forces and torques is studied in robot dynamics. Forward kinematics and inverse kinematics are the main components in robot kinematics.

Forward kinematics (also known as direct kinematics) is the computation of the position and orientation of a robot's end effector as a function of its joint angles. Inverse kinematics is defined as: given the position and orientation of a robot's end-effector, calculate all possible sets of joint motion that could be used to attain this given position and orientation.

From the viewpoint of robot structure, robot can be divided into two basic types: serial robot and parallel robot. Besides, there is a hybrid type, which is the combination of serial and parallel robots. Serial robots have open kinematic chain, which can be further classified as either articulated or cartesian robots.

In the following, the basic mathematical and geometric concepts including position and orientation of a rigid body are presented (Sect. 2.2). Translational coordinate transformation, rotational coordinate transformation and homogeneous transformation are introduced in Sect. 2.3. Denavit–Hartenberg expression of kinematic parameters is discussed in Sect. 2.4. Section 2.5 describes the derivation of Jacobian Matrix. Finally, the conclusions are given in Sect. 2.6.

2.2 Position and Orientation of Rigid Body

2.2.1 Rotation Matrix

To explain the relationship between parts, tools, manipulator etc., some concepts such as position vector, plane, and coordinate frame are utilized.

D. Zhang, *Parallel Robotic Machine Tools*, DOI 10.1007/978-1-4419-1117-9_2,

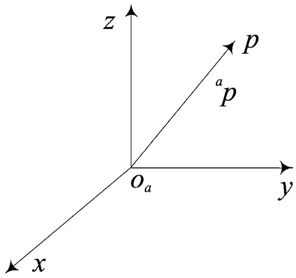

Fig. 2.1 Presentation of position

The motion of a robot can be described by its position and orientation, which is called pose as well. Once the reference coordinate system has been established, any point in the space can be expressed by a (3 × 1) vector. For orthogonal coordinate system $\{O_a - x_a y_a z_a\}$, any point $\boldsymbol{p}$ in the space can be written as follow:

$$^a\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}, \tag{2.1}$$

where p_x, p_y, p_z denote the components of the vector $\boldsymbol{p}$ along the coordinate axis x_a, y_a, z_a, respectively. Here, $\boldsymbol{p}$ is called position vector, which is shown in Fig. 2.1.

To investigate the motion and manipulation of robots, not only the description of position is needed, but also the orientation is likewise important. To define the orientation of point b, we should assume that there is an orthogonal coordinate system $\{O_b - x_b y_b z_b\}$ attached to the point. Here, x_b, y_b, z_b denote the unit vectors of the coordinate axes. With respect to the reference coordinate system $\{O_a - x_a y_a z_a\}$, the orientation of point b is expressed as follow:

$$^a_b\boldsymbol{R} = \begin{bmatrix} ^a\mathbf{x}_b & ^a\mathbf{y}_b & ^a\mathbf{z}_b \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \tag{2.2}$$

where $^a_b\boldsymbol{R}$ is called rotation matrix. $^a_b\boldsymbol{R}$ has nine elements in total, but only three of them are independent. The following constraint conditions should be satisfied by the nine elements:

$$^a\mathbf{x}_b \cdot {^a\mathbf{x}_b} = {^a\mathbf{y}_b} \cdot {^a\mathbf{y}_b} = {^a\mathbf{z}_b} \cdot {^a\mathbf{z}_b} = 1, \tag{2.3}$$

$$^a\mathbf{x}_b \cdot {^a\mathbf{y}_b} = {^a\mathbf{y}_b} \cdot {^a\mathbf{z}_b} = {^a\mathbf{z}_b} \cdot {^a\mathbf{x}_b} = 0. \tag{2.4}$$

It can be concluded that the rotation matrix $^a_b\boldsymbol{R}$ is orthogonal, and the following condition should be satisfied:

$$^a_b\boldsymbol{R}^{-1} = {^a_b\boldsymbol{R}^T}; \quad |^a_b\boldsymbol{R}| = 1. \tag{2.5}$$

The rotation matrix with respect to the rotation transformation by an angle θ about the axis x, y, z, respectively, can be calculated:

$$\boldsymbol{R}(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}, \tag{2.6}$$

$$\boldsymbol{R}(y,\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}, \tag{2.7}$$

$$\boldsymbol{R}(z,\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.8}$$

where $s\theta = \sin\theta$ and $c\theta = \cos\theta$

Suppose that coordinate frames {**B**} and {**A**} have the same orientation. But the original points of the two coordinate frames do not overlap. Using the position vector ${}^{a}\boldsymbol{p}_{O_b}$ to describe the position related to frame {**A**}. ${}^{a}\boldsymbol{p}_{O_b}$ is called the translational vector of frame {**B**} with respect to frame {**A**}. If the position of point $\boldsymbol{p}$ in the coordinate frame {**B**} is written as ${}^{b}\boldsymbol{p}$, then the position vector of $\boldsymbol{p}$ with respect to frame {**A**} can be written as follows:

$${}^{a}\boldsymbol{p} = {}^{b}\boldsymbol{p} + {}^{a}\boldsymbol{p}_{O_b}, \tag{2.9}$$

That is equation of coordinate translation which is shown in Fig. 2.2.

Suppose that coordinate frames {**B**} and {**A**} have the same orientation, but their orientation is different. Using the rotation matrix ${}^{a}_{b}\boldsymbol{R}$ to describe the orientation of

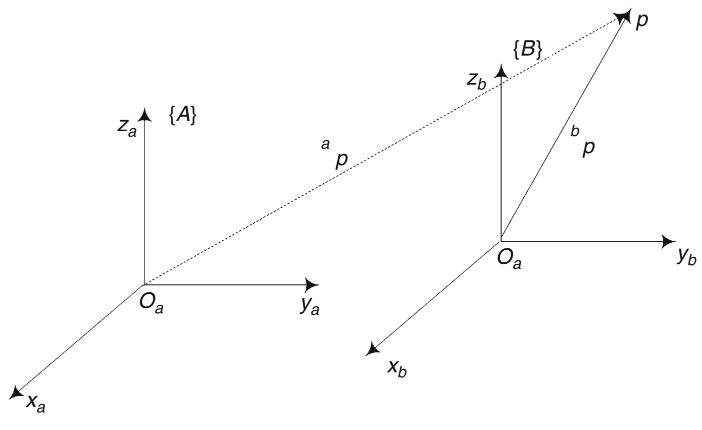

Fig. 2.2 Translational transformation

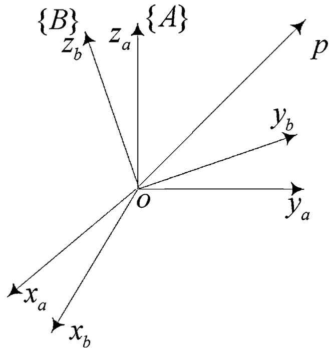

Fig. 2.3 Rotational transformation

frame {**B**} with respect to frame {**A**}, then the transformation of point $\boldsymbol{p}$ in frames {**A**} and {**B**} can be deduced as:

$$^{a}\boldsymbol{p} = {}^{a}_{b}\boldsymbol{R} \cdot {}^{b}\boldsymbol{p}, \tag{2.10}$$

where $^{a}\boldsymbol{p}$ denotes the position $\boldsymbol{p}$ with the reference coordinate system {**A**}, and $^{b}\boldsymbol{p}$ denotes the position $\boldsymbol{p}$ with the reference coordinate system {**B**}. It is called equation of coordinate rotation which is shown in Fig. 2.3.

The following equation can be deduced:

$${}^{b}_{a}\boldsymbol{R} = {}^{a}_{b}\boldsymbol{R}^{-1} = {}^{a}_{b}\mathbf{R}^{-T}. \tag{2.11}$$

For the common condition, neither the original points of frames {**A**} and {**B**} overlap nor they have the same orientation. Use the position vector $^{a}\boldsymbol{p}_{O_b}$ to describe the original point of frame {**B**} with respect to frame {**A**}; use the rotation matrix ${}^{a}_{b}\boldsymbol{R}$ to describe the orientation of frame {**B**} with respect to frame {**A**}. To any point in the space, the transformation can be found:

$$^{a}\boldsymbol{p} = {}^{a}_{b}\boldsymbol{R} \cdot {}^{b}\boldsymbol{p} + {}^{a}\boldsymbol{p}_{O_b}. \tag{2.12}$$

2.2.2 *Euler Angles*

The Euler angle I, shown in Fig. 2.4, defines a rotation angle ϕ around the z-axis, then a rotation angle θ around the new x-axis, and a rotation angle φ around the new z-axis.

$$\boldsymbol{R}_{z\phi} = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{R}_{u'\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}, \quad \boldsymbol{R}_{w''\varphi} = \begin{bmatrix} c\varphi & -s\varphi & 0 \\ s\varphi & c\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.13}$$

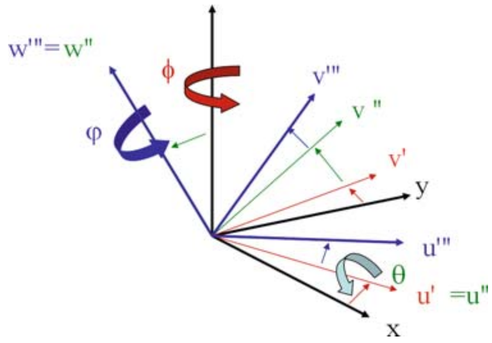

Fig. 2.4 Euler angle I

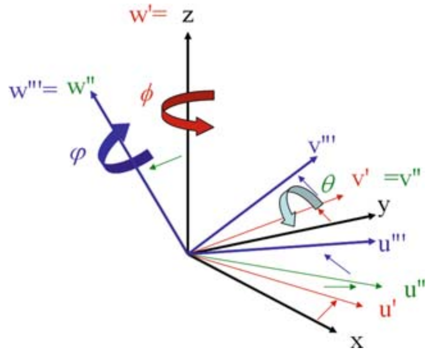

Fig. 2.5 Euler angle II

Resultant Eulerian rotation matrix generates:

$$\boldsymbol{R} = \boldsymbol{R}_{z\phi}\boldsymbol{R}_{u'\theta}\boldsymbol{R}_{w''\varphi} = \begin{bmatrix} c\phi c\varphi - s\phi s\varphi c\theta & -c\phi s\varphi - s\phi c\varphi c\theta & s\varphi s\theta \\ s\phi c\varphi + c\phi s\varphi c\theta & -s\phi s\varphi + c\phi c\varphi c\theta & -c\phi s\theta \\ s\varphi s\theta & c\varphi s\theta & c\theta \end{bmatrix}. \tag{2.14}$$

The Euler angle II, shown in Fig. 2.5, defines a rotation of angle ϕ around the z-axis, then a rotation of angle θ around the new y-axis, and finally a rotation angle φ around the new z-axis.

Note the opposite (clockwise) sense of the third rotation ϕ. Matrix with Euler Angle II generates:

$$\begin{bmatrix} -s\phi s\varphi + c\phi c\varphi c\theta & -s\phi c\varphi - s\phi c\varphi c\theta & c\phi s\theta \\ c\phi s\varphi + s\phi c\varphi c\theta & c\phi c\varphi - s\phi c\varphi c\theta & s\varphi s\theta \\ -c\varphi s\theta & s\varphi s\theta & c\theta \end{bmatrix} \tag{2.15}$$

2.3 Homogeneous Transformation

If the coordinates of any point in an orthogonal coordinate system is given, then the coordinates of this point in another orthogonal coordinate system can be calculated by homogeneous coordinate transformation.

The transformation (2.12) is inhomogeneous to point ${}^{b}\boldsymbol{p}$, but it can be expressed by an equivalent homogeneous transformation:

$$\begin{bmatrix} {}^{a}\boldsymbol{p} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{a}_{b}\boldsymbol{R} & {}^{a}\boldsymbol{p}_{O_b} \\ 0_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} {}^{b}\boldsymbol{p} \\ 1 \end{bmatrix}, \tag{2.16}$$

where the vector (4×1) denotes the coordinates in three-dimensional space. It still can be noted as ${}^{a}\boldsymbol{p}$ or ${}^{b}\boldsymbol{p}$. The above equation can be rewritten in the format of matrix:

$${}^{a}\boldsymbol{p} = {}^{a}_{b}\boldsymbol{T} \cdot {}^{b}\boldsymbol{p} + {}^{a}\boldsymbol{p}_{O_b}, \tag{2.17}$$

where the vector (4×1) of ${}^{a}\boldsymbol{p}$ and ${}^{b}\boldsymbol{p}$ is called homogeneous coordinates, here,

$${}^{a}_{b}\boldsymbol{T} = \begin{bmatrix} {}^{a}_{b}\boldsymbol{R} & {}^{a}\boldsymbol{p}_{O_b} \\ 0_{4\times 1} & 1 \end{bmatrix}. \tag{2.18}$$

In fact, the transformation (2.18) is equivalent to (2.12). The (2.17) can be rewritten as

$${}^{a}\boldsymbol{p} = {}^{a}_{b}\boldsymbol{R} \cdot {}^{b}\boldsymbol{p} + {}^{a}\boldsymbol{p}_{O_b}. \tag{2.19}$$

Suppose vector $ai + bj + ck$ describes one point in the space, where i, j, k are the unit vector of the axes x, y, z, respectively. This point can be expressed by the translational homogeneous transformation matrix.

$$Trans(a,b,c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.20}$$

where Trans denotes translational transformation.

If a rigid body rotates about x, y and z-axis with θ, then the following equations can be obtained:

$$Rot(x, \theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.21}$$

$$Rot(y,\theta) = \begin{bmatrix} c\theta & 0 & s\theta & 0 \\ 0 & 1 & 0 & 0 \\ -s\theta & 0 & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.22}$$

$$Rot(z,\theta) = \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.23}$$

where *Rot* denotes rotational transformation.

As the transformation is based on the fixed reference frame, a left-handed multiplication of transformation sequences is followed. For example, a rigid body rotates 90° about the z-axis of the reference frame, then it rotates another 90° about the y-axis and finally it translates 4 unit lengths along x-axis of the fixed reference frame, the transformation of this rigid body can be described as:

$$\boldsymbol{T} = Trans(4,0,0)Trans(y,90)Trans(z,90) = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.24}$$

The above matrix represents the operations of rotation and translation about the primary reference frame. The six points of the wedge-shaped object (Fig. 2.6(a)) can be expressed as:

$$\begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 6 & 6 & 4 & 4 \\ 4 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{2.25}$$

Figure 2.6(b) shows the result of transformation.

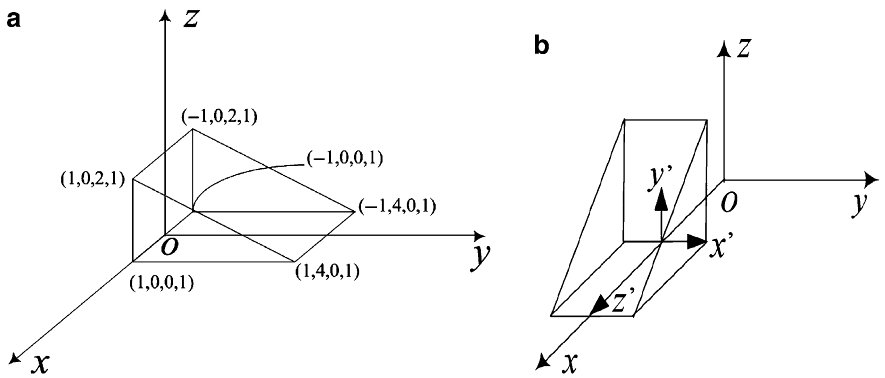

Fig. 2.6 Transformation of wedge-shaped object

In the above sections, the rotational transformation matrix with respect to rotations about x, y and z-axis has been analyzed. Here is the rotation matrix in the common situation: rotation about any vector (axis) with θ.

Suppose **f** is the unit vector of z-axis in coordinate frame C, namely:

$$\mathbf{C} = \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.26}$$

$$\mathbf{f} = a_x i + a_y j + a_z k. \tag{2.27}$$

Therefore, rotation about vector f is equivalent to rotation about z-axis in coordinate frame C, thus one has,

$$Rot(\mathbf{f},\ \theta) = Rot(\mathbf{c},\ \theta). \tag{2.28}$$

If the coordinate frame $\{T\}$ is known with respect to reference coordinate frame, then another coordinate frame $\{S\}$ can be calculated with respect to frame $\{C\}$, because,

$$T = CS, \tag{2.29}$$

Where, S is the relative position of T with respect to C, then,

$$S = C^{-1}T. \tag{2.30}$$

The rotation of T about f is equivalent to the rotation of S about z-axis of frame $\{C\}$,

$$Rot(\mathbf{f},\ \theta)T = CRot(\mathbf{z},\ \theta)S, \tag{2.31}$$

$$Rot(\mathbf{f},\ \theta)T = CRot(\mathbf{z},\ \theta)C^{-1}T. \tag{2.32}$$

Then the following equation can be derived,

$$Rot(\mathbf{f},\ \theta) = CRot(\mathbf{z},\ \theta)C^{-1}. \tag{2.33}$$

As **f** is the z-axis of frame $\{C\}$, then it can be found that $Rot(\mathbf{z},\ \theta)C^{-1}$ is just the function of **f**, because,

$$CRot(\mathbf{z},\ \theta)C^{-1} = \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}. \tag{2.34}$$

Note that $z = a$, $vers\theta = 1 - c\theta$, $f = z$. Equation (2.34) can be simplified as,

$$Rot(\mathbf{f},\ \theta) = \begin{bmatrix} \mathbf{f}_x\mathbf{f}_x vers\theta + c\theta & \mathbf{f}_y\mathbf{f}_x vers\theta - \mathbf{f}_z s\theta & \mathbf{f}_z\mathbf{f}_x vers\theta + \mathbf{f}_y s\theta & 0 \\ \mathbf{f}_x\mathbf{f}_y vers\theta + \mathbf{f}_z s\theta & \mathbf{f}_y\mathbf{f}_y vers\theta + c\theta & \mathbf{f}_z\mathbf{f}_y vers\theta - \mathbf{f}_x s\theta & 0 \\ \mathbf{f}_x\mathbf{f}_z vers\theta + \mathbf{f}_z s\theta & \mathbf{f}_y\mathbf{f}_z vers\theta + \mathbf{f}_x s\theta & \mathbf{f}_z\mathbf{f}_z vers\theta + c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.35}$$

Each basic rotation transformation can be derived from the general rotation transformation, i.e., if $\mathbf{f}_x = 1, \mathbf{f}_y = 0$ and $\mathbf{f}_z = 0$, then $Rot(\mathbf{f},\ \theta) = Rot(\mathbf{x},\ \theta)$. Equation (2.35) yields,

$$Rot(x, \theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.36}$$

which is identical to (2.21).

2.4 Denavit–Hartenberg Representation

Denavit–Hartenberg (DH) representation is a generic and simple method to define the relative motion parameters of two consecutive links and joints. Any arbitrary type of mechanism can be represented using the DH method to relate the position and orientation of the last link to the first one.

In studying the kinematic motion between two jointed links, the DH method defines the position and orientation of two consecutive links in a chain, link i with respect to link $(i-1)$ using a 4×4 homogeneous transformation matrix. With reference to Fig. 2.7, let axis i denotes the axis of the joint connecting link $(i-1)$ to link i. Four parameters should be determined, which are θ_i, d_i, a_i, and α_i. θ_i denotes the rotation angle measured from axis x_{i-1} to x_i with respect to z_i axis. d_n denotes the displacement measured from axis x_{n-1} to x_n with respect to z_n axis. a_i denotes the displacement measured from axis z_i to z_{i+1} with respect to x_i axis. α_i denotes the rotation angle measured from axis z_i to z_{i+1} with respect to x_i axis.

Following steps can help to determine the link and joint parameters of the whole kinematic model.

- Number the joints from 1 to n starting with the base and ending with the end-effecter
- Establish the base coordinate system. Establish a right-handed orthonormal coordinate system (x_0, y_0, z_0) at the supporting base with z_0 axis lying along the axis of motion of joint 1
- Establish joint axis. Align the z_i with the axis of motion (rotary or sliding) of joint $i+1$

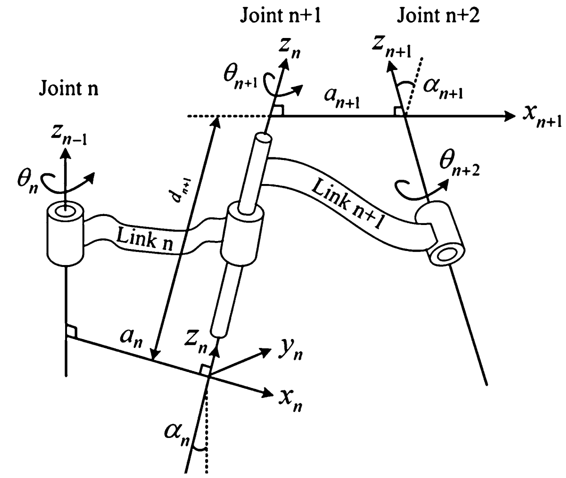

Fig. 2.7 Denavit–Hartenberg kinematic description

- Establish the origin of the ith coordinate system. Locate the origin of the ith coordinate at the intersection of the z_i and z_{i-1} or at the intersection of common normal between the z_i and z_{i-1} axes and the z_i axis
- Establish x_i axis. Establish $x_i = \pm(z_{i-1} \times z_i)/\|z_{i-1} \times z_i\|$ or along the common normal between the z_{i-1} and z_i axes when they are parallel
- Establish y_i axis. Assign $y_i = +(z_i \times x_i)/\|z_i \times x_i\|$ to complete the right-handed coordinate system
- Find the link and joint parameters

With the parameters defined in Fig. 2.5, the DH model transformation matrix can be obtained as follows

$$ {}^{i-1}_{i}\mathbf{T} = \mathbf{A}_i = Rot(z,\ \theta_i) \times Trans(0, 0, d_i) \times Trans(a_i, 0, 0) \times Rot(x, \alpha_i), \quad (2.37)$$

$$\mathbf{A}_i = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2.38)$$

2.5 Jacobian Matrix

To describe a micro-motion of robot, differential coefficient is utilized for coordinate transformation. Given a coordinate frame {T},

$$T + dT = Trans(dx, dy, dz)Rot(f, d\theta)T, \quad (2.39)$$

where $Trans(dx, dy, dz)$ denotes the differential translation of dx, dy, dz, and $Rot(f, d\theta)$ denotes the differential rotation about the vector f. Then dT can be calculated as follows:

$$dT = [Trans(dx, dy, dz)Rot(f, d\theta) - I]T. \tag{2.40}$$

The homogeneous transformation expressing differential translation is

$$Trans(dx, dy, dz) = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.41}$$

For the formula of general rotation transformation

$$Rot(f, d\theta) = \begin{bmatrix} f_x f_x vers\theta + c\theta & f_y f_x vers\theta - f_z s\theta & f_z f_x vers\theta + f_y s\theta & 0 \\ f_x f_y vers\theta + f_z s\theta & f_y f_y vers\theta + c\theta & f_z f_y vers\theta - f_x s\theta & 0 \\ f_x f_z vers\theta + f_z s\theta & f_y f_z vers\theta + f_x s\theta & f_z f_z vers\theta + c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.42}$$

Since $\lim_{\theta \to 0} \sin\theta = d\theta, \lim_{\theta \to 0} \cos\theta = 1, \lim_{\theta \to 0} vers\ \theta = 0$, differential rotational homogeneous transformation can be expressed as,

$$Rot(f, d\theta) = \begin{bmatrix} 1 & -f_z d\theta & f_y d\theta & 0 \\ f_z d\theta & 1 & -f_x d\theta & 0 \\ -f_z d\theta & f_x d\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.43}$$

Since $\Delta = Trans(dx, dy, dz)Rot(f, d\theta)$, it yields,

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -f_z d\theta & f_y d\theta & 0 \\ f_z d\theta & 1 & -f_x d\theta & 0 \\ -f_y d\theta & f_x d\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -f_z d\theta & f_y d\theta & dx \\ f_z d\theta & 0 & -f_x d\theta & dy \\ -f_y d\theta & f_x d\theta & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.44}$$

The differential rotation $d\theta$ about vector f is equivalent to the differential rotation with respect to the x, y and z-axis, namely δ_x, δ_y, and δ_z, respectively. Then $f_x d\theta = \delta_x$, $f_y d\theta = \delta_y$, $f_z d\theta = \delta_z$. Displace the above results into (2.44) yields:

$$\Delta = \begin{bmatrix} 0 & -\delta_z & \delta_y & dx \\ \delta_z & 0 & -\delta_x & dy \\ -\delta_y & \delta_x & 0 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.45}$$

If $d = d_x i + d_y j + d_z k, \delta = \delta_x i + \delta_y j + \delta_z k$, then the differential motion vector of rigid body or coordinate frame can be expressed as follows:

$$\mathbf{D} = \begin{bmatrix} dx & dy & dz & \delta_x & \delta_y & \delta_z \end{bmatrix}^T = \begin{bmatrix} d \\ \delta \end{bmatrix}. \tag{2.46}$$

The linear transformation between motion speed of manipulator and each joint can be defined as the Jacobian matrix of a robot. This Jacobian matrix represents the drive ratio of motion velocity from the space of joints to the space of end-effector. Assume the motion equation of manipulator

$$x = x(q) \tag{2.47}$$

represents the displacement relationship between the space of operation (end-effector) and the space of joints. Differentiating (2.47) with respect to time yields,

$$\dot{x} = J(q)\dot{q}, \tag{2.48}$$

where $\dot{x}$ is the generalized velocity of end-effector in operating space. $\dot{q}$ is the joint velocity. $J(q)$ is $6 \times n$ partial derivative matrix which is called Jacobian Matrix. The component in line i and column j is:

$$\mathbf{J}_{ij}(q) = \frac{\partial x_i(q)}{\partial q_j}, \quad i = 1, 2, \ldots, 6; \quad j = 1, 2, \ldots, n \tag{2.49}$$

From (2.49), it is observed that Jacobian Matrix $J(q)$ is a linear transformation from the velocity of joints space.

The generalized velocity $\dot{x}$ of rigid body or coordinate frame is a six-dimensional column vector composed of linear velocity v and angular velocity w.

$$\dot{x} = \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \begin{bmatrix} d \\ \delta \end{bmatrix}. \tag{2.50}$$

Equation (2.50) can be rewritten as

$$\mathbf{D} = \begin{bmatrix} d \\ \delta \end{bmatrix} = \lim_{\Delta t \to 0} \dot{x} \Delta t. \tag{2.51}$$

Replace (2.48) into (2.50), one has:

$$\mathbf{D} = \lim_{\Delta t \to 0} J(q)\dot{q}\Delta t, \tag{2.52}$$

$$\mathbf{D} = J(q)dq. \tag{2.53}$$

For a robot with n joints, its Jacobian matrix is a $6 \times n$ matrix, in which the first three lines denote the transferring rate of end-effector's linear velocity, and the last three lines denote the transferring rate of end-effector's angular velocity. Jacobian matrix can be expressed as:

$$\begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} J_{l1} & J_{l2} & \cdots & J_{ln} \\ J_{a1} & J_{a2} & \cdots & J_{an} \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}. \tag{2.54}$$

The linear velocity and angular velocity of an end-effector can be expressed as the linear function of each joint velocity $\dot{q}$

$$\boldsymbol{v} = J_{l1}\dot{q}_1 + J_{l2}\dot{q}_2 + \cdots + J_{ln}\dot{q}_n; \ \boldsymbol{w} = J_{a1}\dot{q}_1 + J_{a2}\dot{q}_2 + \cdots + J_{an}\dot{q}_n, \tag{2.55}$$

where J_{li} and J_{ai} means the linear velocity and angular velocity of end-effector resulted in joint i.

2.6 Conclusions

In this chapter, kinematics of robot manipulators is introduced, including the concept of reference coordinate frame, translational transformation, rotational transformation and homogeneous transformation, as well as the basic knowledge in robot kinematics, such as Euler angle, Denavit–Hartenberg representation, and Jacobian matrix of robot. These are the important knowledge in parallel robotic machine design.

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