Robust Controller Synthesis is Convex for Systems without Control Channel Uncertainties

Carsten W. Scherer

Abstract We consider an uncertain generalized plant whose control channel is not affected by uncertainties. It is shown that these configurations emerge in various concrete problems such as robust estimator or feed-forward controller design. Under the mere hypothesis that the (possibly non-linear) uncertainties are described by integral quadratic constraints, we reveal how one can translate robust controller synthesis to a problem of designing parametric and dynamic components in a standard plant configuration, and how this can be turned into a semi-definite program.

1 Introduction

It is well-known that the most general robust output-feedback controller synthesis problem cannot be easily solved by techniques from convex optimization. Still, in recent years, there has been a strong interest in classifying particularly structured robust synthesis problems that do have solutions in terms of semi-definite programming. Let us consider, for example, the intensively studied configuration as depicted in Figure 1 (see references in [20, 19]). Given the uncertain systems $G_1(\Delta)$, $G_2(\Delta)$

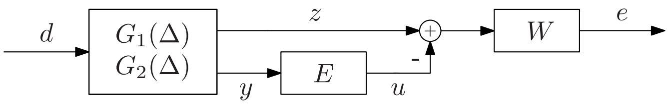

Fig. 1 Interconnection for Robust Estimator Synthesis

and a performance weight W, the problem is to design an estimator E such that

Carsten W. Scherer
Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands, e-mail: c.w.scherer@tudelft.nl

P.M.J. Van den Hof et al. (eds.), *Model-Based Control: Bridging Rigorous Theory and Advanced Technology*, DOI: 10.1007/978-1-4419-0895-7_2,

the worst-case energy gain from the disturbance d to the weighted estimation error e is minimized. Clearly the interconnection in Figure 1 can be represented by the generalized plant configuration

$$\begin{pmatrix} e \\ y \end{pmatrix} = \begin{pmatrix} WG_1(\Delta) & -W \\ G_2(\Delta) & 0 \end{pmatrix} \begin{pmatrix} d \\ u \end{pmatrix}, \quad u = Ey$$

in which the control channel $u \to y$ is not affected by uncertainty.

Dually the interconnection for robust feed-forward control is shown in Figure 2 and relates to the generalized plant description

$$\begin{pmatrix} e \\ y \end{pmatrix} = \begin{pmatrix} G_1(\Delta)W & G_2(\Delta) \\ W & 0 \end{pmatrix} \begin{pmatrix} d \\ u \end{pmatrix}, \quad u = Ky$$

with a similar structural property (see references in [10, 12]).

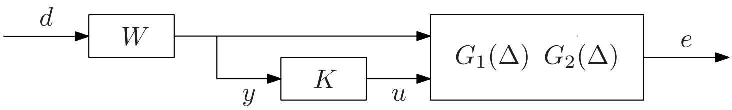

Fig. 2 Interconnection for Robust Feed-Forward Synthesis

Such interconnection structures emerge as well if considering synthesis problems with uncertain input- and output-performance weights as they are used in suppressing specific disturbances with time-varying frequencies or for describing coloring filters that are subject to parameter variations in order to capture disturbance spectrum uncertainty in stochastic control (see [8] and references therein).

The above sketched - and many more - problems can be subsumed to the following general formulation. Consider the uncertain generalized plant

$$\begin{pmatrix} e \\ y \end{pmatrix} = \begin{pmatrix} G_{11}(\Delta_1) & G_{12}(\Delta_2) \\ G_{21}(\Delta_1) & G_{22} \end{pmatrix} \begin{pmatrix} d \\ u \end{pmatrix} \quad \text{with } \Delta_1 \in \boldsymbol{\Delta_1},\ \Delta_2 \in \boldsymbol{\Delta_2} \tag{1}$$

where $d \to e$ is the performance channel, $u \to y$ is the control channel and $\boldsymbol{\Delta_1}, \boldsymbol{\Delta_2}$ are sets of structured stable uncertainties that are star-convex with center zero (which just means $[0,1]\boldsymbol{\Delta_k} \subset \boldsymbol{\Delta_k}$ for $k = 1,2$). As the only essential hypothesis, the control channel of the model is not affected by any uncertainty. In typical applications Δ_1 and Δ_2 are diagonally structured and might include time-invariant or time-varying rate-bounded parametric uncertainties, linear dynamic uncertainties or static and dynamic nonlinearities. Let us refer to Remark 1 for a brief discussion of variants of how the channels $u \to e$, $d \to e$ and $d \to y$ could be affected by uncertainties.

The goal is to design an output-feedback controller which robustly stabilizes (1) and which achieves a robust performance specification on the channel $d \to e$. In this paper both the uncertainties and the performance criterion are described by integral quadratic constraints (IQCs) with [14] being the main reference. If $\|.\|$ denotes the energy gain, this covers e.g. the pretty general robust model matching problem (see [4] and references therein) of designing a stable K which minimizes γ such that

$$\|G_{11}(\Delta_1)+G_{12}(\Delta_2)KG_{21}(\Delta_1)\|<\gamma \ \text{ for all } \ \Delta_k\in\boldsymbol{\Delta_k},\ k=1,2.$$

The paper is structured as follows. In Section 2 we discuss the construction of a suitable linear fractional representation of (1). Section 3 describes how to translate robust performance synthesis into a feasibility problem that involves parametric and dynamic decision variables, and Section 2 is devoted to turning this question into one of semi-definite programming. Further applications of this design framework are briefly sketched in Section 5, while a technical proof is deferred to the Appendix in Section 7.

Notation. We call a problem LMIable if it can be equivalently transformed into a linear matrix inequality (LMI) optimization problem. A set is LMIable if it can be represented as the feasible set of an LMI constraint [2, 3]. For complex matrices M we denote by M^* their conjugate transpose and use the abbreviation $\mathrm{He}(M)=M+M^*$. We adopt the classical notions of interconnection stability as developed in [7]. In the sequel $\mathscr{L}_2$ denotes the space of (vector-valued) finite energy signals on $[0,\infty)$ while $\|d\|$ and $\hat{d}$ are the norm and the Fourier transform of $d\in\mathscr{L}_2$. We use the symbol $\mathbb{C}^0$ for the extended imaginary axis $i\mathbb{R}\cup\{\infty\}$. By abusing notation we do not distinguish between finite-dimensional linear time-invariant systems and their representation in terms of transfer matrices. For transfer matrices P and K the lower linear fractional transformation of P and K is denoted by $\star$ and defined as

$$\begin{pmatrix} P_{11} & P_{12}\\ P_{21} & P_{22}\end{pmatrix}\star K=P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21}$$

if $I-P_{22}K$ has a proper inverse. If Δ is nonlinear, we define $e=(\Delta\star P)(d)$ by

$$\begin{pmatrix} z\\ e\end{pmatrix}=\begin{pmatrix} P_{11} & P_{12}\\ P_{21} & P_{22}\end{pmatrix}\begin{pmatrix} w\\ d\end{pmatrix},\quad w=\Delta(z)$$

if $I-P_{11}\Delta$ has a causal inverse; note that $e=P_{22}d+P_{21}\Delta((I-P_{11}\Delta)^{-1}(P_{21}d))$. Finally $P=\left[\begin{array}{c|c} A & B\\ \hline C & D\end{array}\right]$ means $P(s)=C(sI-A)^{-1}B+D$.

2 System Interconnections and Performance Specification

We assume that (1) admits a linear fractional representation. More precisely let us suppose that

$$\begin{pmatrix} G_{11}(\Delta_1)\\ G_{21}(\Delta_1)\end{pmatrix}=\Delta_1\star\begin{pmatrix} P_{11} & P_{13}\\ P_{31} & P_{33}\\ P_{43} & P_{44}\end{pmatrix} \ \text{ and } \ G_{12}(\Delta_2)=\Delta_2\star\begin{pmatrix} P_{22} & P_{24}\\ P_{32} & P_{34}\end{pmatrix}$$

where we assume that

$$I - P_{11}\Delta_1 \ \text{ and } \ I - P_{22}\Delta_2 \ \text{ have a causal inverse for all } \ \Delta_1 \in \boldsymbol{\Delta_1},\ \Delta_2 \in \boldsymbol{\Delta_2}. \tag{2}$$

With $P_{44} = G_{22}$ this leads to the following generalized plant description of (1):

$$\begin{pmatrix} z_1 \\ z_2 \\ \hline e \\ y \end{pmatrix} = \underbrace{\left(\begin{array}{cc|cc} P_{11} & 0 & P_{13} & 0 \\ 0 & P_{22} & 0 & P_{24} \\ \hline P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & 0 & P_{43} & P_{44} \end{array}\right)}_{P} \begin{pmatrix} w_1 \\ w_2 \\ \hline d \\ u \end{pmatrix}, \ \ w_1 = \Delta_1(z_1),\ \ w_2 = \Delta_2(z_2). \tag{3}$$

As well-known, the closed-loop interconnection of (1) with a dynamic controller $u = Ky$ can be obtained by first computing $P \star K$ and then interconnecting the uncertainties. Note that

$$P \star K = \underbrace{\left(\begin{array}{cc|c} P_{11} & 0 & P_{13} \\ P_{24}K(I-P_{44}K)^{-1}P_{41} & P_{22} & P_{24}K(I-P_{44}K)^{-1}P_{43} \\ \hline P_{31} + P_{34}K(I-P_{44}K)^{-1}P_{41} & P_{32} & P_{33} + P_{34}K(I-P_{44}K)^{-1}P_{43} \end{array}\right)}_{\mathscr{P}}.$$

Hence the controlled uncertain system admits the description

$$\begin{pmatrix} z_1 \\ z_2 \\ \hline e \end{pmatrix} = \underbrace{\left(\begin{array}{cc|c} P_{11} & 0 & P_{13} \\ \mathscr{P}_{21} & P_{22} & \mathscr{P}_{23} \\ \hline \mathscr{P}_{31} & P_{32} & \mathscr{P}_{33} \end{array}\right)}_{\mathscr{P}} \begin{pmatrix} w_1 \\ w_2 \\ \hline d \end{pmatrix}, \ \ w_1 = \Delta_1(z_1),\ \ w_2 = \Delta_2(z_2). \tag{4}$$

Here we have indicated those blocks which explicitly depend on the controller by calligraphic symbols. The particular structure how the controller enters the individual blocks of $\mathscr{P}$ will be of crucial relevance in the sequel.

We assume that there exists some K which (internally) stabilizes P. Clearly P_{11}, P_{13}, P_{22}, P_{32} must hence be stable. If a nominally stabilizing controller K also achieves robust stability, then $d \in \mathscr{L}_2$ implies for the controlled uncertain system (4) that $e \in \mathscr{L}_2$. With some real symmetric matrix

$$\Pi_p = \begin{pmatrix} Q_p & S_p \\ S_p^T & T_p^T T_p \end{pmatrix} \ \text{ where } T_p \text{ has full column rank,} \tag{5}$$

the desired quadratic performance specification is then assumed to be expressed as

$$\int_{-\infty}^{\infty} \begin{pmatrix} \hat{d}(i\omega) \\ \hat{e}(i\omega) \end{pmatrix}^* \Pi_p \begin{pmatrix} \hat{d}(i\omega) \\ \hat{e}(i\omega) \end{pmatrix} d\omega \leq -\varepsilon \|d\|_2^2 \ \text{ for all } \ d \in \mathscr{L}_2 \tag{6}$$

and some $\varepsilon > 0$. Robust quadratic performance is achieved if this inequality holds for all trajectories of the uncertain system (4). Let us recall the special case $Q_p = -I$, $S_p = 0$, $T_p = I$; then (6) means that the $\mathscr{L}_2$-gain of $d \to e$ is bounded by one.

Remark 1. We might be confronted with an uncertain model

$$\begin{pmatrix} e \\ y \end{pmatrix} = \begin{pmatrix} G_{11}(\Delta_1) & G_{12}(\Delta_3) \\ G_{21}(\Delta_2) & G_{22} \end{pmatrix} \begin{pmatrix} d \\ u \end{pmatrix} \tag{7}$$

whose blocks are affected by uncertainties Δ_1, Δ_2 and Δ_3 in the different classes $\boldsymbol{\Delta_1}$, $\boldsymbol{\Delta_2}$ and $\boldsymbol{\Delta_3}$. One would then construct the linear fractional representation

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \hline e \\ y \end{pmatrix} = \left(\begin{array}{ccc|cc} P_{11} & 0 & 0 & P_{14} & 0 \\ 0 & P_{22} & 0 & P_{24} & 0 \\ 0 & 0 & P_{33} & 0 & P_{35} \\ \hline P_{41} & 0 & P_{43} & P_{44} & P_{45} \\ 0 & P_{52} & 0 & P_{54} & G_{22} \end{array}\right) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \hline d \\ u \end{pmatrix}, \quad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \Delta_1(z_1) \\ \Delta_2(z_2) \\ \Delta_3(z_3) \end{pmatrix}.$$

This can be obviously subsumed to (3) and is, therefore, just a special version of the generalized plant considered throughout this paper.

Moreover, all our results could as well be based on a dual construction; one would then combine a linear fractional representation of $e = G_{11}(\Delta_1)(d) + G_{12}(\Delta_1)(u)$ with one of $y = G_{21}(\Delta_2)(d)$.

3 Robust Performance Analysis

For $k = 1,2$ let $\boldsymbol{\Pi_k}$ denote a class of valid IQC for $\boldsymbol{\Delta_k}$ [14]. This just means that any $\Pi_k \in \boldsymbol{\Pi_k}$, which is said to be a multiplier, is a transfer matrix that is bounded and Hermitian-valued on the imaginary axis and for which we have

$$\int_{-\infty}^{\infty} \begin{pmatrix} \widehat{\Delta_k(w)}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix}^* \Pi_k(i\omega) \begin{pmatrix} \widehat{\Delta_k(w)}(i\omega) \\ \hat{w}(i\omega) \end{pmatrix} d\omega \geq 0 \;\text{ for all }\; w \in \mathscr{L}_2 \tag{8}$$

and for all $\Delta_k \in \boldsymbol{\Delta_k}$, $k = 1,2$. In the sequel we sometimes introduce the partition $\Pi_k = \begin{pmatrix} Q_k & S_k \\ S_k^* & R_k \end{pmatrix}$ according to the dimensions of the signals $\Delta_k(w)$ and w in (8). Since $\boldsymbol{\Delta_k}$ is star-convex it contains 0; therefore (8) implies $R_k \succcurlyeq 0$ on $\mathbb{C}^0$ for $k = 1,2$.

If K stabilizes P (implying stability of $\mathscr{P}$), it achieves robust stability and robust performance in case there exists some $\Pi_1 \in \boldsymbol{\Pi_1}$ and $\Pi_2 \in \boldsymbol{\Pi_2}$ such that the following frequency domain inequality (FDI) is satisfied:

$$\begin{pmatrix} I & 0 & 0 \\ P_{11} & 0 & P_{13} \end{pmatrix}^* \Pi_1 \begin{pmatrix} I & 0 & 0 \\ P_{11} & 0 & P_{13} \end{pmatrix} + \begin{pmatrix} 0 & I & 0 \\ \mathscr{P}_{21} & P_{22} & \mathscr{P}_{23} \end{pmatrix}^* \Pi_2 \begin{pmatrix} 0 & I & 0 \\ \mathscr{P}_{21} & P_{22} & \mathscr{P}_{23} \end{pmatrix} + \\ + \begin{pmatrix} 0 & 0 & I \\ \mathscr{P}_{31} & P_{32} & \mathscr{P}_{33} \end{pmatrix}^* \Pi_p \begin{pmatrix} 0 & 0 & I \\ \mathscr{P}_{31} & P_{32} & \mathscr{P}_{33} \end{pmatrix} \prec 0 \;\text{ on }\; \mathbb{C}^0. \tag{9}$$

Indeed, since the right-lower blocks of Π_1, Π_2 and Π_p are positive-semi-definite on $\mathbb{C}^0$, we infer from (9) that

$$\begin{pmatrix} I \\ P_{11} \end{pmatrix}^* \Pi_1 \begin{pmatrix} I \\ P_{11} \end{pmatrix} \prec 0 \text{ and } \begin{pmatrix} I \\ P_{22} \end{pmatrix}^* \Pi_2 \begin{pmatrix} I \\ P_{22} \end{pmatrix} \prec 0 \text{ on } \mathbb{C}^0. \qquad (10)$$

The standard IQC theorem [14] hence implies that the inverses of $I - P_{11}\Delta_1$ and $I - P_{22}\Delta_2$ are stable. In turn this guarantees robust stability of (4). If $\varepsilon > 0$ is sufficiently small, (9) persists to hold if replacing the right-hand side by $-\frac{\varepsilon}{2\pi}I$. If we then choose any trajectory of (4) with $d \in \mathscr{L}_2$, we infer that all other signals are in $\mathscr{L}_2$ as well. Evaluation of the quadratic form (9) for $\mathrm{col}(\widehat{w}_1(i\omega), \widehat{w}_2(i\omega), \widehat{d}(i\omega))$ leads, with the system description (4), to

$$\sum_{k=1}^{2} \begin{pmatrix} \widehat{w}_k(i\omega) \\ \widehat{z}_k(i\omega) \end{pmatrix}^* \Pi_k(i\omega) \begin{pmatrix} \widehat{w}_k(i\omega) \\ \widehat{z}_k(i\omega) \end{pmatrix} + \begin{pmatrix} \widehat{d}(i\omega) \\ \widehat{e}(i\omega) \end{pmatrix}^* \Pi_p \begin{pmatrix} \widehat{d}(i\omega) \\ \widehat{e}(i\omega) \end{pmatrix} \leq -\frac{\varepsilon}{2\pi}\|\hat{d}(i\omega)\|^2$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$. Integration and exploiting (8) implies the validity of (6).

We have sketched the standard arguments which reduce the desired robustness specification on the closed-loop system to an FDI. For fixed mulitpliers Π_1 and Π_2, the search for a stabilizing controller which renders (9) satisfied is well-know to be LMIable. However, for reduced conservatism, one wishes to view Π_1 and Π_2 together with the controller as optimization variables. Even if considering "nicely" (conicly) parameterized classes of multipliers $\boldsymbol{\Pi_k}$, all standard procedures which are convexifying for fixed multipliers lead to bi-linear matrix inequalities if the multipliers are also considered as variables.

As one of the key contributions of this paper we show that one can equivalently rewrite the FDI (9) into one in Π_1 and Π_2^{-1} which requires all elements of $\boldsymbol{\Pi_2}$ to be non-singular. This is formulated in the following auxiliary result whose proof is found in the Appendix (Section 7).

Lemma 1. *Suppose that any $\Pi_2 \in \boldsymbol{\Pi_2}$ is non-singular and that the left-upper block of its inverse is negative semi-definite on $\mathbb{C}^0$. Then (9) is equivalent to*

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Q_p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ P_{11} & P_{13} \end{pmatrix}^* \Pi_1 \begin{pmatrix} I & 0 \\ P_{11} & P_{13} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^T -$$
$$- \begin{pmatrix} 0 & 0 \\ 0 & S_p \\ I & 0 \\ 0 & T_p \end{pmatrix} \begin{pmatrix} P_{22} & -I \\ P_{32} & 0 \end{pmatrix} \Pi_2^{-1} \begin{pmatrix} P_{22} & -I \\ P_{32} & 0 \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ 0 & S_p \\ I & 0 \\ 0 & T_p \end{pmatrix}^T +$$
$$+ He \begin{pmatrix} 0 & 0 \\ 0 & S_p \\ I & 0 \\ 0 & T_p \end{pmatrix} \begin{pmatrix} \mathscr{P}_{21} & \mathscr{P}_{23} \\ \mathscr{P}_{31} & \mathscr{P}_{33} \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix} \prec 0 \ \textit{on}\ \mathbb{C}^0. \qquad (11)$$

In view of the IQC dualization techniques in [12], this novel partial dualization of the robust performance FDI leads to an additive separation of the transfer matrices into a part that depends on the multipliers Π_1 and Π_2^{-1} and another part which is only affected by the controller. This is the essential feature in order to convexify the considered synthesis problem. For actual computations we assume that the multipliers Π_1 and Π_2^{-1} are parameterized as

$$\boldsymbol{\Pi_1} = \{\Psi_1^* M_1 \Psi_1 : \, M_1 \in \mathscr{M}_1\} \;\text{ and }\; \boldsymbol{\Pi_2}^{-1} = \{\Psi_2^* M_2 \Psi_2 : \, M_2 \in \mathscr{M}_2\}$$

with LMIable sets $\mathscr{M}_1$ and $\mathscr{M}_2$ of real symmetric matrices and with some fixed transfer matrices Ψ_1 and Ψ_2 that have no poles on the extended imaginary axis $\mathbb{C}^0$. It is stressed that we do not require stability of these outer factors in the sequel. Note that the parametrization of $\boldsymbol{\Pi_1}$ naturally complies with many multiplier classes that have been suggested in the literature. Since this is not generically true for $\boldsymbol{\Pi_2}^{-1}$, we refer to [12] for a more detailed discussion of this issue. Recall that P_{11}, P_{13} and P_{22}, P_{32} are stable. Therefore one can describe the family obtained by the sum of the first three matrices in (11) as

$$\Psi^* M \Psi \;\text{ with parameter }\; M \in \mathscr{M}$$

where $\mathscr{M}$ is an LMIable set of real symmetric structured matrices and Ψ is a tall transfer matrix without poles in $\mathbb{C}^0$. Let us finally observe that the last term in (11) just equals $\mathrm{He}(H \star K)$ if defining

$$H := \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & S_p & 0 \\ I & 0 & 0 \\ 0 & T_p & 0 \\ \hline 0 & 0 & I \end{array}\right) \begin{pmatrix} 0 & 0 & P_{24} \\ P_{31} & P_{33} & P_{34} \\ P_{41} & P_{43} & P_{44} \end{pmatrix} \left(\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right).$$

This reveals how the considered robust performance synthesis problem subsumes to the generic optimization problem as considered in the subsequent Section 2.

4 Parametric-Dynamic Feasibility Problems

The previous section revealed that a surprisingly large variety of robust synthesis questions can be reduced to the following problem: Let us be given a generalized plant H, a transfer matrix Ψ without poles on the imaginary axis and any LMIable set $\mathscr{M}$ of symmetric matrices of dimension compatible with the number of rows of Ψ. Determine some matrix $M \in \mathscr{M}$ and a controller K which stabilizes H such that

$$\Psi^* M \Psi + (H \star K) + (H \star K)^* \prec 0 \;\text{ on }\; \mathbb{C}^0. \tag{12}$$

We choose to call this a parametric-dynamic feasibility problem since it involves the common search for the parameter $M \in \mathscr{M}$ and the dynamic controller K in order to render (12) satisfied. Observe that the FDI can as well be expressed as

$$\begin{pmatrix} I \\ \Psi \\ H\star K \end{pmatrix}^* \underbrace{\begin{pmatrix} 0 & 0 & I \\ 0 & M & 0 \\ I & 0 & 0 \end{pmatrix}}_{M_e} \begin{pmatrix} I \\ \Psi \\ H\star K \end{pmatrix} \prec 0 \text{ on } \mathbb{C}^0. \tag{13}$$

Despite this relatively simplistic and specific formulation, we will briefly illustrate in Section 5 that it also straightforwardly covers the problem classes in [5, 6, 11, 15].

In the sequel we use the minimal state-space realizations

$$\Psi = \left[\begin{array}{c|c} A_\Psi & B_\Psi \\ \hline C_\Psi & D_\Psi \end{array}\right] \text{ and } H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \left[\begin{array}{c|cc} A_H & B_{H1} & B_H \\ \hline C_{H1} & D_{H1} & E_H \\ C_H & F_H & 0 \end{array}\right],$$

where the partition of H is induced by the dimension of K. Note that A_Ψ has no eigenvalues on the imaginary axis. Moreover (A_H, B_H) and (A_H, C_H) are assumed to be stabilizable and detectable such that the existence of a stabilzing controller for H is guaranteed. Realizations of controllers K are denoted as

$$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right].$$

Then the closed-loop transfer matrix admits the natural state-space description

$$H\star K = H_{cl} = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] = \left[\begin{array}{cc|c} A_H + B_H D_K C_H & B_H C_K & B_{H1} + B_H D_K F_H \\ C_H B_K & A_K & B_K F_H \\ \hline C_{H1} + E_H D_K C_H & E_H C_K & D_{H1} + E_H D_K F_H \end{array}\right].$$

This leads to the following realization of the dynamics in the outer factor of (13):

$$\begin{pmatrix} \Psi \\ H\star K \end{pmatrix} = \begin{pmatrix} \Psi \\ H_{cl} \end{pmatrix} = \left[\begin{array}{cc|c} A_\Psi & 0 & B_\Psi \\ 0 & A_{cl} & B_{cl} \\ \hline C_\Psi & 0 & D_\Psi \\ 0 & C_{cl} & D_{cl} \end{array}\right].$$

The very same transfer matrix is obtained by interconnecting K with

$$\begin{pmatrix} \Psi & 0 \\ H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \left[\begin{array}{cc|cc} A_\Psi & 0 & B_\Psi & 0 \\ 0 & A_H & B_{H1} & B_{H2} \\ \hline C_\Psi & 0 & D_\Psi & 0 \\ 0 & C_{H1} & D_{H11} & D_{H12} \\ 0 & C_{H2} & D_{H21} & 0 \end{array}\right] =: \left[\begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & D_1 & E \\ C & F & 0 \end{array}\right] \tag{14}$$

by a lower linear fractional transformation as

$$\begin{pmatrix} \Psi \\ H_{cl} \end{pmatrix} = \begin{pmatrix} \Psi & 0 \\ H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \star K = \left[\begin{array}{cc|c} A+BD_KC & BC_K & B_1+BD_KF \\ CB_K & A_K & B_KF \\ \hline C_1+ED_KC & EC_K & D_1+ED_KF \end{array}\right]. \tag{15}$$

Let us finally introduce a shorthand notation for the closed-loop realization matrices in terms of calligraphic symbols and observe that they can be written in the following three different ways:

$$\left(\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \end{array}\right) := \left(\begin{array}{cc|c} A_\Psi & 0 & B_\Psi \\ 0 & A_{cl} & B_{cl} \\ \hline C_\Psi & 0 & D_\Psi \\ 0 & C_{cl} & D_{cl} \end{array}\right) = \left(\begin{array}{cc|c} A+BD_KC & BC_K & B_1+BD_KF \\ B_KC & A_K & B_KF \\ \hline C_1+ED_KC & EC_K & D_1+ED_KF \end{array}\right) =$$
$$= \left(\begin{array}{cc|c} A & 0 & B_1 \\ 0 & 0 & 0 \\ \hline C_1 & 0 & D_1 \end{array}\right) + \left(\begin{array}{cc} B & 0 \\ 0 & I \\ \hline E & 0 \end{array}\right) \begin{pmatrix} D_K & C_K \\ B_K & A_K \end{pmatrix} \left(\begin{array}{cc|c} C & 0 & F \\ 0 & I & 0 \end{array}\right). \tag{16}$$

4.1 Analysis

Let us now suppose that K stabilizes H and leads to (13). This means that A_{cl} is Hurwitz and that

$$\left[\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline 0 & I \\ \mathscr{C} & \mathscr{D} \end{array}\right]^* M_e \left[\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline 0 & I \\ \mathscr{C} & \mathscr{D} \end{array}\right] \prec 0 \text{ on } \mathbb{C}^0. \tag{17}$$

Since A_Ψ does not have eigenvalues on the imaginary axis, the same holds for $\mathscr{A}$. We can apply the Kalman Yakubovich Popov lemma [1] to (17) in order to infer that its validity is equivalent to the existence of a symmetric solution $\mathscr{X}$ of the LMI

$$\begin{pmatrix} \mathscr{A}^T\mathscr{X} + \mathscr{X}\mathscr{A} + \mathscr{C}^T \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \mathscr{C} & \mathscr{X}\mathscr{B} + \mathscr{C}^T \begin{pmatrix} MD_\Psi \\ I \end{pmatrix} \\ \mathscr{B}^T\mathscr{X} + \begin{pmatrix} D_\Psi^T M & I \end{pmatrix} \mathscr{C} & \begin{pmatrix} 0 & I \end{pmatrix} \mathscr{D} + \mathscr{D}^T \begin{pmatrix} 0 \\ I \end{pmatrix} + D_\Psi^T M D_\Psi \end{pmatrix} \prec 0. \tag{18}$$

In view of (16) let us note that the left-upper block of this inequality reads as

$$\begin{pmatrix} A_\Psi & 0 \\ 0 & A_{cl} \end{pmatrix}^T \begin{pmatrix} \mathscr{X}_{11} & \mathscr{X}_{12} \\ \mathscr{X}_{21} & \mathscr{X}_{22} \end{pmatrix} + \begin{pmatrix} \mathscr{X}_{11} & \mathscr{X}_{12} \\ \mathscr{X}_{21} & \mathscr{X}_{22} \end{pmatrix} \begin{pmatrix} A_\Psi & 0 \\ 0 & A_{cl} \end{pmatrix} + \begin{pmatrix} C_\Psi^T M C_\Psi & 0 \\ 0 & 0 \end{pmatrix} \prec 0$$

with a partition of $\mathscr{X}$ compatible with that of $\mathrm{diag}(A_\Psi, A_{cl})$. Since M is indefinite, stability of A_{cl} does in general *not* translate into the simple constraint $\mathscr{X} \succ 0$ on the solution of (18). However, we observe that (18) always implies

$$A_{cl}^T \mathscr{X}_{22} + \mathscr{X}_{22} A_{cl} \prec 0. \tag{19}$$

Since A_{cl} is Hurwitz we infer $\mathscr{X}_{22} \succ 0$. If we define

$$\mathscr{I} = \begin{pmatrix} I \\ 0 \end{pmatrix} \text{ in the partition of } \mathscr{A} = \begin{pmatrix} A_\Psi & 0 \\ 0 & A_{cl} \end{pmatrix},$$

it is easy to verify that $\mathscr{X}_{22} \succ 0$ is equivalent to the existence of some $\hat{M} = \hat{M}^T$ with

$$\mathscr{X} + \mathscr{I} \hat{M} \mathscr{I}^T \succ 0. \tag{20}$$

Conversely, let there exist $\mathscr{X}$, $\hat{M}$ with (18) and (20). Then $\mathscr{X}_{22} \succ 0$ and the consequence (19) of (18) reveals that A_{cl} is Hurwitz; hence K stabilizes H. Moreover, (18) leads to the validity of the FDI (13). This proves the following intermediate result.

Lemma 2. *The controller K stabilizes H and renders (13) satisfied iff there exist solutions $\mathscr{X}$ and $\hat{M}$ of the inequalities (18) and (20).*

Most parts of this result are standard. The key twist is the very elementary characterization of the fact that K stabilizes H by the inequality (20). This formulation of the analysis result is tailored towards a convenient proof for synthesis.

4.2 Synthesis

On the basis of Lemma 2 let us now develop the required transformation of $\mathscr{X}$ and the state-space description of K which convexifies both (18) and (20).

For this purpose we assume that there exists a controller K for which one can find M, $\hat{M}$, $\mathscr{X}$ that satisfy (18) and (20). According to the sizes of A and A_K in (16) let us partition $\mathscr{X}$ as

$$\mathscr{X} = \begin{pmatrix} X & U \\ U^T & \hat{X} \end{pmatrix}.$$

Note that the dimension of A_K can be taken at least as large as that of A (by adding uncontrollable/uobservalbe modes in the controller if necessary). Hence U is a wide matrix. By perturbation (if necessary) we can then assume without loss of generality that $\mathscr{X}$ and $\hat{X}$ are non-singular and that U has full row rank. This implies that $X - U\hat{X}^{-1}U^T$ is invertible. Let us now partition this latter matrix as A in (14). An additional perturbation allows to make sure that the right-lower block of $X - U\hat{X}^{-1}U^T$ is non-singular as well. Hence this block can be written as W_{22}^{-1} for some non-singular real symmetric matrix W_{22}, and we can then sequentially determine real matrices W_{21} and $W_{11} = W_{11}^T$ with

$$X - U\hat{X}^{-1}U^T = \begin{pmatrix} W_{11} + W_{21}^T W_{22}^{-1} W_{21} & W_{21}^T W_{22}^{-1} \\ W_{22}^{-1} W_{21} & W_{22}^{-1} \end{pmatrix}. \tag{21}$$

Let us collect the blocks W_{11}, W_{21}, W_{22} into W and introduce the structured matrices Y and Z as follows:

$$W := \begin{pmatrix} W_{11} & W_{21}^T \\ W_{21} & W_{22} \end{pmatrix},\ \ Y := \begin{pmatrix} I & -W_{21}^T \\ 0 & W_{22} \end{pmatrix},\ \ Z := \begin{pmatrix} W_{11} & 0 \\ W_{21} & I \end{pmatrix}. \tag{22}$$

Due to (21) one easily checks that $Z = Y(X - U\hat{X}^{-1}U^T)$ and hence

$$Z(X - U\hat{X}^{-1}U^T)^{-1} = Y. \tag{23}$$

Note that both Y and Z are non-singular. Then the matrix

$$\mathscr{Y} = \begin{pmatrix} Z & 0 \\ X & U \end{pmatrix} \mathscr{X}^{-1}$$

has full row rank. Moreover, just because $\begin{pmatrix} X & U \end{pmatrix}$ is the upper block row of $\mathscr{X}$, the lower block row of $\mathscr{Y}$ must equal $\begin{pmatrix} I & 0 \end{pmatrix}$. Since $(X - U\hat{X}^{-1}U^T)^{-1}$ is the left-upper block of $\mathscr{X}^{-1}$, (23) implies that the left-upper block of $\mathscr{Y}$ must equal Y. Therefore we arrive at the following structure of $\mathscr{Y}$ together with a crucial factorization of $\mathscr{X}$:

$$\mathscr{Y} = \begin{pmatrix} Y & V \\ I & 0 \end{pmatrix} \ \text{ and } \ \mathscr{Y}\mathscr{X} = \begin{pmatrix} Z & 0 \\ X & U \end{pmatrix}. \tag{24}$$

Since $\mathscr{Y}$ has full row rank, (18) implies that

$$\begin{pmatrix} \mathrm{He}(\mathscr{Y}\mathscr{X}\mathscr{A}\mathscr{Y}^T) + \mathscr{Y}\mathscr{C}^T \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \mathscr{C}\mathscr{Y}^T & \mathscr{Y}\mathscr{X}\mathscr{B} + \mathscr{Y}\mathscr{C}^T \begin{pmatrix} MD_\Psi \\ I \end{pmatrix} \\ \mathscr{B}^T\mathscr{X}\mathscr{Y}^T + \begin{pmatrix} D_\Psi^T M & I \end{pmatrix} \mathscr{C}\mathscr{Y}^T & \mathrm{He}\left(\begin{pmatrix} 0 & I \end{pmatrix}\mathscr{D}\right) + D_\Psi^T M D_\Psi \end{pmatrix} \prec 0. \tag{25}$$

With the last relation in (16) and with (24) we obtain

$$\left(\begin{array}{c|c} \mathscr{Y}\mathscr{X}\mathscr{A}\mathscr{Y}^T & \mathscr{Y}\mathscr{X}\mathscr{B} \\ \hline \mathscr{C}\mathscr{Y}^T & \mathscr{D} \end{array}\right) = \left(\begin{array}{c|c} \mathscr{Y}\mathscr{X} & 0 \\ \hline 0 & I \end{array}\right) \left(\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \end{array}\right) \left(\begin{array}{c|c} \mathscr{Y}^T & 0 \\ \hline 0 & I \end{array}\right) =$$

$$= \left(\begin{array}{cc|c} ZAY^T & ZA & ZB_1 \\ XAY^T & XA & XB_1 \\ \hline C_1Y^T & C_1 & D_1 \end{array}\right) + \left(\begin{array}{cc} ZB & 0 \\ XB & U \\ \hline E & 0 \end{array}\right) \begin{pmatrix} D_K & C_K \\ B_K & A_K \end{pmatrix} \left(\begin{array}{cc|c} CY^T & C & F \\ V^T & 0 & 0 \end{array}\right) =$$

$$= \left(\begin{array}{cc|c} ZAY^T & ZA & ZB_1 \\ 0 & XA & XB_1 \\ \hline C_1Y^T & C_1 & D_1 \end{array}\right) + \left(\begin{array}{cc} ZB & 0 \\ 0 & I \\ \hline E & 0 \end{array}\right) N \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F \end{array}\right)$$

in case we introduce the new variable

$$N = \begin{pmatrix} 0 & 0 \\ XAY^T & 0 \end{pmatrix} + \begin{pmatrix} I & 0 \\ XB & U \end{pmatrix} \begin{pmatrix} D_K & C_K \\ B_K & A_K \end{pmatrix} \begin{pmatrix} CY^T & I \\ V^T & 0 \end{pmatrix}. \tag{26}$$

This motivates the definition

$$\begin{pmatrix} \boldsymbol{A}(N,W,X) & \boldsymbol{B}(N,W,X) \\ \boldsymbol{C}(N,W) & \boldsymbol{D}(N) \end{pmatrix} := \left(\begin{array}{cc|c} ZAY^T & ZA & ZB_1 \\ 0 & XA & XB_1 \\ \hline C_1Y^T & C_1 & D_1 \end{array}\right) + \left(\begin{array}{cc} ZB & 0 \\ 0 & I \\ \hline E & 0 \end{array}\right) N \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F \end{array}\right)$$

with which (25) just reads as

$$\begin{pmatrix} \mathrm{He}(\boldsymbol{A}(N,W,X)) + \boldsymbol{C}(N,W)^T \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{C}(N,W) & * \\ \boldsymbol{B}(N,W,X)^T + \left(D_\Psi^T M \; I \right) \boldsymbol{C}(N,W) & \mathrm{He}\left(\left(0 \; I \right) \boldsymbol{D}(N) \right) + D_\Psi^T M D_\Psi \end{pmatrix} \prec 0. \tag{27}$$

If we zoom in and exploit the special structure of the matrices in (14) it is not hard to see that this is actually an LMI in M and (N,W,X). Indeed with (22) we have

$$\begin{pmatrix} ZAY^T & ZB \\ C_1Y^T & 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} W_{11}A_\Psi & 0 \\ W_{21}A_\Psi - A_H W_{21} & A_H W_{22} \end{pmatrix} & \begin{pmatrix} 0 \\ B_H \end{pmatrix} \\ \begin{pmatrix} C_\Psi & 0 \\ -C_{H1}W_{21} & C_{H1}W_{22} \end{pmatrix} & 0 \end{pmatrix}$$

which reveals *affine dependence* of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ on (N,W,X). Moreover, again with (14) we observe that

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \left(E \;\; 0 \right) N \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ D_{H12} & 0 \end{pmatrix} N \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix} = 0$$

and hence

$$\boldsymbol{C}(N,W)^T \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{C}(N,W) = \begin{pmatrix} C_\Psi^T M C_\Psi & 0 & C_\Psi^T M C_\Psi & 0 \\ 0 & 0 & 0 & 0 \\ C_\Psi^T M C_\Psi & 0 & C_\Psi^T M C_\Psi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is actually *independent from* (N,W) and, clearly, still affine in M. Similarly

$$\begin{aligned} &\left(D_\Psi^T M \;\; I \right) \boldsymbol{C}(N,W) = \\ &= \left(D_\Psi^T M C_\Psi - C_{H1}W_{21} \;\; C_{H1}W_{22} \;\; D_\Psi^T M C_\Psi \;\; C_{H1} \right) + \left(D_{H12} \;\; 0 \right) W \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix} \end{aligned} \tag{28}$$

is affine in M and (N,W). All this shows that the left-hand side of (27) depends affinely on M and (N,W,X) as claimed.

Let us now consider (20). Again since $\mathscr{Y}$ has full row rank this inequality implies

$$\mathscr{Y}\mathscr{X}\mathscr{Y}^T + \mathscr{Y}\mathscr{J}\hat{M}(\mathscr{Y}\mathscr{J})^T \succ 0. \tag{29}$$

It is easy to check that (29) reads explicitly as

$$\begin{pmatrix} \begin{pmatrix} W_{11}+\hat{M} & 0 \\ 0 & W_{22} \end{pmatrix} & \begin{pmatrix} W_{11}+\hat{M} & 0 \\ W_{21} & I \end{pmatrix} \\ \begin{pmatrix} W_{11}+\hat{M} & W_{21}^T \\ 0 & I \end{pmatrix} & \begin{pmatrix} \hat{M} & 0 \\ 0 & 0 \end{pmatrix} + X \end{pmatrix} \succ 0. \tag{30}$$

By taking the Schur complement, (30) is equivalent to $W_{11}+\hat{M} \succ 0$ and

$$\begin{pmatrix} W_{22} & \begin{pmatrix} W_{21} & I \end{pmatrix} \\ \begin{pmatrix} W_{21}^T \\ I \end{pmatrix} & X - \begin{pmatrix} W_{11} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \succ 0 \tag{31}$$

which is clearly an LMI constraint on (W,X). All this proves necessity in the following main synthesis result of this paper.

Theorem 1. *There exists a controller K which stabilizes H and for which there are $M \in \mathscr{M}$ and symmetric $\mathscr{X}$, $\hat{M}$ with (18) and (20) if and only if the LMIs (27) and (31) admit a solution $M \in \mathscr{M}$ and (N,W,X).*

To conclude the proof let us suppose that $M \in \mathscr{M}$ and (N,W,X) satisfy (27) and (31). Define Y and Z according to (22). Taking the Schur complement shows that (31) implies $W_{22} \succ 0$ and

$$X - \begin{pmatrix} W_{11} + W_{21}^T W_{22}^{-1} W_{21} & W_{21}^T W_{22}^{-1} \\ W_{22}^{-1} W_{21} & W_{22}^{-1} \end{pmatrix} \succ 0. \tag{32}$$

Therefore, W_{22} and hence also Y are non-singular. The non-singular matrix on the left in (32) is nothing but $X - Y^{-1}Z$. Therefore $Z - YX$ is non-singular. This allows to choose square and non-singular matrices U and V with $VU^T = Z - YX$. We can then define the square and non-singular matrix $\mathscr{Y}$ as in (24) and solve the second equation in (24) for the (necessarily symmetric) matrix $\mathscr{X}$. Similarly we can solve (26) for A_K, B_K, C_K and D_K which specify a controller K. With these definitions all the relations used in the necessity part of the proof hold true, and we can just reverse the arguments. Indeed (27) is identical to (25); since $\mathscr{Y}$ is square and non-singular, a congruence transformation with $\operatorname{diag}(\mathscr{Y}, I)$ leads to (18). Moreover, there exists an $\hat{M}$ with $W_{11} + \hat{M} \succ 0$; then (31) implies (30) and thus (29) and hence, by non-singularity of $\mathscr{Y}$, again (20). This completes the proof of Theorem 1.

Note that the last part of the proof provides a constructive procedure for designing a controller. It is easily verified that the McMillan degree of K is not larger than the dimension of A which is the sum of the dimensions of A_H (the degree of H) and of A_Ψ (the degree of Ψ).

We stress that the proposed convexifying transformation can be viewed as a one-shot combination of those in [18, 13] and [16] as already exploited in [8] and [21]. It is worthwhile to emphasize that the transformation of [18, 13] directly applies without complication if Ψ involves no dynamics. Due to our generic problem formulation we fully cover weighted robust estimator [19] and weighted robust feed-forward synthesis involving dynamic IQCs in a unifying manner. In comparison to this previous work, we would like to highlight that we made effective use of the fact that the the (possibly unstable) dynamics in Ψ are outside the control loop which leads to a considerable simplifications of the proofs in this paper under minimal hypotheses.

4.3 Elimination

This little section serves to illustrate how we can routinely eliminate (possibly high-dimensional) variables in our general synthesis result. For this purpose let us note that, due to (16),

$$\begin{pmatrix} 0 & I \end{pmatrix} \boldsymbol{D}(N) = \begin{pmatrix} 0 & I \end{pmatrix} \left(D_1 + \begin{pmatrix} E & 0 \end{pmatrix} N \begin{pmatrix} 0 \\ F \end{pmatrix} \right) = D_{H1} + \begin{pmatrix} D_{H12} & 0 \end{pmatrix} N \begin{pmatrix} 0 \\ F \end{pmatrix}.$$

If we also recall (28) and $ZB = B$ we conclude that (27) can be written as

$$\boldsymbol{E}(M,W,X) + \mathrm{He}\left(\begin{array}{cc} B & 0 \\ 0 & I \\ \hline D_{H12} & 0 \end{array} \right) N \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F \end{array} \right) \prec 0$$

with some easily determined map $\boldsymbol{E}$ that is affine in (M,W,X). This allows us to apply the projection lemma [9] and eliminate N. In order to formulate the resulting synthesis LMIs we introduce basis matrices $\boldsymbol{F}$ and $\boldsymbol{G}$ of, respectively, the kernels of

$$\begin{pmatrix} B^T & 0 & D_{H12}^T \\ 0 & I & 0 \end{pmatrix} \text{ and } \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F \end{array} \right). \tag{33}$$

This leads to the following synthesis result with reduced computational complexity.

Corollary 1. *There exists $M \in \mathscr{M}$ and solutions (N,W,X) of the LMIs (27) and (31) if and only if there exist $M \in \mathscr{M}$ and (W,X) satisfying the LMIs (31) as well as*

$$\boldsymbol{F}^T \boldsymbol{E}(M,W,X) \boldsymbol{F} \prec 0 \; \textit{ and } \; \boldsymbol{G}^T \boldsymbol{E}(M,W,X) \boldsymbol{G} \prec 0.$$

For actual coding, these LMIs can be rendered somewhat more explicit by exploiting the particular structures of B, C, $F = D_{H21}$ and of the matrices (33) in order to determine tailored basis matrices of the kernels of (33).

Technically this leads to the results of [11] (for stable Ψ) and similar ones in [17] that have been obtained independently. Conceptually it fully covers the robust feedforward synthesis results in [12] with a substantially simplified proof and for more general configurations. Note that a similar unifying approach is possible for H_2-synthesis which is not pursued in this paper for reasons of space.

5 A Sketch of Further Applications

Generalized l_2-synthesis has been introduced in [5] in order to handle nominal performance synthesis with independently norm-bounded disturbances or robust synthesis against full but element-by-element bounded uncertainties. With LMIable sets of real matrices $\mathscr{L}$ and $\mathscr{R}$ this has been translated to the question of finding a controller K which stabilizes H and such that there exist $L \in \mathscr{L}$ and $R \in \mathscr{R}$ with

$$\|L^{-1/2}(H \star K)R^{-1/2}\|_\infty < 1 \;\text{ or }\; \begin{pmatrix} L & H \star K \\ (H \star K)^* & R \end{pmatrix} \succ 0 \;\text{ on }\; \mathbb{C}^0.$$

Convexification by controller transformation can be achieved with [18, 13] while [5] presents the LMI synthesis condition obtained after elimination. An extension to cost criteria and disturbances described by dynamic IQCs has been discussed in [11, 6], with applications to e.g. the suppression of disturbances with specific spectral contents. In [6] it is shown that this amounts to the above described scenario for LMIable sets $\mathscr{L}$ and $\mathscr{R}$ of transfer matrices, and inner-outer factorizations are employed to reduce the synthesis problem to one for static multipliers. More generally, [11] proposes a more direct solution to synthesizing a controller K which stabilizes H and for which there exists some $M \in \mathscr{M}$ with

$$\Psi^* M \Psi + \begin{pmatrix} I \\ H \star K \end{pmatrix}^* \Pi_p \begin{pmatrix} I \\ H \star K \end{pmatrix} \prec 0 \;\text{ on }\; \mathbb{C}^0.$$

This FDI translates into

$$\begin{pmatrix} Q_p + \Psi^* M \Psi & 0 \\ 0 & -I \end{pmatrix} + \mathrm{He}\left(\begin{pmatrix} S_p \\ T_p \end{pmatrix} (H \star K) \begin{pmatrix} I & 0 \end{pmatrix} \right) \prec 0 \;\text{ on }\; \mathbb{C}^0,$$

which clarifies that all these design questions subsume to the more specifically looking but as general problem in Section 2.

Techniques that are directly related to the ones presented here have been applied in [15]. Given stable transfer matrices H_0, H_1, H_2 and a generalized plant H, this paper covers the design of a controller K which stabilizes H and for which there exists some $M \in \mathscr{M}$ with

$$\|H_0 + H_1 M H_2 - H \star K\|_\infty < \gamma. \tag{34}$$

It is shown that this encompasses a rich class of questions related to structured controller synthesis and multi-objective control, see also [17]. Mathematically it corresponds to the projection of the transfer matrix of a controlled system onto an affinely parameterized set of transfer matrices in the H_∞-norm. Since (34) is nothing but

$$\begin{pmatrix} -\gamma I & G_0 + G_1 M G_2 \\ (G_0 + G_1 M G_2)^* & -\gamma I \end{pmatrix} + \mathrm{He} \begin{pmatrix} -I \\ 0 \end{pmatrix} (G \star K) \begin{pmatrix} 0 & I \end{pmatrix} \prec 0 \;\text{ on }\; \mathbb{C}^0,$$

this is yet another variant of the problem in Section 2.

6 Conclusions

In this paper we have provided a complete solution to a general parametric dynamic frequency-domain feasibility problem under minimal hypothesis. We have as well worked out various relations to nominal and robust controller synthesis problems in the literature. In particular it has been shown that robust synthesis for generalized plants whose control channel is not affected by uncertainties can be turned into a semi-definite optimization problem.

Acknowledgements The author appreciates many stimulating discussions with Jan Willems (Katholieke Universiteit Leuven), Emre Köse (Boğazici University Istanbul), Hakan Köroğlu and Sjoerd Dietz (Delft University of Technology). Moreover, financial support from MicroNed, a program to promote micro-systems-technology in The Netherlands, is gratefully acknowledged.

7 Appendix: Proof of Lemma 1

Let us partition Π_2^{-1} similarly as Π_2 into the blocks $\tilde{Q}_2$, $\tilde{R}_2$ and $\tilde{S}_2$ and recall that

$$\begin{pmatrix} \tilde{Q}_2 & \tilde{S}_2 \\ \tilde{S}_2^* & \tilde{R}_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -R_2^{-1}S_2 & I \end{pmatrix} \begin{pmatrix} (Q_2 - S_2R_2^{-1}S_2^*)^{-1} & 0 \\ 0 & R_2^{-1} \end{pmatrix} \begin{pmatrix} I & -S_2^*R_2^{-1} \\ 0 & I \end{pmatrix}.$$

Let us remember that $R_2 \succcurlyeq 0$ on $\mathbb{C}^0$. We first give the proof under the assumption $R_2 \succ 0$ on $\mathbb{C}^0$. If we abbreviate

$$T_1 := \begin{pmatrix} I & 0 & 0 \\ P_{11} & 0 & P_{13} \end{pmatrix}^* \Pi_1 \begin{pmatrix} I & 0 & 0 \\ P_{11} & 0 & P_{13} \end{pmatrix}$$

and recall (5), the inequality (9) can be written as

$$T_1 + \begin{pmatrix} 0 & I & 0 \\ \mathscr{P}_{21} & P_{22} + R_2^{-1}S_2 & \mathscr{P}_{23} \end{pmatrix}^* \begin{pmatrix} \tilde{Q}_2^{-1} & 0 \\ 0 & R_2 \end{pmatrix} \begin{pmatrix} 0 & I & 0 \\ \mathscr{P}_{21} & P_{22} + R_2^{-1}S_2 & \mathscr{P}_{23} \end{pmatrix} +$$
$$+ \begin{pmatrix} 0 & 0 & \mathscr{P}_{31}^*S_p^T \\ 0 & 0 & P_{32}^*S_p^T \\ S_p\mathscr{P}_{31} & S_pP_{32} & S_p\mathscr{P}_{33} + Q_p + \mathscr{P}_{33}^*S_p^T \end{pmatrix} +$$
$$+ \begin{pmatrix} \mathscr{P}_{31} & P_{32} & \mathscr{P}_{33} \end{pmatrix}^* T_p^T T_p \begin{pmatrix} \mathscr{P}_{31} & P_{32} & \mathscr{P}_{33} \end{pmatrix} \prec 0$$

and thus (Schur)

$$\begin{pmatrix} T_1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{Q}_2^{-1} & P_{32}^* S_p^T \\ 0 & S_p P_{32} & Q_p \end{pmatrix} & \begin{pmatrix} 0 \\ P_{22}^* - \tilde{Q}_2^{-1}\tilde{S}_2 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ P_{32}^* T_p^T \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & P_{22} - \tilde{S}_2^* \tilde{Q}_2^{-1} & 0 \end{pmatrix} & -R_2^{-1} & 0 \\ \begin{pmatrix} 0 & T_p P_{32} & 0 \end{pmatrix} & 0 & -I \end{pmatrix} + T_2 \prec 0 \quad (35)$$

with

$$T_2 := \mathrm{He} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & S_p \\ I & 0 \\ 0 & T_p \end{pmatrix} \begin{pmatrix} \mathscr{P}_{21} & \mathscr{P}_{23} \\ \mathscr{P}_{31} & \mathscr{P}_{33} \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{pmatrix}.$$

Note that the second block rows and columns of T_1 and T_2 vanish; therefore we can eliminate all non-zero off-diagonal blocks in the second row/column of (35) without affecting T_1 and T_2 at all; this elimination amounts to performing a congruence transformation with a transfer matrix that is upper block-triangular and has identity blocks on its diagonal. For

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{Q}_2^{-1} & P_{32}^* S_p^T \\ 0 & S_p P_{32} & Q_p \end{pmatrix} & \begin{pmatrix} 0 \\ P_{22}^* - \tilde{Q}_2^{-1}\tilde{S}_2 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ P_{32}^* T_p^T \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & P_{22} - \tilde{S}_2^* \tilde{Q}_2^{-1} & 0 \end{pmatrix} & -R_2^{-1} & 0 \\ \begin{pmatrix} 0 & T_p P_{32} & 0 \end{pmatrix} & 0 & -I \end{pmatrix}$$

this operation is easily seen to result in

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{Q}_2^{-1} & 0 & 0 & 0 \\ 0 & 0 & Q_p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & S_p \\ I & 0 \\ 0 & T_p \end{pmatrix} \begin{pmatrix} P_{22} & -I \\ P_{32} & 0 \end{pmatrix} \tilde{\Pi}_2 \begin{pmatrix} P_{22} & -I \\ P_{32} & 0 \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & S_p \\ I & 0 \\ 0 & T_p \end{pmatrix}^T.$$

After canceling the second row and column, we have proved that (9) implies (11).

Now consider the case that R_2 is not positive definite on $\mathbb{C}^0$. Then (9) persists to hold if replacing the right-hand side of (9) by $-\delta I$ for some sufficiently small $\delta > 0$. We can then repeat all arguments after perturbing R_2 to $R_2 + \varepsilon I$ for any $\varepsilon > 0$. Let us denote the corresponding multiplier by Π_2^ε and apply the above described procedure. It is crucial to observe that $(\Pi_2^\varepsilon)^{-1}$ converges to Π_2^{-1} for $\varepsilon \to 0$ since Π_2 is non-singular. Denote the upper triangular elimination matrix appearing in the course of the arguments by T_ε; observe that T_ε has ones on its diagonal and that its upper-diagonal elements which depend on ε only involve blocks from $(\Pi_2^\varepsilon)^{-1}$; hence T_ε is guaranteed to converge, for $\varepsilon \to 0$, to a matrix T_0 which is non-singular on $\mathbb{C}^0$. The given proof then leads to (11) with Π_2^{-1} replaced by $(\Pi_2^\varepsilon)^{-1}$ and the right-hand side being $-\delta T_\varepsilon^* T_\varepsilon$. This allows to take the limit $\varepsilon \to 0$ and we obtain (11) again.

The converse statement is proved by first assuming $\tilde{Q}_2 \prec 0$ and then using a perturbation argument similar to the one just provided.

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Model-Based Control:
Bridging Rigorous Theory and Advanced Technology
van den Hof, P.M.J.; Scherer, C.; Heuberger, P.S.C.
(Eds.)
2009, XV, 239 p., Hardcover
ISBN: 978-1-4419-0894-0