Chapter 4

Further Topics in First-Order Logic

§40. Duality

We introduce this subject with a parable about two scholars, named Oren and Nero, who were visiting an archaeologist and were shown a recently discovered tablet, the contents of which are reproduced in Figure 4.1. They soon realized that the figures on the tablet were truth tables, and they set about translating them into more familiar notations. (Before proceeding further, the reader is advised to do this for himself for at least a few of the tables.) Oren produced the translation in Figure 4.2, and Nero produced that in Figure 4.3. As they started to show their translations to the archaeologist, Nero modestly remarked "That was really quite easy, as soon as I realized that $\oplus$ denoted conjunction". "But you're wrong!" exclaimed Oren. "$\oplus$ denoted disjunction!" They argued for some time, and neither was able to persuade the other that he was wrong. All they could agree on was that $\ominus$ denoted negation.

Then the archaeologist showed them an inscription which had been found on another tablet. "We've figured out how to translate most of the language they apparently used in everyday affairs", he said, "but this inscription seems to be in a mixture of two languages. Here's how the inscription looks under the translation we've achieved so far:"

> 'The value of $(\boxplus\ \mathbf{x})\mathbf{A}(\mathbf{x})$ is $\triangledown$ iff there is an n such that the value of $\mathbf{A}(n)$ is $\triangledown$.'

Oren and Nero recognized that this was another fragment of a text on logic,

○	△	▽
△	△	△
▽	△	△

□	▽	△
▽	▽	▽
△	▽	▽

⊕	△	▽
△	△	△
▽	△	▽

⊞	▽	△
▽	▽	▽
△	▽	△

⦸	△	▽
△	△	△
▽	▽	△

⧅	▽	△
▽	▽	▽
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⊘	△	▽
△	△	▽
▽	△	△

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▽	▽	△
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⊗	△	▽
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⊠	▽	△
▽	▽	△
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⊙	△	▽
△	▽	▽
▽	▽	△

⊡	▽	△
▽	△	△
△	△	▽

	⊖
△	▽
▽	△

Figure 4.1: The tablet

and each soon produced a translation of it. Oren's translation was:

> 'The value of $\forall \mathbf{x} \mathbf{A}(\mathbf{x})$ is falsehood iff there is an n such that the value of $\mathbf{A}(n)$ is falsehood.'

Nero's translation was:

> 'The value of $\exists \mathbf{x} \mathbf{A}(\mathbf{x})$ is truth iff there is an n such that the value of $\mathbf{A}(n)$ is truth.'

Again, they could find no way to resolve their disagreement.

t	T	F
T	T	T
F	T	T

f	F	T
F	F	F
T	F	F

$\vee$	T	F
T	T	T
F	T	F

$\wedge$	F	T
F	F	F
T	F	T

$\subset$	T	F
T	T	T
F	F	T

$\not\supset$	F	T
F	F	F
T	T	F

$\supset$	T	F
T	T	F
F	T	T

$\not\subset$	F	T
F	F	T
T	F	F

$\equiv$	T	F
T	T	F
F	F	T

$\not\equiv$	F	T
F	F	T
T	T	F

$\downarrow$	T	F
T	F	F
F	F	T

$\mid$	F	T
F	T	T
T	T	F

	$\sim$
T	F
F	T

Figure 4.2: Oren's translation of the tablet

f	F	T
F	F	F
T	F	F

t	T	F
T	T	T
F	T	T

$\wedge$	F	T
F	F	F
T	F	T

$\vee$	T	F
T	T	T
F	T	F

$\not\supset$	F	T
F	F	F
T	T	F

$\subset$	T	F
T	T	T
F	F	T

$\not\subset$	F	T
F	F	T
T	F	F

$\supset$	T	F
T	T	F
F	T	T

$\not\equiv$	F	T
F	F	T
T	T	F

$\equiv$	T	F
T	T	F
F	F	T

$\mid$	F	T
F	T	T
T	T	F

$\downarrow$	T	F
T	F	F
F	F	T

	$\sim$
F	T
T	F

Figure 4.3: Nero's translation of the tablet

"There's one more fragment of a tablet you might be interested in", said the archeologist. "We managed to translate the first word on it as '**AXIOMS**', and there's a line under that which looks like this:"

$$(\boxplus p)[p \oplus \ominus p].$$

"That settles it!" exclaimed Oren. "That translates to '$\forall p[p \vee \sim p]$', which is a fine axiom. Clearly my method of translation is correct." "Not at all", replied Nero. "It translates to '$\exists p[p \wedge \sim p]$'. The people who wrote these tablets were known throughout the ancient world as deceitful, treacherous, inveterate liars. Apparently they even axiomatized their lies!"

We leave it to the reader to decide what the moral of this story is.

Clearly there is a symmetry, or *duality*, in logic induced by systematically interchanging truth and falsehood. We shall show how this duality can be used. The ideas in this section apply to any sound and complete formulation of first-order logic. We shall present them in terms of $\mathcal{F}$, but they apply equally well to a system in which both $\forall$ and $\exists$, and all binary propositional connectives, are primitive symbols.

When $\subset$, $\not\supset$, $\not\subset$, $\not\equiv$, $\downarrow$, and $\mid$ occur in abbreviations of wffs of $\mathcal{F}$, they are to be regarded as having been introduced by the following definitions, which we hereby add to the definitions of $\wedge$, $\supset$, and $\equiv$ in §10, and of $\exists$ in §20:

$[\mathbf{A} \subset \mathbf{B}]$	stands for	$[\mathbf{A} \vee \sim \mathbf{B}]$.
$[\mathbf{A} \not\supset \mathbf{B}]$	stands for	$\sim [\sim \mathbf{A} \vee \mathbf{B}]$.
$[\mathbf{A} \not\subset \mathbf{B}]$	stands for	$\sim [\mathbf{A} \vee \sim \mathbf{B}]$.
$[\mathbf{A} \not\equiv \mathbf{B}]$	stands for	$\sim [\mathbf{A} \equiv \mathbf{B}]$.
$[\mathbf{A} \downarrow \mathbf{B}]$	stands for	$\sim [\mathbf{A} \vee \mathbf{B}]$.
$[\mathbf{A} \mid \mathbf{B}]$	stands for	$\sim [\mathbf{A} \wedge \mathbf{B}]$.

Definition. Let $\mathbf{A}$ be a wff of $\mathcal{F}$. A wff $\mathbf{B}$ is a *dual* of $\mathbf{A}$ iff there is an abbreviation $\mathbf{C}$ for $\mathbf{A}$ such that $\mathbf{B}$ is the result of interchanging $\forall$ with $\exists$, $\vee$ with $\wedge$, $\supset$ with $\not\subset$, $\subset$ with $\not\supset$, $\equiv$ with $\not\equiv$, and $\downarrow$ with $\mid$ everywhere in $\mathbf{C}$ (See Figure 4.4.) The *principal dual* $\mathbf{A}^d$ of $\mathbf{A}$ is the result of making these interchanges in the unabbreviated form of $\mathbf{A}$.

$\forall$	$\exists$
$\wedge$	$\vee$
$\supset$	$\not\subset$
$\subset$	$\not\supset$
$\equiv$	$\not\equiv$
$\downarrow$	$\mid$

Figure 4.4: Dual Pairs

4000 Proposition. If $\mathbf{B}$ and $\mathbf{D}$ are duals of $\mathbf{A}$, then $\vdash \mathbf{B} \equiv \mathbf{D}$.

Proof: We show that if $\mathbf{B}$ is any dual of $\mathbf{A}$, then $\vdash \mathbf{B} \equiv \mathbf{A}^d$. Clearly this will suffice to prove the proposition. We show, by induction on the construction of $\mathbf{D}$, that if $\mathbf{D}$ is any abbreviation for a wff $\mathbf{D}_0$ of $\mathcal{F}$, and $\mathbf{D}'$ is the result of interchanging $\forall$ with $\exists$, $\vee$ with $\wedge$, $\supset$ with $\not\subset$, etc., in $\mathbf{D}$, then $\vdash \mathbf{D}' \equiv \mathbf{D}_0^d$.

Of course, this is trivially true if $\mathbf{D}$ is atomic. Now consider the following cases:

(a) $\mathbf{D}$ is $\forall \mathbf{xM}$. Then $\mathbf{D}' = \exists \mathbf{xM}'$, but $\vdash \mathbf{M}' \equiv \mathbf{M}_0^d$ by inductive hypothesis, so $\vdash \mathbf{D}' \equiv \exists \mathbf{xM}_0^d$. Also $\mathbf{D}_0^d = (\forall \mathbf{xM}_0)^d = \exists \mathbf{xM}_0^d$, so $\vdash \mathbf{D}' \equiv \mathbf{D}_0^d$.
(b) $\mathbf{D}$ is $\exists \mathbf{xM}$. Then $\mathbf{D}' = \forall \mathbf{xM}'$, but $\vdash \mathbf{M}' \equiv \mathbf{M}_0^d$ by inductive hypothesis, so $\vdash \mathbf{D}' \equiv \forall \mathbf{xM}_0^d$. Also $\mathbf{D}_0^d = (\sim \forall \mathbf{x} \sim \mathbf{M}_0)^d = \ \sim \exists \mathbf{x} \sim \mathbf{M}_0^d$, so $\vdash \mathbf{D}' \equiv \mathbf{D}_0^d$.
(c) $\mathbf{D}$ is $[\mathbf{M} \vee \mathbf{N}]$. Then $\mathbf{D}' = [\mathbf{M}' \wedge \mathbf{N}']$, but $\vdash \mathbf{M}' \equiv \mathbf{M}_0^d$ and $\vdash \mathbf{N}' \equiv \mathbf{N}_0^d$ by inductive hypothesis, so $\vdash \mathbf{D}' \equiv [\mathbf{M}_0^d \wedge \mathbf{N}_0^d]$. Also $\mathbf{D}_0^d = [\mathbf{M}_0 \vee \mathbf{N}_0]^d = [\mathbf{M}_0^d \wedge \mathbf{N}_0^d]$, so $\vdash \mathbf{D}' \equiv \mathbf{D}_0^d$.
(d) $\mathbf{D}$ is $[\mathbf{M} \wedge \mathbf{N}]$. Then $\mathbf{D}' = [\mathbf{M}' \vee \mathbf{N}']$, but $\vdash \mathbf{M}' \equiv \mathbf{M}_0^d$ and $\vdash \mathbf{N}' \equiv \mathbf{N}_0^d$ by inductive hypothesis, so $\vdash \mathbf{D}' \equiv [\mathbf{M}_0^d \vee \mathbf{N}_0^d]$. Also $\mathbf{D}_0^d = (\sim [\sim \mathbf{M}_0 \vee \sim \mathbf{N}_0])^d = \ \sim [\sim \mathbf{M}_0^d \wedge \sim \mathbf{N}_0^d] = \ \sim\sim [\sim\sim \mathbf{M}_0^d \vee \sim\sim \mathbf{N}_0^d]$, so $\vdash \mathbf{D}' \equiv \mathbf{D}_0^d$.

Since the pattern of proof is now clear, we leave it to the reader to check the remaining cases (Exercise X4000). Note that there is some purpose to this exercise, since it should be verified that the duals of the remaining connectives are defined correctly. ■

4001 Corollary. For any wff $\mathbf{A}$ of $\mathcal{F}$, $\vdash \mathbf{A}^{dd} \equiv \mathbf{A}$.

Proof: It is easy to see that $\mathbf{A}$ is a dual of $\mathbf{A}^d$, so $\vdash \mathbf{A}^{dd} \equiv \mathbf{A}$ by 4000. ■

REMARK. Before looking at the next theorem, the reader should test his understanding of the basic idea of duality by trying to figure out what the theorem should say. The theorem has the form $\mathcal{V}_\varphi^\mathcal{M} \mathbf{A}^d = ?$, and essentially says that one can compute $\mathcal{V}_\varphi^\mathcal{M} \mathbf{A}^d$ if one knows $\mathcal{V}_\varphi^\mathcal{N} \mathbf{A}$ for an appropriate interpretation $\mathcal{N}$.

4002 Theorem. Let $\mathcal{M} = \langle \mathcal{D}, \mathcal{J} \rangle$ be any interpretation. Define $\mathcal{J}'$ to agree with $\mathcal{J}$ on all individual and function constants, and for all n-ary predicate constants $\mathbf{P}$, let $(\mathcal{J}'\mathbf{P})d_1 \ldots d_n = \ \sim (\mathcal{J}\mathbf{P})d_1 \ldots d_n$ for all $d_1, \ldots, d_n \in \mathcal{D}$. Let $\mathcal{M}' = \langle \mathcal{D}, \mathcal{J}' \rangle$. Then for any wff $\mathbf{A}$ and assignment φ, $\mathcal{V}_\varphi^\mathcal{M} \mathbf{A} = \ \sim \mathcal{V}_\varphi^{\mathcal{M}'} \mathbf{A}$.

NOTES:

(1) In accord with the convention introduced in §25, for simplicity we are

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