Preface to the Second Edition

1. The first edition of this book was published in 1977. The text has been well received and is still used, although it has been out of print for some time.

In the intervening three decades, a lot of interesting things have happened to mathematical logic:

(i) Model theory has shown that insights acquired in the study of formal languages could be used fruitfully in solving old problems of conventional mathematics.

(ii) Mathematics has been and is moving with growing acceleration from the set-theoretic language of structures to the language and intuition of (higher) categories, leaving behind old concerns about infinities: a new view of foundations is now emerging.

(iii) Computer science, a no-nonsense child of the abstract computability theory, has been creatively dealing with old challenges and providing new ones, such as the P/NP problem.

Planning additional chapters for this second edition, I have decided to focus on model theory, the conspicuous absence of which in the first edition was noted in several reviews, and the theory of computation, including its categorical and quantum aspects.

The whole Part IV: Model Theory, is new. I am very grateful to Boris I. Zilber, who kindly agreed to write it. It may be read directly after Chapter II.

The contents of the first edition are basically reproduced here as Chapters I–VIII. Section IV.7, on the cardinality of the continuum, is completed by Section IV.7.3, discussing H. Woodin’s discovery.

The new Chapter IX: Constructive Universe and Computation, was written especially for this edition, and I tried to demonstrate in it some basics of categorical thinking in the context of mathematical logic. More detailed comments follow.

I am grateful to Ronald Brown and Noson Yanofsky, who read preliminary versions of new material and contributed much appreciated criticism and suggestions.

2. Model theory grew from the same roots as other branches of logic: proof theory, set theory, and recursion theory. From the start, it focused on language and formalism. But the attention to the foundations of mathematics in model...
theory crystallized in an attempt to understand, classify, and study models of theories of real-life mathematics.

One of the first achievements of model theory was a sequence of local theorems of algebra proved by A. Maltsev in the late 1930s. They were based on the compactness theorem established by him for this purpose. The compactness theorem in many of its disguises remained a key model-theoretic instrument until the end of the 1950s. We follow these developments in the first two sections of Chapter X, which culminate with a general discussion of nonstandard analysis discovered by A. Robinson. The third section introduces basic tools and concepts of the model theory of the 1960s: types, saturated models, and modern techniques based on these.

We try to illustrate every new model-theoretic result with an application in “real” mathematics. In Section 4 we discuss an algebro-geometric theorem first proved by J. Ax model-theoretically and re-proved by G. Shimura and A. Borel. Moreover, we explain an application of the Tarski–Seidenberg quantifier elimination for $\mathbb{R}$ due to L. Hörmander. A real gem of model-theoretic techniques of the 1980s is the calculation by J. Denef of the Poincaré series counting $p$-adic points on a variety based on A. Macintyre’s quantifier elimination theorem for $\mathbb{Q}_p$.

In the last two sections we present a survey of classification theory, which started with M. Morley’s analysis of theories categorical in uncountable powers in 1964, and was later expanded by S. Shelah and others to a scale that no one could have envisaged.

The striking feature of these developments is the depth of the very abstract “pure” model theory underlying the classification, in combination with the diversity of mathematical theories affected by it, from algebraic and Diophantine geometry to real analysis and transcendental number theory.

3. The formal languages with which we work in the first, and in most of the second, edition of this book are exclusively linear in the following sense. Having chosen an alphabet consisting of letters, we proceed to define classes of well-formed expressions in this alphabet that are some finite sequences of letters. At the next level, there appear well-formed sequences of words, such as deductions and descriptions. Church’s $\lambda$-calculus furnishes a good example of strictures imposed by linearity.

Nonlinear languages have existed for centuries. Geometers and composers could not perform without using the languages of drawings, resp. musical scores; when alchemy became chemistry, it also evolved its own two-dimensional language. For a logician, the basic problem about nonlinear languages is the difficulty of their formalization.

This problem is addressed nowadays by relegating nonlinear languages of contemporary mathematics to the realm of more conventional mathematical objects, and then formally describing such languages as one would describe any other structure, that is, linearly.
Such a strategy probably cannot be avoided. But one must be keenly aware that some basic mathematical structures are “linguistic” at their core. Recognition or otherwise of this fact influences the problems that are chosen, the questions that are asked, and the answers that are appreciated.

It would be difficult to dispute nowadays that category theory as a language is replacing set theory in its traditional role as the language of mathematics. Basic expressions of this language, commutative diagrams, are one-dimensional, but nonlinear: they are certain decorated graphs, whose topology is that of 1-dimensional triangulated spaces.

When one iterates the philosophy of category theory, replacing sets of morphisms by objects of a category of the next level, commutative diagrams become two-dimensional simplicial sets (or cell complexes), and so on. Arguably, in this way the whole of homotopy topology now develops into the language of contemporary mathematics, transcending its former role as an important and active, but reasonably narrow research domain. Much remains to be recognized and said about this emerging trend in foundations of mathematics.

The first part of Chapter IX in this edition is a very brief and tentative introduction to this way of thinking, oriented primarily to some reshuffling of classical computability theory, as was explained in the Part II of the first edition.

4. The second part of the new Chapter IX is dedicated to some theoretical problems of classical and quantum computing. It introduces the P/NP problem, classical and quantum Boolean circuits, and presents several celebrated results of this early stage of theoretical quantum computing, such as Shor’s factoring and Grover’s search algorithms.

The main reason to include these topics is my conviction that at least some theoretical achievements of modern computer science must constitute an organic part of contemporary mathematical logic.

Already in the first edition, the manuscript for which was completed in September 1974, “quantum logic” was discussed at some length; cf. Section II.12.

A Russian version of the Part II of first edition was published as a separate book, Computable and Uncomputable, by “Soviet Radio” in 1980. For this Russian publication, I had written a new introduction, in which, in particular, I suggested that quantum computers could be potentially much more powerful than classical ones, if one could use the exponential growth of a quantum phase space as a function of the number of degrees of freedom of the classical system.

When a mathematical implementation of this idea, massive quantum parallelism, made possible by quantum entanglement, gradually matured, I gave a talk at a Bourbaki seminar in June 1999, explaining the basic ideas and results.

Chapter IX is a revised and expanded version of this talk.

5. Finally, a few words about the last digression in Chapter II, “Truth as Value and Duty: Lessons of Mathematics.”
“Mathematical truth” was the central concept of the first part of the book, “Provability.” Writing this part, I felt that if I did not compensate somehow the aridity and sheer technicality of the analysis of formal languages, I would not be able to convince people—the readers that I imagined, working mathematicians like me—that it is worth studying at all. The literary device I used to struggle with this feeling of helplessness was this: from time to time I allowed myself free associations, and wrote the outcome in a series of six digressions, with which the first two Chapters were interspersed.

By the end of the second chapter, I realized that I was finally on the fertile soil of “real mathematics,” and the need for digressions faded away.

Nevertheless, the whole of Part I was left without proper summary.

Its role is now played by the “Last Digression,” published here for the first time. It is a slightly revised text of the talk prepared for a Balzan Foundation International Symposium on “Truth in the Humanities, Science and Religion” (Lugano, 2008), where I was the only mathematician speaker among philosophers, historians, lawyers, theologians, and physicists. I was confronted with the task to explain to a distinguished “general audience” what is so different about mathematical truth, and what light the usage of this word in mathematics can throw on its meaning in totally foreign environments.

The main challenge was this: avoid sounding ponderous.

Yu. Manin, Bonn

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