# Preface

*Numbers imitate space, which is of such a different nature*
—Blaise Pascal

It is fair to date the study of the foundation of mathematics back to the ancient Greeks. The urge to understand and systematize the mathematics of the time led Euclid to postulate axioms in an early attempt to put geometry on a firm footing. With roots in the *Elements*, the distinctive methodology of mathematics has become *proof*. Inevitably two questions arise: *What are proofs?* and *What assumptions are proofs based on?*

The first question, traditionally an internal question of the field of *logic*, was also wrestled with in antiquity. Aristotle gave his famous syllogistic systems, and the Stoics had a nascent propositional logic. This study continued with fits and starts, through Boethius, the Arabs and the medieval logicians in Paris and London. The early germs of logic emerged in the context of philosophy and theology.

The development of analytic geometry, as exemplified by Descartes, illustrated one of the difficulties inherent in founding mathematics. It is classically phrased as the question of *how one reconciles the arithmetic with the geometric.* Are numbers one type of thing and geometric objects another? What are the relationships between these two types of objects? How can they interact? Discovery of new types of mathematical objects, such as imaginary numbers and, much later, formal objects such as free groups and formal power series make the problem of finding a common playing field for all of mathematics importunate.

Several pressures made foundational issues urgent in the 19th century. The development of alternative geometries created doubts about the view that mathematical truth is part of an absolute all-encompassing logic and caused it to evolve towards one in which mathematical propositions follow logically from assumptions that may vary with context.

Mathematical advances involving the understanding of the relationship between the completeness of the real line and the existence of solutions to equations led inevitably to anxieties about the role of infinity in mathematics.

These too had antecedents in ancient history. The Greeks were well aware of the scientific importance of the problems of the infinite which were put forth, not only in the paradoxes of Zeno, but in the work of Eudoxus, Archimedes and others. Venerable concerns about resolving infinitely divisible lines into individual points and what is now called "Archimedes' Axiom" were recapitulated in 19th century mathematics.

In response, various "constructions" of the real numbers were given, such as those using Cauchy sequences and Dedekind cuts, as a way of understanding the relationship between discrete entities, such as the integers or the rationals and the continuum. Even simple operations, such as addition of arbitrary real numbers began to be understood as infinitary operations, defined by some kind of limiting process. The notion of *function* was liberalized beyond those that can be written in closed form. Sequences and series became routine tools for solving equations.

The situation was made acute when Cantor, working on natural problems involving trigonometric series, discovered the existence of different magnitudes of infinity. The widespread use of inherently infinitary techniques, such as the use of the Baire Category Theorem to prove the existence of important objects, became deeply embedded in standard mathematics, making it impossible to simply reject infinity as part of mathematics.

In parallel 19th century developments, through the work of Boole and others, logic became once again a mathematical study. Boole's algebraization of logic made it grist for mathematical analysis and led to a clear understanding of propositional logic. Dually, logicians such as Frege viewed mathematics as a special case of logic. Indeed a very loose interpretation of the work of Frege is that it is an attempt to base mathematics on a broad notion of logic that subsumed all mathematical objects.

With Russell's paradox and the failure of Frege's program, a distinction began to be made between *logic* and *mathematics.* Logic began to be viewed as a formal epistemological mechanism for exploring mathematical truth, barren of mathematical content and in need of assumptions or axioms to make it a useful tool.

Progress in the 19th and 20th centuries led to the understanding of logics involving quantifiers as opposed to propositional logic and to distinctions such as those between first and second-order logic. With the semantics developed by Tarski and the compactness and completeness theorems of Gödel, first-order logic has become widely accepted as a well-understood, unproblematic answer to the question *What is a proof?*

The desirable properties of first-order logic include:

- Proofs and propositions are easily and uncontroversially recognizable.
- There is an appealing semantics that gives a clear understanding of the relationship between a mathematical structure and the formal propositions that hold in it.
- It gives a satisfactory model of what mathematicians actually do: the "rigorous" proofs given by humans seem to correspond exactly to the

> "formal" proofs of first-order logic. Indeed formal proofs seem to provide a normative ideal towards which controversial mathematical claims are driven as part of their verification process.

While there are pockets of resistance to first-order logic, such as constructivism and intuitionism on the one hand and other alternatives such as second-order logic on the other, these seem to have been swept aside, if simply for no other reason than their comparative lack of mathematical fruitfulness.

To summarize, a well-accepted conventional view of foundations of mathematics has evolved that can be caricatured as follows:

$$\text{Mathematical Investigation} = \text{First-Order Logic} + \text{Assumptions}$$

This formulation has the advantage that it segregates the difficulties with the foundations of mathematics into discussions about the underlying assumptions rather than into issues about the nature of reasoning.

So what are the appropriate assumptions for mathematics? It would be very desirable to find assumptions that:

1. involve a simple primitive notion that is easy to understand and can be used to "build" or develop all standard mathematical objects,

2. are *evident*,

3. are *complete* in that they settle all mathematical questions,

4. can be easily recognized as part of a recursive schema.

Unfortunately Gödel's incompleteness theorems make item 3 impossible. *Any* recursive consistent collection $\mathcal{A}$ of mathematical assumptions that are strong enough to encompass the simple arithmetic of the natural numbers will be *incomplete*; in other words there will be mathematical propositions $P$ that cannot be settled on the basis of $\mathcal{A}$. This inherent limitation is what has made the foundations of mathematics a lively and controversial subject.

Item 2 is also difficult to satisfy. To the extent that we understand mathematics, it is a difficult and complex business. The Euclidean example of a collection of axioms that are easily stated and whose content is simple to appreciate is likely to be misleading. Instead of simple, distinctly conceived and obvious axioms, the project seems more analogous to specifying a complicated operating system in machine language. The underlying primitive notions used to develop standard mathematical objects are combined in very complicated ways. The axioms describe the operations necessary for doing this and the test of the axioms becomes how well they code higher level objects as manipulated in ordinary mathematical language so that the results agree with educated mathematicians' sense of correctness.

Having been forced to give up 3 and perhaps 2, one is apparently left with the alternatives:

2′. Find assumptions that are in accord with the intuitions of mathematicians well versed in the appropriate subject matter.

3′. Find assumptions that *describe* mathematics to as large an extent as is possible.

With regard to item 1, there are several choices that could work for the primitive notion for developing mathematics, such as *categories* or *functions*. With no *a priori* reason for choosing one over another, the standard choice of *sets* (or set membership) as the basic notion is largely pragmatic. Taking sets as the primitive, one can easily do the traditional constructions that "build" or "code" the usual mathematical entities: the empty set, the natural numbers, the integers, the rationals, the reals, $\mathbb{C}$, $\mathbb{R}^n$, manifolds, function spaces—all of the common objects of mathematical study.

In the first half of the 20th century a standard set of assumptions evolved, the axiom system called the Zermelo-Fraenkel axioms with the Axiom of Choice (ZFC). It is pragmatic in spirit; it posits sufficient mathematical strength to allow the development of standard mathematics, while explicitly rejecting the type of objects held responsible for the various paradoxes, such as Russell's.

While ZFC is adequate for most of mathematics, there are many mathematical questions that it does not settle. Most prominent among them is the first problem on Hilbert's celebrated list of problems given at the 1900 International Congress of Mathematicians, the *Continuum Hypothesis*.

In the jargon of logic, a question that cannot be settled in a theory $T$ is said to be *independent* of $T$. Thus, to give a mundane example, the property of being Abelian is independent of the axioms for group theory. It is routine for normal axiomatizations that serve to synopsize an abstract concept internal to mathematics to have independent statements, but more troubling for axiom systems intended to give a definitive description of mathematics itself. However, independence phenomena are now known to arise from many directions; in essentially every area of mathematics with significant infinitary content there are natural examples of statements independent of ZFC.

This conundrum is at the center of most of the chapters in this Handbook. Its investigation has left the province of abstract philosophy or logic and has become a primarily mathematical project. The intent of the Handbook is to provide graduate students and researchers access to much of the recent progress on this project. The chapters range from expositions of relatively well-known material in its mature form to the first complete published proofs of important results. The introduction to the Handbook gives a thorough historical background to set theory and summaries of each chapter, so the comments here will be brief and general.

The chapters can be very roughly sorted into four types. The first type consists of chapters with theorems demonstrating the independence of mathematical statements. Understanding and proving theorems of this type require a thorough understanding of the mathematics surrounding the source of the problem in question, reducing the ambient mathematical constructions to combinatorial statements about sets, and finally using some method (primarily forcing) to show that the combinatorial statements are independent.

A second type of chapter involves delineating the edges of the independence phenomenon, giving proofs in ZFC of statements that on first sight would be suspected of being independent. Proofs of this kind are often extremely subtle and surprising; very similar statements are independent and it is hard to detect the underlying difference.

The last two types of chapters are motivated by the desire to *settle* these independent statements by adding assumptions to ZFC, such as large cardinal axioms. Proposers of augmentations to ZFC carry the burden of marshaling sufficient evidence to convince informed practitioners of the reasonableness, and perhaps truth, of the new assumptions as descriptions of the mathematical universe. (Proposals for axiom systems intended to *replace* ZFC carry additional heavier burdens and appear in other venues than the Handbook.)

One natural way that this burden is discharged is by determining what the supplementary axioms *say*; in other words by investigating the consequences of new axioms. This is a strictly mathematical venture. The theory is assumed and theorems are proved in the ordinary mathematical manner. Having an extensive development of the consequences of a proposed axiom allows researchers to see the overall picture it paints of the set-theoretic universe, to explore analogies and disanalogies with conventional axioms, and judge its relative coherence with our understanding of that universe. Examples of this include chapters that posit the assumption that the Axiom of Determinacy holds in a model of Zermelo-Fraenkel set theory that contains all of the real numbers and proceed to prove deep and difficult results about the structure of definable sets of reals.

Were there an obvious and compelling unique path of axioms that supplement ZFC and settle important independent problems, it is likely that the last type of chapter would be superfluous. However, historically this is not the case. Competing axioms systems have been posited, sometimes with obvious connections, sometimes appearing to have nothing to do with each other.

Thus it becomes important to compare and contrast the competing proposals. The Handbook includes expositions of some stunningly surprising results showing that one axiom system actually implies an apparently unrelated axiom system. By far the most famous example of this are the proofs of determinacy axioms from large cardinal assumptions.

Many axioms or independent propositions are not related by implication, but rather by *relative consistency* results, a crucial idea for the bulk of the Handbook. A remarkable meta-phenomenon has emerged. There appears to be a central spine of axioms to which all independent propositions are comparable in consistency strength. This spine is delineated by large cardinal axioms. There are no known counterexamples to this behavior.

Thus a project initiated to understand the relationships between disparate axiom systems seems to have resulted in an understanding of most known natural axioms as somehow variations on a common theme—at least as far as consistency strength is concerned. This type of unifying deep structure is taken as strong evidence that the axioms proposed reflect some underlying reality and is often cited as a primary reason for accepting the existence of large cardinals.

The methodology for settling the independent statements, such as the Continuum Hypothesis, by looking for evidence is far from the usual deductive paradigm for mathematics and goes against the rational grain of much philosophical discussion of mathematics. This has directed the attention of some members of the philosophical community towards set theory and has been grist for many discussions and message boards. However interpreted, the investigation itself is entirely mathematical and many of the most skilled practitioners work entirely as mathematicians, unconcerned about any philosophical anxieties their work produces.

Thus set theory finds itself at the confluence of the foundations of mathematics, internal mathematical motivations and philosophical speculation. Its explosive growth in scope and mathematical sophistication is testimony to its intellectual health and vitality.

The Handbook project has some serious defects, and does not claim to be a remotely complete survey of set theory; the work of Shelah is not covered to the appropriate extent given its importance and influence and the enormous development of classical descriptive set theory in the last fifteen years is nearly neglected. While the editors regret this, we are consoled that those two topics, at least, are well documented elsewhere. Other parts of set theory are not so lucky and we apologize.

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