Preface

About three hundred years before our era, the great Greek geometer Euclid[1] attempted to compile and logically order the many results of geometry known by that time.

The Greeks were well acquainted with the notion of proof; logical arguments were used to obtain new results from old. However, these old theorems had to be derived from other known theorems, which in turn were to be deduced from more primitive results, and so on. Hence, in order to give a logically coherent exposition of geometry, avoiding an infinite regress, it became necessary to posit a number of *first theorems*, that is, to decide at what point one must begin the chain of reasoning that allows new theorems to be proved from those that have already been established. Results were sought that were considered to be so self-evident that it would not be necessary to prove them.

Euclid began his great masterpiece, *The Elements*, with a list of five postulates that play the role of *first theorems* and which are considered to be so obvious as to be accepted without proof.

Explicitly, these postulates are:

1. A straight line can be drawn from any point to any point.
2. A finite straight line can be produced continuously in a straight line.
3. A circle can be described with any center and distance.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced

[1] Very little is known about Euclid. We cannot even be certain of his existence. *The Elements* is a monumental achievement, perhaps more than one man could conceivably have produced, so it is possible that Euclid simply directed the work.

indefinitely, meet on that side on which the angles are less than the two right angles.

Prior to this, in order to clarify what was being discussed, Euclid gave twenty-three definitions (a *point* is that which has no parts; a *line* is a breadthless length, etc.) together with five logical rules or common notions (things which are equal to the same thing are also equal to one another; if equals be added to equals, the wholes are equals, etc.). See, for instance, [13] or [26].

From this, with a level of rigor that was considered exemplary until the 19th century, Euclid recovered all the theorems of elementary geometry. Some more involved topics, such as conic sections, for example, although known in Euclid's time, do not appear in *The Elements*.[2]

In the paper [12], A. Dou makes a remark concerning *The Elements* that is worth quoting here:

> The geometry of *The Elements* is a geometry that today should be called physical geometry, because for Euclid and Aristotle the terms of the propositions in *The Elements* refer precisely to the real natural bodies of the physical world [. . .] *It is a geometry that tries to study the structure of physical space.*

Euclid's points and straight lines are not physical points and straight lines, but an abstraction of them. When one tries to give a rigorous foundation of geometry, some fundamental questions arise, such as: What is a point? What is a straight line? The definitions given by Euclid are not satisfactory, they must be considered only as descriptions, and it is clear that the definition of a point (that which has no parts) and straight line (a length without width) in *The Elements* do not provide much in the way of rigor.

For more than two thousand years after Euclid many mathematicians tried to prove the fifth postulate. Unable to find a direct proof, they instead tried an indirect approach, hoping to arrive at a contradiction by assuming its negation. Curiously, the assumption that the fifth postulate is false does not lead to a contradiction, but rather opens the door to new and marvelous geometries.

The first to develop this approach, and realize that it does not lead to contradiction, were N. Lobatchevski and J. Bolyai, who independently[3] discovered

[2] *The Elements* is composed of thirteen books; books I, III, IV, XI, XII and XIII are concerned with geometry, while books VII, VIII and IX cover arithmetic. The remaining four books are dedicated to algebra. The results of the first four books, the seventh and the ninth, are principally due to the Pythagorean school, the fifth and the sixth are due to Eudoxus (4th century BC), and the tenth and thirteenth are due to Teatetus (368 BC). *The Elements* contain a total of 131 definitions and 465 propositions.

[3] Lobatchevski's first publications on non-Euclidean Geometry appear in the *Kazan Messenger* (1829–1830). The work of Bolyai, known as the *Appendix*, because it

Hyperbolic Geometry (a totally coherent geometry that does not satisfy the fifth postulate).

To establish these new geometries rigorously it was necessary to reformulate Euclid's *Elements*, removing all ambiguities such as that of the definition of a straight line mentioned above. Several works appeared in this direction and the efforts culminated in 1899 with D. Hilbert's *Grundlagen der Geometrie*.

Hilbert's new axiomatization was a synthesis of the work of many mathematicians (including, for instance, M. Pasch). The main novelty of Hilbert's approach with respect to Euclid's work was that, at the outset, he assumed only a pair of sets (here we restrict our discussion to plane geometry) whose elements are *not defined* but that are called respectively *points* and *straight lines*. Among the elements of these sets there are certain *relations*[4] that again are *not defined*, but which satisfy some axioms or properties: incidence, order, continuity and congruence.

The *incidence* axioms describe the fact that through any two points only one straight line can pass. The *order* axioms enable us to talk about line segments, and those of *continuity* permit the construction of the real numbers. The *congruence* axioms assert, essentially, that "given a line segment AB and a straight half-line with origin C, there exists a unique point D on this straight half-line such that the line segment AB is congruent to the line segment CD", together with an analogous assertion for angles.

From this one can reproduce the results of *The Elements*. A simplified axiomatic development of Euclidean Plane Geometry can be found in [24].

This approach immediately suggests some natural questions: What happens if we do not include the order axioms, or the axioms of continuity? Is it essential to identify the points of a straight line with the set of real numbers? Can we "do geometry" identifying, for instance, the points of straight lines with the complex numbers?

We shall see that, in fact, we can do geometry identifying the points of straight lines with the elements of an arbitrary field.

Next we state the axioms of Affine Plane Geometry as they appear in [1].

Recall that we only have a pair of sets whose elements are undefined but which we suggestively call, respectively, *points* and *straight lines*. Let us assume that there is a relation, called *incidence*, among the elements of the first set and the elements of the second. When an element P of the first set is *related*

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was published as an addendum to his father's book *Tentamen Juventutem Studiosam in Elementa Matheseos Purae Introducendi*, dates from 1831, but the year of publication of the *Tentamen* itself was 1829. Bolyai elaborated his geometry in outline in 1823 and the final work, in German, was completed in 1826.

[4] A *relation* among the elements of two sets is a subset of the Cartesian product of these two sets.

to an element l of the second, we shall write $P \in l$, and we shall say that P *belongs to* l, or that the straight line l *passes through* the point P.

We further assume that this (undefined) relation satisfies the following axioms:

Axiom 1

Given two distinct points P and Q, there exists a unique straight line l such that $P \in l$ and $Q \in l$.

Axiom 2

Given a point P and a straight line l, there exists a unique straight line m such that $P \in m$ and $l \| m$.

The notation $l \| m$ means that either $l = m$ or there is no point P satisfying both $P \in l$ and $P \in m$. One says that l is *parallel* to m.

Axiom 3

There are three non-collinear points.

The next axioms are more easily stated if we first define the concepts of *dilation* and *translation*.

Definition

A map σ from the set of points into itself is called a *dilation* if it has the following property: Let P, Q be distinct points and let l be the straight line they determine. Then the straight line l' determined by $\sigma(P)$ and $\sigma(Q)$ is parallel to l. A dilation without fixed points is called a *translation*.

The bundle of straight lines, each line determined by a point and its image under a given translation, is called the *direction* of the translation.

We shall postulate that there must be 'sufficiently many' dilations and translations. More precisely:

Axiom 4

Given two distinct points P and Q there exists a translation τ such that $\tau(P) = Q$.

Axiom 5

Given three distinct collinear points P, Q and R there exists a dilation σ such that $\sigma(P)=P$ and $\sigma(Q)=R$.

With these last two axioms we can define *coordinates* in the plane in such a way that each point has two coordinates and each straight line is given by a *linear equation*. These coordinates are not necessarily real numbers, but elements of some field k. This field can be constructed from axioms 1, 2, 3 and 4. In fact k is formed by certain morphisms of the group T of translations. Concretely (for details, see [1]) we have

$$k=\{\alpha:T\longrightarrow T:\alpha \text{ satisfies the following two properties}\}$$

1. For all $\tau,\sigma\in T$, $\alpha(\tau\circ\sigma)=\alpha(\tau)\circ\alpha(\sigma)$, i.e. α is a group homomorphism.
2. For all $\tau\in T$, τ and $\alpha(\tau)$ have the same direction.

All axiomatic theories face the problem of *consistency*. A system is consistent if no contradiction can be derived within it. To prove that a theory is consistent, one provides a mathematical model in which all the axioms are satisfied. If this model is consistent, so too is the axiomatic system. The axiomatic system is, at least, as consistent as the model.

This is what we are going to do in this book. We shall construct an algebraic model, based on the concept of a vector space over a field, which, with suitable definitions, will satisfy all the axioms of Affine and Euclidean Geometry.

Organization In Chapter 1 we introduce the most fundamental concept of these notes: affine space. This is a natural generalization of the concept of vector space but with a clear distinction between points and vectors. This distinction is not often made in vector spaces: it is commonplace, for example, not to distinguish between the point $(1,2)\in\mathbb{R}^2$ and the vector $v=(1,2)\in\mathbb{R}^2$. The problem is that $\mathbb{R}^2$ is both a set of points and a vector space.

In the study of vector spaces, the vector subspaces and the relations among them (the Grassmann formulas) play a central role. In the same way, in the study of affine spaces, the affine subspaces and the relations among them (the affine Grassmann formulas) also play an important role.

The simplest figure that we can form with points and straight lines is the triangle. In this chapter we shall meet two important results that refer to triangles and the incidence relation: the theorems of Menelaus and Ceva.

In Exercise 1.5 of this chapter, page 38, we verify Axioms 1, 2 and 3 of Affine Geometry.

In Chapter 2 we introduce a class of maps between affine spaces: affine maps, or *affinities*. The definition is a natural one, indeed affinities are revealed to be simply those maps that take collinear points to collinear points.

We shall also see that there are 'enough' affinities. In fact, in an affine space of dimension n, given two subsets of $n+1$ points, there exists an affinity which maps the points of the first subset onto the points of the second.

In Exercise 2.9 of this chapter, page 88, we verify Axioms 4 and 5 of Affine Geometry.

In Chapters 3 and 4 we answer the natural question of how many affinities there are. To do so we first define an equivalence relation between affinities and study its equivalence classes. In low dimensions the problem is not too hard and is solved explicitly in Chapter 3. However, in arbitrary dimensions the problem is rather involved, since it depends upon the classification of endomorphisms, and in particular on the Jordan normal form of a matrix. In Chapter 4 we provide the full details of this classification, since we have not been able to find it in the literature.

In Chapter 5 we consider affine spaces on which a distance has been defined. Thus we have a model of classical Euclidean Geometry, where, for instance, Pythagoras' Theorem holds.

In Chapter 6 we study distance preserving maps, that is, the *Euclidean motions*. Since there are fewer Euclidean motions than affinities, their classification is simpler. We also introduce a natural equivalence relation among Euclidean motions, similar to that for affinities, and we characterize each equivalence class by a finite sequence of numbers (the coefficients of a polynomial and a *metric invariant*).

In Chapter 7 we study Euclidean motions in dimensions 1, 2 and 3. In dimension three, for instance, there are only three types of Euclidean motion: *helicoidals* (which include rotations, translations and the identity mapping), *glide reflections* (which include mirror symmetries) and *anti-rotations*.

In Chapter 8 we study *quadrics*, two quadrics being considered equivalent if there is an affinity that maps one onto the other. Quadrics are the zeros of quadratic polynomials, and therefore they are the most natural objects to consider after straight lines (the zeros of linear polynomials). From this perspective, there are only three inequivalent quadrics (conics) in the plane: the ellipse, the hyperbola and the parabola. We also give the complete list of quadrics in three dimensions.

In Chapter 9 we study quadrics, this time considering two quadrics to be equivalent if there is a Euclidean motion that maps one onto the other. From this perspective there are infinitely many quadrics (conics) in the plane, since ellipses, parabolas and hyperbolas of different sizes are inequivalent to one another. Nevertheless, we shall give the classification in dimensions two and three, representing the quadrics by a finite sequence of real numbers. For example, in the plane, there are as many ellipses as pairs (a,b) of real numbers, with $0 < a \leq b$ and as many hyperbolas as pairs (a,b) of real numbers, with $0 < a$, $0 < b$.

We give a faithful list of all quadrics in arbitrary dimensions. For this we need to introduce a suitable definition of a *good order* among various real numbers. Most textbooks are not concerned with the faithfulness of this list, that is, that each quadric appears in the list once and only once; for this reason this notion of good order is, as far as we know, new in this context.

Finally we have collected together in the appendices the results from linear algebra that we have used in the text: bilinear maps and their diagonalization, isometries, the classification of isometries, diagonalization of symmetric bilinear maps, the method of completing the squares, orthogonal diagonalization, simultaneous diagonalization (the spectral theorem), the Nullstellensatz, and so on.

There are many interesting books on Affine Geometry, treated in many cases from the viewpoint of projective geometry. In order to complement the presentation given in this book the reader may also wish to consult, for example, [2–5, 11, 14–16, 18, 19, 22, 28, 29, 31–34] or [36].

Barcelona, Spain Agustí Reventós Tarrida

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Reventós Tarrida, A.
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