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# INCONSISTENCIES IN THE HISTORY OF MATHEMATICS

## *The Case of Infinitesimals**

### 1. INTRODUCTION

In this paper I will not confine myself exclusively to historical considerations. Both philosophical and technical matters will be raised, all with the purpose of trying to understand (better) what Newton, Leibniz and the many precursors (might have) meant when they talked about infinitesimals. The technical part will consist of an analysis why apparently infinitesimals have resisted so well to be formally expressed. The philosophical part, actually the most important part of this paper, concerns a discussion that has been going on for some decennia now. After the Kuhnian revolution in philosophy of science, notwithstanding Kuhn's own suggestion that mathematics is something quite special, the question was nevertheless asked how mathematics develops. Are there revolutions in mathematics? If so, what do we have to think of? If not, why do they not occur? Is mathematics the so often claimed totally free creation of the human spirit? As usual, there is a continuum of positions, but let me sketch briefly the two extremes: the *completists* (as I call them) on the one hand, and the *contingents* (as I call them as well) on the other hand.

A completist basically defends the thesis that mathematics has a definite, if not a "forced" route to follow in its development. As a matter of fact, mathematics strives for its own completion. In that sense, there can be no revolutions, because the outcome of such a revolution is not settled beforehand. Whereas in the completist's case, there is a definite direction. Usually, this implies a (large-scale)[1] linear growth and a cumulative growth of mathematical knowledge. The deeper philosophical

* My most sincere thanks to Joke Meheus and Diderik Batens for thoroughly reading the paper and for indicating some grave errors that, I hope, have been set right in this new version. Without going into too much detail, apparently, although I reject the completist's option, unconsciously I was still thinking and acting as one. Thanks also to Bart Van Kerkhove for helpful suggestions and remarks.

[1] I emphasize large-scale because one cannot exclude human error in the short term. It is very instructive to have a look at, e.g., Altmann's analysis of the development of quaternions in the work of Hamilton (1992, chapter 2). Not exactly a comedy of errors, but rather an amusing play how to arrive at the right notion using the wrong picture.



*J. Meheus (ed.), Inconsistency in Science,* 43—57.

justification of this set of beliefs is some form of Platonism. As there is a mathematical universe somewhere out there and the aim of mathematicians is to produce a full description of that universe, there can necessarily only be one and precisely one correct description that can be considered complete once the entire mathematical universe has been described.

The example that is always given is the development of the notion of number. You start with the natural numbers and for addition everything is fine. But subtraction is a problem, so you extend the natural numbers to get the whole numbers. No problem with multiplication, but division, there is a problem. Another extension is needed to go to the rationals and the problem is solved. But then you still have the square roots and things like that and before you know it there you have the reals. But reals are ugly in a certain sense. Take an equation of second degree? Either it has two solutions, one solution or no solutions at all. But complex numbers, an extension of the real numbers, solve the problem: an equation of degree $n$ has (quite nicely) $n$ (not necessarily different) solutions. But complex numbers are not the final word: quaternions are the next step. And there it stops, because of the following theorem: "If $D$ is a finite-dimensional vector space over $\mathbf{R}$ (the reals), then it must be isomorphic to the quaternions, or to the complex numbers, or to the reals themselves" (MacLane 1986, 121). As MacLane makes clear, one of the basic forces of this process is related to the fact that "each extension of the number system to a larger system is driven by the need to solve questions which the smaller system cannot always answer" (MacLane 1986, 114). I will not burden this paper with a host of quotes showing that completists exist and (perhaps more implicitly than explicitly) form a large, if not the major part of the mathematical community.

The contingents on the other hand stress the fact that mathematics is basically a human enterprise and that, hence, it has all the corresponding characteristics. The best known example of a contingent approach is Imre Lakatos's *Proofs and Refutations*. Ever since this seminal book, many authors have further elaborated these ideas in one direction or another.[2] Mathematics now appears fallible, subject to change, in need of guidance (think about the important role of Hilbert's famous lecture in 1900 presenting his list of 23 problems for the coming century), and definitely not as "forced" as the completists would have it.

It is clear that the discussion (if any) between completists and contingents is basically a discussion about certainty, reliability on the one hand and fallibility on the other. Common sense knowledge has been knocked down, scientific knowledge in the post-Kuhnian and post-modernist atmosphere has received some serious blows, but mathematical knowledge still stands strong. Or does it?

## 2. THE CONSEQUENCE OF CONTINGENCY

Why do I believe that these philosophical considerations are so important for (an understanding of) the history of infinitesimal calculus? For the simple reason that from the completist's point of view, infinitesimal calculus, as conceived before and

[2] It is not my intention to present a full overview of the contingents. As a first guide, the reader can consult Restivo *et al.* 1993, Gillies 1992 or Hersh 1997.

during the Newton-Leibniz era, had to be a temporary stage in the development of mathematics. As is well known, there were very serious problems with the notion of an infinitesimal number, and, when in the 19th century we had full-fledged limit analysis or ε-δ-analysis (as it is sometimes called), it became clear what the "solution" was. Infinitesimals were "monsters" or ill-conceived concepts and now that they have been eliminated, all is well again.

Suppose now that it were possible to show that other routes were open to solve the problems posed by the infinitesimals, then this would surely constitute an argument for the contingents. It would show that several routes were possible and that, for a number of reasons, one particular route was chosen, ignoring the others. Against this background, I can reformulate the aim of this paper in the following way: first, to show what an alternative route could possibly look like, second, to show that it could avoid some of the serious problems, third, to explain why this route has not been explored. I will therefore proceed as follows. In paragraph 3, I present the main problems; in paragraph 4, I indicate the outlines of what could be elaborated into an alternative approach; in paragraph 5, I show how some of the problems stated are in principle solvable, in paragraph 6 I try to figure out why this alternative view has been "missed" by the mathematicians, and, finally in paragraph 7, I look briefly at some other existing alternatives in order to show that, even if the ideas presented in this paper fail to do what I expect them to do, these alternatives can be explored in their own right. In other words, even if this paper does not survive, the contingent's case is certainly not lost.

## 3. THE PROBLEMS WITH INFINITESIMALS

In *Paraconsistent Logic. Essays on the Inconsistent* Priest and Routley write in chapter V, "Systems of Paraconsistent Logic" the following general considerations concerning infinitesimal calculus:

> Another group of examples of inconsistent but non-trivial theories derive from the history of science. Consider, for example, the Newton-Leibniz versions of the calculus. Let us concentrate on the Leibniz version. This was inconsistent since it required division by infinitesimals. Hence if $\alpha$ is any infinitesimal, $\alpha \neq 0$. Yet it also required that infinitesimals and their products be neglected in the final value of the derivative. Thus $\alpha = 0$. [...] Despite this the calculus was certainly non-trivial. None of Newton, Leibniz, the Bernoullis, Euler, and so on, would have accepted that Inf $\int_0^1 x dx = \pi$. [p. 152]

and, in chapter XVIII, "The Philosophical Significance and Inevitability of Paraconsistency":

> Similar points apply as regards the infinitesimal calculus. This was inconsistent and widely recognized as such. In this case various attempts were made to rework the theory in a consistent way. ... However, the attempts did not meet with a great deal of success. Moreover these attempts confirm the fact that the theory and certain of its parts, e.g. the Newtonian theory of fluxions, were inconsistent. If they were not, attempted consistentizations would hardly have been necessary. [p. 495]

Both quotes show that to make sense of infinitesimals will not be an easy task as the contradictions seem to be plentiful and very hard to get rid of, as they apparently touch the heart of the theory. In fact, in chapter XIII, "Applications of Paraconsistent

Logic", p. 376, they present a very simple, yet devastating argument that shows that a *naive* formalisation of (inconsistent) infinitesimals leads to triviality (or to something quite close). In that sense it pleads against the use, if not the existence of infinitesimals. I will therefore call this counterargument[3] $A$:

> Suppose that we have a theory of infinitesimals such that we both have an axiom stating that infinitesimals are equal to zero—$dx = 0$—and an axiom stating that the opposite—$dx \neq 0$—holds. If the usual rules for the real numbers apply, the following argument can be built up:

| | | |
|---|---|---|
| (A1) | $0 + 0 = 0$ | arithmetical truth |
| (A2) | $dx + dx = dx$ | axiom: $dx = 0$ |
| (A3) | $2 \cdot dx = dx$ | arithmetical truth |
| (A4) | $2 = 1$ | axiom: $dx \neq 0$ and division |

A second counterargument $B$ basically shows the same thing, but does so in terms of sets. The focus in this argument is on the question how to keep (standard) real numbers distinguished from infinitesimals, given that we want the same calculating rules to apply to both of them. If we approach the matter rather naively, then problems appear very quickly:

> Suppose we have two sets of numbers $\mathbf{R} = \{r, r', ...\}$ (standard real numbers) and $\mathrm{Inf} = \{\varepsilon, \varepsilon', ... \}$ (specific infinitesimals), such that $\mathbf{R} \cap \mathrm{Inf} = \varnothing$. Suppose further that arithmetical operations, such as addition and subtraction, are defined over $\mathbf{R} \cup \mathrm{Inf}$. On the one hand, we do want that for an arbitrary real number $r$, it is the case that for an arbitrary infinitesimal $\varepsilon$, $r \neq \varepsilon$. But it is easy to show that the contrary statement holds as well. It is sufficient to ask the question whether $r + \varepsilon \in \mathbf{R}$ or $r + \varepsilon \in \mathrm{Inf}$, for any pair $r$ and $\varepsilon$?
>
> (B1) If $r + \varepsilon \in \mathbf{R}$, then $(r + \varepsilon) - r \in \mathbf{R}$, hence $\varepsilon \in \mathbf{R}$, thus every infinitesimal equals some real number.
>
> (B2) If $r + \varepsilon \in \mathrm{Inf}$, then $(r + \varepsilon) - \varepsilon \in \mathrm{Inf}$, hence $r \in \mathrm{Inf}$, thus every real number equals some infinitesimal.
>
> (B3) It follows that the distinction between $\mathbf{R}$ and Inf is completely lost, contradicting (at least) $\mathbf{R} \cap \mathrm{Inf} = \varnothing$.

When we look (even briefly) at the historical material, e.g., Boyer 1959, then it is clear that the same questions bothered everybody at the time:

[3] An important distinction must be made here. It is a counterargument for the view that in present-day mathematics infinitesimals can still (and perhaps should) play an important part as they are easier and simplier to handle than limit operations. It is not a counterargument for the historical problem: what is it that Newton, Leibniz and others had in mind, when they used infinitesimals? Here we can only conclude that whatever it is, it must be a complex thing, which is actually what Priest and Routley claim.

(Q1) What do we mean when we say that infinitesimals are infinitely small? Are they finite numbers or are they quite simply zero? Newton was not at all clear about the matter and the many metaphors Leibniz used to clarify his ideas, show more often than not that he considered them to be finite (a short line compared to a long line or a sphere one holds in one hand compared to the earth one is standing on).

(Q2) How do infinitesimals compare to "standard" numbers? Are they comparable, for that matter? Are there differences among infinitesimals? If $dx$ is an infinitesimal how does it compare to $(dx)^2$ or $(dx)^n$?

## 4. THE HIGH COST OF INFINITESIMALS OR A MODEST PROPOSAL (MP)

In this paragraph I will outline how one can have "genuine" infinitesimals on condition that one is willing to accept the following:

(a) in terms of models, only local models (in a sense to be specified in what follows) are considered, or, alternatively, there are no global models[4],

(b) all local models are essentially finite.

I realize that these conditions run counter to almost anything that is cherished by logicians, mathematicians and philosophers. I will return to this point in paragraph 6.

If, however, one is willing to make these "sacrifices", then matters become rather easy, if somewhat tedious. What follows presents a rough outline and not a full-blown theory. Let us start with the standard theory $T$ of real numbers. The first change that has to be made is that two sets of distinct variables will be used:

(i) variables for "standard" real numbers: $x, y, z, \ldots$

(ii) variables for "infinitesimals": $\varepsilon, \varepsilon', \varepsilon'', \ldots$

Suppose we now have a finite series of formulas $\mathbf{F} = \{F_1, F_2, \ldots, F_n\}$, all expressed in the language of $T$. The intuitive idea is that $\mathbf{F}$ could, e.g., represent a calculation of the value of a function in a particular point. Further suppose that if all formulas are interpreted in $\mathbf{R}$ such that all variables are treated in the same way, then they are true in the standard model of real numbers.

*Example*: $\mathbf{F} = \{F_1, F_2, F_3, F_4, F_5\}$

$F_1$: $((x+\varepsilon)^3 - x^3)/\varepsilon = ((x^3 + 3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3) - x^3)/\varepsilon$

$F_2$: $((x^3 + 3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3) - x^3)/\varepsilon = (3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3)/\varepsilon$

$F_3$: $(3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3)/\varepsilon = 3x^2 + 3x\varepsilon + \varepsilon^2$

$F_4$: $3x^2 + 3x\varepsilon + \varepsilon^2 = 3x^2 + (3x+\varepsilon)\cdot\varepsilon$

$F_5$: $((x+\varepsilon)^3 - x^3)/\varepsilon = 3x^2 + (3x+\varepsilon)\cdot\varepsilon$

(I consider all the formulas universally quantified both over $x$ and $\varepsilon$, taking into account that $\varepsilon \neq 0$, i.e., every $F_i$ is preceded by $(\forall x)(\forall\varepsilon)(\varepsilon \neq 0 \supset \ldots))$

Obviously, *if* $\mathbf{F}$ *is finite, then so are the number of variables*, both "standard" and "infinitesimal", so is the number of constants, and so is the set of terms occurring in the members of $\mathbf{F}$.

[4] Yet another formulation is that one should be willing to give up compactness.

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