# Chapter 2
# Isometrics in Non-Archimedean Strictly Convex and Strictly 2-Convex 2-Normed Spaces

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*Dedicated to the memory of Professor George Isac*

**Abstract** In this paper, we present a Mazur–Ulam type theorem in non-Archimedean strictly convex 2-normed spaces and present some properties of mappings on non-Archimedean strictly 2-convex 2-normed spaces.

## 2.1 Introduction and Preliminaries

A non-Archimedean field is a field $\mathscr{K}$ equipped with a function (valuation) $|\cdot|$ : $\mathscr{K} \to [0,\infty)$ such that for all $r,s \in \mathscr{K}$:
(i) $|r| = 0$ if and only if $r = 0$,
(ii) $|rs| = |r|\,|s|$,
(iii) $|r+s| \le \max\{|r|,|s|\}$.
Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbf{N}$. An example of a non-Archimedean valuation is the mapping

$$|x| = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This valuation is called trivial.
In 1897, Hensel [4] discovered the $p$-adic numbers. Fix a prime number $p$. For any nonzero rational number $x$, there exists a unique integer $n_x \in \mathbf{Z}$ such that $x = \frac{a}{b}p^{n_x}$,

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where neither of the integers $a$ and $b$ is divisible by $p$. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbf{Q}$. The completion of $\mathbf{Q}$ with respect to the metric $d(x,y) = |x-y|_p$ is denoted by $\mathbf{Q}_p$ which is called the $p$-adic number field; cf. [14].

Let $\mathscr{X}$ be a vector space over a field $\mathscr{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : \mathscr{X} \to [0,\infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions for all $x,y \in \mathscr{X}$ and $r \in \mathscr{K}$:

(i) $\|x\| = 0$ if and only if $x = 0$,
(ii) $\|rx\| = |r|\|x\|$,
(iii) $\|x+y\| \leq \max\{\|x\|, \|y\|\}$.

Then $(\mathscr{X}, \|\cdot\|)$ is called a non-Archimedean normed space. A non-Archimedean normed space is called strictly convex if $\|x+y\| = \max\{\|x\|,\|y\|\}$ and $\|x\| = \|y\|$ imply $x = y$. Theory of non-Archimedean normed spaces is not trivial, for instance there may not be any unit vector; see [9] and references therein.

**Definition 2.1.** Let $\mathscr{X}$ be a vector space of dimension greater than 1 over a field $\mathscr{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot,\cdot\| : \mathscr{X} \times \mathscr{X} \to \mathbf{R}$ is said to be a non-Archimedean 2-norm if it satisfies the following conditions:

(i) $\|x,y\| = 0$ if and only if $x,y$ are linearly dependent,
(ii) $\|x,y\| = \|y,x\|$,
(iii) $\|rx,y\| = |r|\|x,y\| \quad (r \in \mathscr{K}, x,y \in \mathscr{X})$,
(iv) the strong triangle inequality

$$\|x,y+z\| \leq \max\{\|x,y\|, \|x,z\|\} \quad (x,y,z \in X).$$

Then $(\mathscr{X}, \|\cdot,\cdot\|)$ is called a non-Archimedean 2-normed space.

Let $\|x,y\| \neq \|x,z\|$ in inequality (iv). Without loss of generality we may assume that $\|x,y\| > \|x,z\|$, then we have

$$\|x,y+z\| \leq \|x,y\| \tag{2.1}$$

and

$$\|x,y\| \leq \max\{\|x,y+z\|, \|x,z\|\} = \|x,y+z\|. \tag{2.2}$$

Inequalities (2.1) and (2.2) imply that

$$\|x,y+z\| = \max\{\|x,y\|, \|x,z\|\},$$

which is interesting in its own right.

*Example 2.2.* Let $(\mathscr{X}, \|\cdot\|)$ be a non-Archimedean normed space over a valued field $\mathscr{K}$. Then $\|\cdot,\cdot\|$ on $\mathscr{X}^2$ defined by

$$\|x,y\| = \begin{cases} \|x\|\,\|y\| & x \text{ and } y \text{ are linearly independent} \\ 0 & \text{otherwise} \end{cases}$$

is a 2-norm and $(\mathscr{X}, \|\cdot,\cdot\|)$ is a non-Archimedean 2-normed space.

**Definition 2.3.** Let $\mathscr{X}$ and $\mathscr{Y}$ be non-Archimedean 2-normed spaces and $f : \mathscr{X} \to \mathscr{Y}$ be a mapping. Then $f$ is called a 2-isometry if

$$\|x-z, y-z\| = \|f(x)-f(z), f(y)-f(z)\|$$

for all $x, y$ and $z$ in $\mathscr{X}$; cf. [2].

For non-zero vectors $x, y$ in $\mathscr{X}$, let $\mathscr{V}(x,y)$ denote the subspace of $\mathscr{X}$ generated by $x$ and $y$.

**Definition 2.4.** Suppose $\mathscr{X}$ is a non-Archimedean 2-normed space over a valued field $\mathscr{K}$ with $|2| = 1$, then $\mathscr{X}$ is called strictly convex if $\|x+y, z\| = \max\{\|x,z\|, \|y,z\|\}$, $\|x,z\| = \|y,z\| \neq 0$ and $z \notin \mathscr{V}(x,y)$ imply that $x = y$.

The theory of isometric mappings had its beginning in the classical paper [7] by S. Mazur and S. Ulam, who proved that every isometry of a normed real vector space onto another normed real vector space is a linear mapping up to translation. A number of mathematicians have had deal with Mazur–Ulam theorem; see [5, 10, 11, 12, 13, 15, 16] and references therein. Mazur–Ulam theorem is not valid in the contents of non-Archimedean normed spaces, in general. As a counterexample take $\mathbf{R}$ with the trivial non-Archimedean valuation and define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^3$. Then $f$ is clearly a surjective isometry and $f(0) = 0$, but $f$ is not linear; cf. [8]. H.Y. Chu [2] studied the notation of 2-isometry and proved the Mazur–Ulam problem in 2-normed spaces.

*Example 2.5.* Suppose $\mathbf{R}^2$ is the vector space over field $\mathbf{R}$ with non-Archimedean trivial valuation $|.|$. Then the function $\|.,.\| : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ is defined by

$$\|(x_1,x_2),(y_1,y_2)\| = |x_1,x_2||y_1,y_2|,$$

where

$$|x,y| = \begin{cases} 1 & x \neq 0 \text{ and } y \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is defined as a non-Archimedean 2-norm. Define $f : \mathbf{R}^2 \to \mathbf{R}^2$ by $f(x,y) = (x^3, y^3)$. Then $f$ is clearly a 2-isometry and $f(0,0) = 0$, but $f$ is not additive.

In this paper, by using the terminology and some ideas of [1], [2], [3], [6], and [8], we establish a Mazur–Ulam type theorem in the framework of non-Archimedean 2-normed spaces.

## 2.2 Non-Archimedean Strictly Convex 2-Normed Spaces

We begin this section with the following useful lemma.

**Lemma 2.6.** *Let* $(\mathscr{X}, \|\cdot,\cdot\|)$ *be a non-Archimedean* 2*-normed space over a valued field* $\mathscr{K}$. *Then* $\|x,y\| = \|x, y+rx\|$ *for all* $x, y \in \mathscr{X}$ *and all* $r \in \mathscr{K}$.

*Proof.* Let $x, y \in \mathscr{X}$, then

$$\|x, y + rx\| \leq \max\{\|x, y\|, \|x, rx\|\} = \|x, y\| \tag{2.3}$$

and

$$\begin{aligned}\|x, y\| = \|x, y + rx - rx\| &\leq \max\{\|x, y + rx\|, \|x, -rx\|\} \\ &= \|x, y + rx\|.\end{aligned} \tag{2.4}$$

It follows from inequalities (2.3) and (2.4) that $\|x, y + rx\| = \|x, y\|$. □

**Definition 2.7.** Let $\mathscr{X}$ be a non-Archimedean linear space over a valued field $\mathscr{K}$ and $x, y, z$ be mutually disjoint of $\mathscr{X}$. Then $x, y, z$ are said to be collinear if $x - y = r(x - z)$ for some $r \in \mathscr{K}$.

**Lemma 2.8.** *Let $(\mathscr{X}, \|.,.\|)$ be a non-Archimedean 2-normed space over a valued field $\mathscr{K}$ which is strictly convex and let $x, y \in \mathscr{X}$. Then $\frac{x+y}{2}$ is the unique member $t$ of $\mathscr{X}$, collinear with $x, y$ satisfying*

$$\|x - z, x - t\| = \|y - t, y - z\| = \|x - z, y - z\|$$

*for all $z \in \mathscr{X}$ with $\|x - z, y - z\| \neq 0$.*

*Proof.* Set $t = \frac{x+y}{2}$. Then $t, x, y$ are collinear. By Lemma 2.6 we have

$$\begin{aligned}\|x - z, x - t\| &= \|x - z, x - \frac{x+y}{2}\| = \|x - z, \frac{x-y}{2}\| \\ &= \frac{1}{|2|}\|x - z, x - y\| = \|x - z, z - y\| = \|x - z, y - z\|;\end{aligned}$$

$$\begin{aligned}\|y - z, y - t\| &= \|y - z, y - \frac{x+y}{2}\| = \|y - z, \frac{y-x}{2}\| \\ &= \frac{1}{|2|}\|y - z, y - x\| = \|y - z, z - x\| = \|x - z, y - z\|.\end{aligned}$$

Assume that another $s \in \mathscr{X}$, collinear with $x, y$ satisfies,

$$\|x - z, x - s\| = \|y - s, y - z\| = \|x - z, y - z\|.$$

Then

$$\begin{aligned}\|x - z, x - \frac{t+s}{2}\| &\leq \max\left\{\|x - z, \frac{x-t}{2}\|, \|x - z, \frac{x-s}{2}\|\right\} \\ &= \|x - z, y - z\|.\end{aligned} \tag{2.5}$$

Similarly

$$\|y - z, y - \frac{t+s}{2}\| \leq \|x - z, y - z\|. \tag{2.6}$$

we have

$$\begin{aligned}\|x-z,y-z\| &= \|x-z,y-x\| \\ &\leq \max\left\{\|x-z,x-\frac{t+s}{2}\|,\|x-z,y-\frac{t+s}{2}\|\right\}.\end{aligned}\tag{2.7}$$

It follows from (ii) and (iv) that

$$\begin{aligned}\|x-z,y-\frac{t+s}{2}\| &= \|y-z+x-y,y-\frac{t+s}{2}\| \\ &\leq \max\left\{\|y-z,y-\frac{t+s}{2}\|,\|x-y,y-\frac{t+s}{2}\|\right\}.\end{aligned}\tag{2.8}$$

Since $t,x,y$ and $s,x,y$ are collinear we have

$$\|x-y,y-\frac{t+s}{2}\| \leq \max\left\{\|x-y,\frac{y-t}{2}\|,\|x-y,\frac{y-s}{2}\|\right\} = 0.\tag{2.9}$$

It follows from (2.8) and (2.9) that

$$\|x-z,y-\frac{t+s}{2}\| \leq \|y-z,y-\frac{t+s}{2}\|.\tag{2.10}$$

If both inequalities (2.5) and (2.6) were strict, then by inequalities (2.7) and (2.10) we would have

$$\|x-z,y-z\| \leq \max\left\{\|x-z,x-\frac{t+s}{2}\|,\|y-\frac{t+s}{2},y-z\|\right\} < \|x-z,y-z\|,$$

a contradiction. So at least one of the equalities holds in (2.5) and (2.6). Without lose of generality assume that equality holds in (2.5). Then

$$\|x-z,\frac{x-t}{2}+\frac{x-s}{2}\| = \max\left\{\|x-z,\frac{x-t}{2}\|,\|x-z,\frac{x-s}{2}\|\right\}.$$

By the strict convexity we obtain $\frac{x-t}{2}=\frac{x-s}{2}$, that is $t=s$. □

**Theorem 2.9.** *Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are non-Archimedean 2-normed spaces over a valued field $\mathscr{K}$ such that $\mathscr{Y}$ is strictly convex. Assume that $f(x), f(y)$, and $f(z)$ are collinear when $x, y$, and $z$ are collinear. If $f:\mathscr{X}\to\mathscr{Y}$ is a 2-isometry, then $f-f(0)$ is an additive mapping.*

*Proof.* Let $g(x)=f(x)-f(0)$. Then $g$ is an 2-isometry and $g(0)=0$. Let $x,y\in\mathscr{X}$ with $x\neq y$. Since $\dim\mathscr{X}>1$, there exists an element $z\in\mathscr{X}$ such that $\|x-z,y-z\|\neq 0$. Since $g$ is a 2-isometry, we have

$$\|g(x)-g(z),g(x)-g\left(\frac{x+y}{2}\right)\| = \|x-z,x-\frac{x+y}{2}\|$$

$$\begin{aligned}
&= \|x-z, \frac{x-y}{2}\| \\
&= \|x-z, x-y\| = \|x-z, y-z\| \\
&= \|g(x)-g(z), g(y)-g(z)\|
\end{aligned}$$

and similarly we obtain

$$\begin{aligned}
\|g(y)-g\left(\frac{x+y}{2}\right), g(y)-g(z)\| &= \|y-\frac{x+y}{2}, y-z\| \\
&= \|y-x, y-z\| = \|x-z, y-z\| \\
&= \|g(x)-g(z), g(y)-g(z)\|.
\end{aligned}$$

Since $\frac{x+y}{2}$, $x$ and $y$ collinear, $g\left(\frac{x+y}{2}\right)$, $g(x)$ and $g(y)$ are also collinear. It follows from Lemma 2.8 that

$$g\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2},$$

for all $x, y \in \mathscr{X}$. Hence $g = f - f(0)$ is additive since $g(0) = 0$. □

*Example 2.10.* Consider $\mathbf{Q}_2 \times \mathbf{Q}_2$ with norm $|(\alpha,\beta)| = \max\{|\alpha|_2, |\beta|_2\}$. Then $\mathbf{Q}_2 \times \mathbf{Q}_2$ is a non-Archimedean normed space such that is not strictly convex. Define 2-norm $|\cdot,\cdot|$ on $\mathbf{Q}_2 \times \mathbf{Q}_2$ by

$$|u,v| = \begin{cases} |u||v| & u \text{ and } v \text{ are linearly independent} \\ 0 & \text{otherwise} \end{cases}$$

Now define the mapping $f : \mathbf{Q}_2 \times \mathbf{Q}_2 \to \mathbf{Q}_2 \times \mathbf{Q}_2$ by

$$f(\alpha,\beta) = \begin{cases} (\alpha^2,\beta^2) & \alpha = \frac{a}{b}2^0 \text{ and } \beta = \frac{c}{d}2^0 \\ (\alpha,\beta) & \text{otherwise.} \end{cases}$$

Then $f$ is a 2-isometry and $f(0,0) = 0$ but $f$ is not additive. Therefore the assumption that $\mathscr{Y}$ is strictly convex cannot be omitted in Theorem 2.9.

## 2.3 Non-Archimedean Strictly 2-Convex 2-Normed Spaces

**Definition 2.11.** Suppose $\mathscr{X}$ is a non-Archimedean 2-normed space over a valued field $\mathscr{K}$ with $|3| = 1$, then $\mathscr{X}$ is called strictly 2-convex if for all $x, y, z \in \mathscr{X}$ which $\|x+z, y+z\| = \max\{\|x,y\|, \|x,z\|, \|y,z\|\}$ and $\|x,y\| = \|x,z\| = \|y,z\| \neq 0$, we have $z = x+y$.

**Definition 2.12.** Suppose $\mathscr{X}$ is a non-Archimedean 2-normed space over a valued field $\mathscr{K}$ and $a, b, c$ are three non-collinear points in $\mathscr{X}$, then

$$\begin{aligned}
T(a,b,c) = \{x \in \mathscr{X} : \|a-b, b-c\| \\
= \max\{\|a-x, b-x\|, \|a-c, x-c\|, \|x-c, b-c\|\}\}
\end{aligned}$$

is called the triangle with vertices $a, b$, and $c$.

**Definition 2.13.** A point $p$ of a 2-normed space $(\mathscr{X}, \|.,.\|)$ is called 2-normed midpoint of three non-collinear points $a, b$, and $c$ of $\mathscr{X}$ ($\|a-c, b-c\| \neq 0$) if

$$\|a-p, b-p\| = \|a-c, p-c\| = \|p-c, b-c\| = \|a-b, b-c\|.$$

If $p$ is a 2-normed midpoint of $a, b$, and $c$, then $p$ is called a center of $T(a,b,c)$

**Lemma 2.14.** *Let $\mathscr{X}$ be a non-Archimedean 2-normed space over a valued field $\mathscr{K}$ with $|3| = 1$ and $x, y, z \in \mathscr{X}$, are non-collinear. Then $u = \frac{x+y+z}{3}$ is the element of $\mathscr{X}$, collinear with $x, y, z$ satisfying*

$$\|x-z, y-z\| = \|x-u, y-u\| = \|x-z, u-z\| = \|u-z, y-z\|.$$

*Proof.*

$$\begin{aligned}\|x-u, y-u\| &= \|x - \frac{x+y+z}{3}, y - \frac{x+y+z}{3}\| = \|2x-y-z, 2y-x-z\| \\ &= \|2x-y-z, 3y-3x\| = \|2x-y-z, y-x\| \\ &= \|y-x, x-z\| = \|x-z, y-z\|\end{aligned}$$

and

$$\|x-z, u-z\| = \|x-z, \frac{x+y+z}{3} - z\| = \|x-z, x+y-2z\| = \|x-z, y-z\|$$

$$\|u-z, y-z\| = \|\frac{x+y+z}{3} - z, y-z\| = \|x+y-2z, y-z\| = \|x-z, y-z\|.$$

□

**Theorem 2.15.** *Let $\mathscr{X}$ be a non-Archimedean 2-normed space over a valued field $\mathscr{K}$ with $|3| = 1$. Then the following statements are equivalent for all $a, b, c \in \mathscr{X}$,*

*(i) $(\mathscr{X}, \|.,.\|)$ is strictly 2-convex.*

*(ii) $T(a,b,c)$ has a unique center.*

*Proof.* Let $(\mathscr{X}, \|.,.\|)$ be strictly 2-convex. By Lemma 2.14 it is a clear that $\frac{a+b+c}{3}$ is a center of $T(a,b,c)$. If $x$ is a center of $T(a,b,c)$, then we have

$$\|a-x, b-x\| = \|a-c, x-c\| = \|x-c, b-c\| = \|a-c, b-c\|$$

or

$$\|a-x, b-x\| = \|a-x, x-c\| = \|x-c, b-x\| = \|a-c, b-c\|.$$

Put $X = a-x, Y = b-x$ and $Z = x-c$. Then

$$\|X+Z, Y+Z\| = \|X, Y\| = \|X, Z\| = \|Y, Z\| \neq 0.$$

$\mathscr{X}$ is a strictly 2-convex, then $Z = X+Y$, namely, $x = \frac{a+b+c}{3}$.

Suppose that $T(a,b,c)$ has a unique center, and also $(\mathscr{X}, \|.,.\|)$ is not strictly 2-convex, then there exist $x, y$, and $z$ in $\mathscr{X}$ such that

$$\|x+y,y+z\| = \max\{\|x,y\|,\|x,z\|,\|y,z\|\}$$

and $\|x,y\| = \|x,z\| = \|y,z\| \neq 0$ such that $x+y \neq z$. Thus $x \neq z-y$ and $y \neq z-x$. Put $a = z-y, b = z-x$ and $c = z$ in $T(a,b,c)$. Then $\|a-c,b-c\| \neq 0$. We will show that $t = z-(x+y) \in T(a,b,c)$.

$$\|a-c,b-c\| = \|z-y-z,z-x-z\| = \|x,y\|$$

$$\|a-t,b-t\| = \|z-y-z+x+y,z-x-z+x+y\| = \|x,y\|$$

$$\|a-c,t-c\| = \|z-y-z,z-x-y-z\| = \|y,x+y\| = \|x,y\|$$

and

$$\|t-c,b-c\| = \|z-x-y-z,z-x-z\| = \|x+y,y\| = \|x,y\|.$$

On the other hand, $t$ is a center of $T(a,b,c)$. Hence $t = \frac{a+b+c}{3}$. Therefore $2(x+y) = 0$ or $\|x,y\| = 0$, which is a contradiction. □

**Theorem 2.16.** *Let $\mathscr{X}$ and $\mathscr{Y}$ be non-Archimedean 2-normed spaces over a valued field $\mathscr{K}$ and $\mathscr{Y}$ is strictly 2-convex. Assume that $f(x), f(y)$, and $f(z)$ are collinear when $x,y$, and $z$ are collinear .If $f : \mathscr{X} \to \mathscr{Y}$ is an 2-isometry, then for all $x,y$, and $z$ in $\mathscr{X}$ we have*

$$f\left(\frac{x+y+z}{3}\right) = \frac{f(x)+f(y)+f(z)}{3}.$$

*Proof.* Let $g(x) = f(x) - f(0)$. Then $g$ is an 2-isometry and $g(0) = 0$. Let $x,y \in \mathscr{X}$ with $x \neq y$. Since $\dim \mathscr{X} > 1$, there exists an element $z \in \mathscr{X}$ such that $\|x-z,y-z\| \neq 0$. Since $g$ is a 2-isometry, we have

$$\begin{aligned} &\left\|g(x) - g\left(\frac{x+y+z}{3}\right) \; , \; g(y) - g\left(\frac{x+y+z}{3}\right)\right\| \\ &\qquad = \left\|x - \frac{x+y+z}{3}, y - \frac{x+y+z}{3}\right\| \quad \text{(by Lemma 2.14)} \\ &\qquad = \|x-z,y-z\| = \|g(x)-g(z),g(y)-g(z)\| \end{aligned}$$

and

$$\begin{aligned} \left\|g(x)-g(z), g\left(\frac{x+y+z}{3}\right) - g(z)\right\| &= \left\|x-z, \frac{x+y+z}{3} - z\right\| \quad \text{(by Lemma 2.14)} \\ &= \|x-z,y-z\| = \|g(x)-g(z),g(y)-g(z)\| \end{aligned}$$

$$\begin{aligned} \left\|g\left(\frac{x+y+z}{3}\right) - g(z), g(y)-g(z)\right\| &= \left\|\frac{x+y+z}{3} - z, y-z\right\| \quad \text{(by Lemma 2.14)} \\ &= \|x-z,y-z\| = \|g(x)-g(z),g(y)-g(z)\|. \end{aligned}$$

Therefore $g(\frac{x+y+z}{3})$ is a center of $T(g(x),g(y),g(z))$. Since $\mathscr{Y}$ is strictly 2-convex by Theorem 2.15 we have

$$g\left(\frac{x+y+z}{3}\right)=\frac{g(x)+g(y)+g(z)}{3}.$$

□

*Example 2.17.* Consider $\mathbf{Q}_3\times\mathbf{Q}_3$ with norm $|(\alpha,\beta)|=\max\{|\alpha|_3,|\beta|_3\}$. Then $\mathbf{Q}_3\times\mathbf{Q}_3$ is a non-Archimedean normed space such that it is not strictly 2-convex. Define the 2-norm $|\cdot,\cdot|$ on $\mathbf{Q}_3\times\mathbf{Q}_3$ by

$$|u,v|=\begin{cases}|u||v| & u \text{ and } v \text{ are linearly independent}\\ 0 & \text{otherwise.}\end{cases}$$

Now define the mapping $f:\mathbf{Q}_3\times\mathbf{Q}_3\to\mathbf{Q}_3\times\mathbf{Q}_3$ by

$$f(\alpha,\beta)=\begin{cases}(2\alpha,2\beta) & \alpha=\frac{a}{b}3^0\text{and}\beta=\frac{c}{d}3^0\\ (\alpha,\beta) & \text{otherwise.}\end{cases}$$

Then $f$ is a 2-isometry and $f(0,0)=0$, but if we put $x=(1,1), y=(3,3)$, and, $z=(0,0)$, then

$$f\left(\frac{x+y+z}{3}\right)\neq\frac{f(x)+f(y)+f(z)}{3}.$$

Therefore the assumption that $\mathscr{Y}$ is strictly 2- convex cannot be omitted in Theorem 2.16.

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