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MATHEMATICAL CHANGE AND INCONSISTENCY

*A Partial Structures Approach**

1. INTRODUCTION

Our understanding of mathematics arguably increases with an examination of its growth, that is with a study of how mathematical theories are articulated and developed in time. This study, however, cannot proceed by considering particular mathematical statements in isolation, but should examine them in a broader context. As is well known, the outcome of the debates in the philosophy of science in the last few decades is that the development of science cannot be properly understood if we focus on isolated theories (let alone isolated statements). On the contrary, we ought to consider broader epistemic units, which may include *paradigms* (Kuhn 1962), *research programmes* (Lakatos 1978a), or *research traditions* (Laudan 1977). Similarly, the first step to be taken by any adequate account of mathematical change is to spell out what is the appropriate epistemic unit in terms of which the evaluation of scientific change is to be made. If we can draw on the considerations that led philosophers of science to expand the epistemic unities they use, and adopt a similar approach in the philosophy of mathematics, we shall also conclude that mathematical change is evaluated in terms of a 'broader' epistemic unit than the one that is often used, such as, statements or theories.

But a second consideration emerges at this point. It may be argued that mathematical change does not provide us with any insight into the nature of mathematics or of mathematical knowledge, since this change is often the result of inconsistent theories, of 'contradictory' views about a particular mathematical domain. For instance, if we consider the development of set theory, it becomes clear that by the time of its formulation, there were quite different views about the concept of function (in particular, with regard to arbitrary functions), different understandings of Cantor's diagonal proof, and conflicting proposals about the distinction between membership and inclusion. Therefore, so goes the argument, given the ubiquity of these inconsistencies, nothing can be learnt about mathematics by focusing on how a mathematical theory has evolved in time (that is, by considering mathematical change).

* Many thanks to Newton da Costa, Steven French, Joke Meheus, and Sarah Kattau for helpful discussions on the issues examined here.

J. Meheus (ed.), Inconsistency in Science, 59—79.

In my view, there are at least two different reactions to this position. The first stresses that once appropriate patterns of mathematical change are spelled out, all the apparent inconsistencies among mathematical theories are dissolved—one should ultimately find consistent formulations of these theories. Thus, once these formulations are found, we are entitled to learn from the study of the dynamics of mathematics. Of course, this reaction adopts a fairly conservative view towards inconsistencies, and basically assumes that all reliable information should be, at least, *consistent*.

The second reaction will challenge this assumption. The existence of inconsistencies within bodies of mathematical information does not necessarily preclude us from learning from such bodies. In particular, even if proposed patterns of mathematical change involve inconsistencies, we cannot conclude from this that such patterns are hopelessly unreliable as a way of providing information about the nature of mathematics. Notice that we have here, in fact, two different levels of inconsistency. The first is concerned with inconsistencies at the *theoretical* level, involving particular mathematical theories; the second concerns inconsistencies at the *metatheoretical* level, and this involves our representations of mathematical change. What the second reaction countenances is a unified strategy of accommodating inconsistencies—at face value—at *both* levels.

The constraint of taking inconsistencies *at face value* is important. The idea is *not* to claim that, once mathematical theories are appropriately reformulated, those inconsistencies can either be shown to be innocuous, or simply are dissolved. This would simply turn the second reaction into a variation of the first. As opposed to this, the second reaction urges one to find an appropriate epistemic role for inconsistencies. Once we have an appropriate framework in terms of which mathematical change can be modelled, the second reaction will also have to provide a clear strategy that allows us to accommodate inconsistencies in a more 'positive' way, i.e. that assigns a positive role for inconsistencies within mathematics and mathematical change. The existence of important mathematical theories that are inconsistent (such as the calculus and naive set theory) is enough to demand such a view.

In this paper, I am concerned with articulating and defending this second reaction. In order to do so, I shall first put forward an account of what sort of epistemic unit should be taken as basic. As we shall see in sections 2 and 3, I take it that two independent proposals found in Kitcher 1984 and Laudan 1984, if adequately combined, provide the first step towards an appropriate framework to do so. This move, however, is not sufficient as a defence of the second reaction, since the resulting framework cannot accommodate inconsistencies. In order to pursue *this* task—which is the chief aim of section 3—I propose that we adopt the conceptual resources provided by da Costa and French's partial structures approach (see, for instance, da Costa and French 1989, 1990, 1993a, and 1995). As we shall see in section 3.1, three important features are brought by this view: a broader notion of structure (partial structure), a weaker notion of truth (quasi-truth), and a paraconsistent setting (in terms of the underlying logic of quasi-truth: a convenient version of Jaśkowski's logic; see da Costa, Bueno and French 1998). In this way, or so I shall argue in section 3.2, we can put forward an account of inconsistencies involved in mathematical change, without triviality. Finally, in section 3.3, I shall provide a case-study, showing how the framework introduced helps us to understand some aspects of the development of set theory, including a consideration of the role played by some recent paraconsistent versions of this theory in spelling out the nature of the research on set-theoretical structures.

2. MODELLING MATHEMATICAL CHANGE: A CONSISTENCY-PRESERVING PATTERN

As is well known, there is an old tradition of interpretation of science that adopts the strategy of putting forward a view about the ways in which science evolves in order to draw general conclusions concerning the nature of scientific knowledge. This tradition takes the notion of scientific change as basic for the understanding of the nature of science and the knowledge it supplies. Of course, this is not to say that problems unrelated to the dynamics of scientific knowledge are not to be considered within such a view; they are. But the analysis proposed is (to be) articulated in terms of an account of scientific change. Indeed, it is at this level that the 'explanatory power' of this tradition can be found.

Kitcher's account of the nature of mathematical knowledge (as presented in Kitcher 1984) certainly belongs to this tradition. The role that Kitcher assigns to scientific change—in particular, to mathematical change—parallels in importance that found within, for instance, either Popper's or Lakatos's proposals: the task of taking into account scientific (or mathematical) knowledge is to be achieved mainly by the formulation of a theory of scientific (or mathematical) change.[1]

However, Kitcher's view of mathematical change is embedded in a general epistemological framework, and much of the motivation for his proposals rests on his critical appraisal of previous epistemological accounts of mathematics. The relevance of mathematical change is presented in the context of a critique of two interrelated doctrines that have characterised such previous accounts, namely, mathematical apriorism (the claim that mathematical knowledge is a priori), and mathematical 'individualism' (a lack of consideration of the role of mathematical community in the development of mathematics). As opposed to the former, Kitcher puts forward his *mathematical empiricism*, rejecting earlier empiricist interpretations of mathematics, such as Lakatos's, Putnam's and Quine's (see Kitcher 1984, 4). As opposed to the latter, he advances a theory of *mathematical practice*, motivated by Kuhn's view, but stripped of his general framework,[2] in order to claim that the development of mathematical knowledge occurs in terms of a 'rational modification

[1] Actually, one of Kitcher's main points consists in spelling out the similarities (without disregarding the differences) between scientific and mathematical changes, in order to motivate his account of the methodology of mathematics (see Kitcher 1984, 150, and the rest of Chapter 7). Indeed, one of his claims is that 'the growth of mathematical knowledge is far more similar to the growth of scientific knowledge than is usually appreciated' (Kitcher 1984, 8). Of course, such a claim clearly echoes Lakatos's view about the connection between scientific and mathematical knowledge (for details, see Lakatos 1976 and Lakatos 1978b).

[2] In particular, the Kuhnian notion of paradigm is not accepted (see Kitcher 1984, 163-164).

of mathematical practices' (Kitcher 1984, 165). These practices, which are the basic epistemic unit of mathematical change, have five components: a language, certain accepted (mathematical) statements, certain accepted types of reasoning (an underlying logic), certain questions considered as important and certain metamathematical standards with regard to proofs, definitions and so on (Kitcher 1984, 163). The introduction of this notion of mathematical practice is Kitcher's way of assigning a function to the *mathematical community* within his proposal, even though in a rather idealised way.

As a result of the shift to mathematical empiricism and the emphasis on mathematical practice, Kitcher's account of mathematics acquires a *historicist* outlook: mathematical knowledge is obtained by extending the knowledge produced by previous generations (see Kitcher 1984, 4-5). At this point, it is natural to ask, working backwards in the chain of generations, how the knowledge of the first generations was produced at all? To this question, Kitcher formulates a surprising answer: he traces back this knowledge to 'ordinary perception' (1984, 5). In order to explain the possibility of obtaining mathematical knowledge from perception, Kitcher devises a theory of mathematical reality, of 'what mathematics is about' (1984, 101-148). In order then to show how from such perceptual origins mathematics can be developed into such an impressive body of theories, techniques and results that one finds today, he puts forward his account of the growth of mathematical knowledge (1984, 149-228). These two doctrines constitute the core of Kitcher's mathematical empiricism.

Thus, Kitcher's account of mathematical change has two important features. Firstly, it represents mathematical change in terms of broad epistemic units, *mathematical practices* (in Kitcher's sense). Secondly, because of this feature, it allows one to formally accommodate the role of mathematical activity and practice in the changes of mathematical theories. As a result, so Kitcher argues, a more faithful account of the dynamics of mathematics, *vis-à-vis* the historical record, is put forward.

In this picture, each particular historical moment is characterised by a given mathematical practice, involving a community of mathematicians who adopt the same *language* to investigate a given mathematical domain, who share a common core of accepted mathematical *statements* and open *questions*, and who, in trying to settle the latter, adopt the same *metamathematical standards* and the same *logic*. Mathematical change results from the interplay of each of these five components, as well as changes in them. For instance, new, more demanding patterns of rigour (a change in metamathematical standards) may lead to the rejection of a previously accepted proof and to the demand for a new proof of a given result. Similarly, the rejection of a previously accepted mathematical statement S will bring an epistemic change in all results which depend on S, requiring new proofs of them independent of S. Moreover, by considering some open questions as no longer important, changes are introduced in the way mathematics is practised. This may result from the acceptance of new mathematical statements, which change the relative importance of the open questions, since they change the direction of the research in a particular mathematical domain. Furthermore, by changing the underlying logic of a given mathematical practice, new inferential patterns that have been previously rejected can be accommodated—and this may increase the set of accepted statements. However, depending on the logic in question (suppose classical logic is replaced by intuitionist logic), previously accepted inferential patterns may be rejected (for instance, the law of excluded middle would not be valid anymore). This may reduce the accepted statements, given that we are no longer able to derive, on the basis of the underlying logic, any statement depending on the excluded middle. Finally, changes in the language used by a given mathematical community also play a role in mathematical changes.

Two questions immediately arise at this point. The first concerns the 'completeness' of Kitcher's account. How can one guarantee that all different kinds of mathematical change will be accommodated in terms of this pattern? If there is a case that cannot be so explained, Kitcher's proposal will be clearly 'incomplete'. The point, however, is not so decisive, given that Kitcher may legitimately relinquish the provision of a 'complete' account of mathematical change: he may be only concerned with spelling out *some* factors involved in this change, without aiming at characterising *all* of them. Although each factor he presents is clearly *sufficient* to characterise a circumstance in which a change has occurred, it does not seem to be *necessary* for it. From the fact that, in Kitcher's proposal, there are several components in a mathematical practice, we can already suppose that each of them is not taken to be necessary. For example, Kitcher will certainly accept that there might be changes in mathematics without any change in metamathematical patterns, but only, say, in the underlying logic. But provided one is able to acknowledge this point (that if taken in isolation, the components of a mathematical practice are not necessary for explaining mathematical change), this first remark may not be problematic. What one would have abandoned is the idea that the framework is capable of accommodating any kind of mathematical change. After all, if none of the components is necessary for the explanation, there are no grounds to claim that all possible types of change will be accommodated. So, the claim of 'completeness' is given up. But it should be acknowledged that any such claim is certainly too strong. In fact, it is most unlikely that an account of mathematical change will be able to accommodate not only every single change in past mathematical theories, but also in those theories which are yet to be constructed. What we should strive for is an account as complete as possible, able to save the phenomena as comprehensively as we can.

The second point, which is far more important, concerns two requirements involved in an account of theory change in mathematics. Patterns of mathematical change should provide: (i) an *aim* in terms of which the various changes can be evaluated, and (ii) an overall *framework* to formally represent the changes in question. Such a framework should accommodate three further issues: (a) *epistemic changes* associated with changes of mathematical theories, (b) *formal interaction* between these different theories, and in particular (c) the existence of *inconsistencies* in theory change in mathematics.

Two questions raise themselves at this point: (1) What is the importance of each of these requirements? Why should an account of mathematical change deal with these issues? (2) Can Kitcher's account accommodate them? As for the first question, the requirements are introduced because, in providing an account of

Springer

http://www.springer.com/978-1-4020-0630-2

Inconsistency in Science
Meheus, J. (Ed.)
2002, IX, 223 p., Hardcover
ISBN: 978-1-4020-0630-2