2
Affinities

2.1 Introduction

From the geometrical point of view, the most natural maps between affine spaces are those taking collinear points to collinear points. We shall see that these maps are also the most natural from the algebraic point of view: they essentially coincide with the affinities.

2.2 Definition of Affinity

Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be affine spaces over the k-vector spaces E_1 and E_2, respectively.

Let us fix a point $P \in \mathbb{A}_1$. Then every map $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ induces a map

$$\tilde{f}_P : E_1 \longrightarrow E_2$$

defined by the formula

$$\tilde{f}_P(v) = \overrightarrow{f(P)f(Q)},$$

where $Q \in \mathbb{A}_1$ is the unique point such that $\overrightarrow{PQ} = v$.

This map $\tilde{f}_P$ between the vector spaces E_1 and E_2 is not in general linear. We shall say that $\tilde{f}_P$ is the map induced by the map f and the point P.

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Definition 2.1

A map $f:\mathbb{A}_1 \longrightarrow \mathbb{A}_2$ between two affine spaces is called an *affinity* if the map $\tilde{f}_P : E_1 \longrightarrow E_2$ induced by f and a point $P \in \mathbb{A}_1$ on the corresponding k-vector spaces is a linear map.

In this case $\tilde{f}_P$ does not depend on the point, that is, $\tilde{f}_P = \tilde{f}_Q$ for all $P, Q \in \mathbb{A}_1$. In fact, if $v = \overrightarrow{QR}$, we have

$$\begin{aligned}\tilde{f}_Q(v) &= \overrightarrow{f(Q)f(R)} \\ &= \overrightarrow{f(Q)f(P)} + \overrightarrow{f(P)f(R)} \\ &= -\tilde{f}_P(\overrightarrow{PQ}) + \tilde{f}_P(\overrightarrow{PR}) \\ &= -\tilde{f}_P(\overrightarrow{PQ}) + \tilde{f}_P(\overrightarrow{PQ} + \overrightarrow{QR}) \\ &= \tilde{f}_P(\overrightarrow{QR}) \\ &= \tilde{f}_P(v).\end{aligned}$$

Note that we have only used the fact that $\tilde{f}_P$ preserves vector addition. If $\tilde{f}_P$ preserves vector addition but does not preserve scalar multiplication, i.e., $\tilde{f}_P(\lambda v) \neq \lambda \tilde{f}_P(v)$, we still have $\tilde{f}_P = \tilde{f}_Q$ for all $P, Q \in \mathbb{A}_1$, but f is not an affinity.

Since all of the linear maps $\tilde{f}_P$ are equal for an affinity f, i.e. they do not depend on the point P, we shall denote this map simply by $\tilde{f}$.

Thus, if $f:\mathbb{A}_1 \longrightarrow \mathbb{A}_2$ is an affinity, there is a linear map $\tilde{f}: E_1 \longrightarrow E_2$ such that, for every pair of points $P, Q \in \mathbb{A}_1$,

$$\boxed{\tilde{f}(\overrightarrow{PQ}) = \overrightarrow{f(P)f(Q)}}$$

Since

$$f(Q) = f(P) + \overrightarrow{f(P)f(Q)},$$

we have

$$f(P + \overrightarrow{PQ}) = f(Q) = f(P) + \tilde{f}(\overrightarrow{PQ}),$$

and, since $\overrightarrow{PQ}$ is an arbitrary vector, we have, for every point $P \in \mathbb{A}_1$ and for every vector $v \in E_1$,

$$\boxed{f(P+v) = f(P) + \tilde{f}(v)}$$

This equality is equivalent to the commutativity of the following diagram

$$\begin{array}{ccc} \mathbb{A}_1 \times E_1 & \longrightarrow & \mathbb{A}_1 \\ {\scriptstyle f\times\tilde{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbb{A}_2 \times E_2 & \longrightarrow & \mathbb{A}_2 \end{array}$$

We summarize these comments in the following proposition.

Proposition 2.2

A map between two affine spaces $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ is an affinity if and only if there exists a linear map $\tilde{f} : E_1 \longrightarrow E_2$ between the corresponding k-vector spaces such that

$$f(P+v) = f(P) + \tilde{f}(v) \quad \text{for all } P \in \mathbb{A}_1 \text{ and } v \in E_1.$$

Proof

If f is an affinity, we take $\tilde{f} = \tilde{f}_P$, for some P, and we are done.

Conversely, if there exists a linear map $\tilde{f}$ with this property, since $Q = P + \overrightarrow{PQ}$, for all $P, Q \in \mathbb{A}_1$, we have

$$f(Q) = f(P) + \tilde{f}(\overrightarrow{PQ}),$$

and hence, $\tilde{f}(\overrightarrow{PQ}) = \overrightarrow{f(P)f(Q)} = f_P(\overrightarrow{PQ})$, that is, $\tilde{f}_P = \tilde{f}$. Thus, $\tilde{f}_P$ is linear, and f is an affinity. □

It is clear that if such an $\tilde{f}$ exists, it is unique.

Note that if $f = \mathrm{id}$, that is, $f(P) = P$, for all $P \in \mathbb{A}$, then f is an affinity with $\tilde{f} = \mathrm{id}$, that is, $\tilde{f}(v) = v$, for all $v \in E$. This can only happen when we have $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_1$, that is, when the source and target affine spaces are the same: the same set, the same associated vector space and the same action. For instance, the identity map $\mathrm{id} : \mathbb{R}(x) \longrightarrow \mathbb{R}(x)$ is not an affinity between the affine spaces k_1 and k_2 considered in Observation 1.3, page 5.

2.3 First Properties

In this section $\mathbb{A}_1$ and $\mathbb{A}_2$ are affine spaces over the k-vector spaces E_1 and E_2, respectively.

Proposition 2.3 (Uniqueness)

Let $f, g : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ be affinities that coincide on some point $P \in \mathbb{A}_1$, that is $f(P) = g(P)$, and which have the same associated linear map, $\tilde{f} = \tilde{g}$. Then $f = g$.

Proof

Let $Q \in \mathbb{A}_1$. Then

$$\begin{aligned} f(Q) &= f(P + \overrightarrow{PQ}) \\ &= f(P) + \tilde{f}(\overrightarrow{PQ}) \\ &= g(P) + \tilde{g}(\overrightarrow{PQ}) \\ &= g(P + \overrightarrow{PQ}) \\ &= g(Q). \end{aligned}$$

Hence, $f = g$. □

Proposition 2.4 (Existence)

Let $\phi : E_1 \longrightarrow E_2$ be a linear map and suppose given two points $P \in \mathbb{A}_1$ and $Q \in \mathbb{A}_2$. Then there exists a unique affinity $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ such that $f(P) = Q$ and $\tilde{f} = \phi$.

Proof

Uniqueness follows from the above proposition.

Let us prove the existence. Define $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ by

$$f(X) = Q + \phi(\overrightarrow{PX}) \quad \text{for all } X \in \mathbb{A}_1.$$

Then, clearly $f(P) = Q$. Moreover, $\tilde{f}_P = \phi$. In fact,

$$\tilde{f}_P(\overrightarrow{PX}) = \overrightarrow{f(P)f(X)} = \overrightarrow{Qf(X)} = \phi(\overrightarrow{PX}).$$

In particular, $\tilde{f}_P$ is linear, and hence f is an affinity with associated linear map $\tilde{f} = \phi$. □

Theorem 2.5 (Transitivity)

Let $P_1, \dots, P_r$ be affinely independent points in an affine space $\mathbb{A}_1$. Let $Q_1, \dots, Q_r$ be points in an affine space $\mathbb{A}_2$. Then there exists an affinity $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ such that $f(P_i) = Q_i$, for $i = 1, \dots, r$. If $r = \dim \mathbb{A}_1 + 1$, then this affinity is unique.

Proof

Since the points $P_1, \dots, P_r$ are affinely independent, the vectors $\overrightarrow{P_1P_2}, \dots, \overrightarrow{P_1P_r}$ are linearly independent. We know (see, for instance, [8], page 288) that there exists a linear map $\phi : E_1 \longrightarrow E_2$ such that

$$\phi(\overrightarrow{P_1P_i}) = \overrightarrow{Q_1Q_i}, \quad i = 1, \dots, r.$$

(Notice that the points Q_i are neither necessarily affinely independent nor distinct.)

We take, using Proposition 2.4, the unique affinity $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ such that $f(P_1) = Q_1$ and such that $\tilde{f} = \phi$. Clearly

$$f(P_i) = f(P_1 + \overrightarrow{P_1P_i}) = f(P_1) + \tilde{f}(\overrightarrow{P_1P_i}) = Q_1 + \overrightarrow{Q_1Q_i} = Q_i.$$

If $r = \dim \mathbb{A}_1 + 1$, the vectors $\overrightarrow{P_1P_2}, \dots, \overrightarrow{P_1P_r}$ form a basis of E_1. In this case there exists (see, for instance, [8], page 288) a unique linear map $\phi : E_1 \longrightarrow E_2$ such that

$$\phi(\overrightarrow{P_1P_i}) = \overrightarrow{Q_1Q_i}, \quad i = 1, \dots, r.$$

But any affinity taking the points P_i to the points Q_i has the above linear map ϕ as associated linear map. Hence, by Proposition 2.3, this affinity is unique. □

Proposition 2.6

Let $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ be an affinity with associated linear map $\tilde{f}$. Then f is injective if and only if $\tilde{f}$ is injective and f is surjective if and only if $\tilde{f}$ is surjective.

Proof

Let us assume that f is injective, and suppose $\tilde{f}(v) = \vec{0}$. Let $v = \overrightarrow{PQ}$. We have

$$0 = \tilde{f}(\overrightarrow{PQ}) = \overrightarrow{f(P)f(Q)},$$

and hence $f(P) = f(Q)$. This implies $P = Q$ and $v = \overrightarrow{PQ} = \vec{0}$, that is, $\tilde{f}$ is injective.

Assume now that $\tilde{f}$ is injective and suppose $f(P) = f(Q)$. Then $\tilde{f}(\overrightarrow{PQ}) = \overrightarrow{f(P)f(Q)} = \vec{0}$, and hence, $\overrightarrow{PQ} = \vec{0}$, that is, $P = Q$, and f is injective.

Assume now that f is surjective and let $v \in E_2$. Let $v = \overrightarrow{P'Q'}$ and choose P, Q such that $f(P) = P'$ and $f(Q) = Q'$. Then $\tilde{f}(\overrightarrow{PQ}) = \overrightarrow{P'Q'} = v$, and hence $\tilde{f}$ is surjective.

Assume now that $\tilde{f}$ is surjective and let $Q \in \mathbb{A}_2$. Take any point $P \in \mathbb{A}_1$ and a vector $v \in E_1$ such that $\tilde{f}(v) = \overrightarrow{f(P)Q}$. Then $f(P + v) = f(P) + \tilde{f}(v) = f(P) + \overrightarrow{f(P)Q} = Q$, and hence f is surjective. □

In particular, f is bijective if and only if $\tilde{f}$ is bijective.

Observation 2.7

Let $P_1, \dots, P_{n+1} \in \mathbb{A}$ and $Q_1, \dots, Q_{n+1} \in \mathbb{A}$ be, respectively, affinely independent points. We know, from Theorem 2.5, that there exists a unique affinity f such that $f(P_i) = Q_i$ and a unique affinity g such that $g(Q_i) = P_i$, $i = 1, \dots, n$.

But, as we saw in the proof of Theorem 2.5, $\tilde{f} = \tilde{g}^{-1}$. Hence, by Proposition 2.6, f is bijective and $f = g^{-1}$.

Observation 2.8

Let $\mathbb{A}$ be an affine space over a k-vector space E. To give an affine frame $\mathcal{R} = \{P; (e_1, \dots, e_n)\}$ in $\mathbb{A}$ is equivalent to giving a bijective affinity between $\mathbb{A}$ and k^n. For this reason we say that an affine space is "essentially" k^n.

In fact, this bijective affinity is given simply by *taking coordinates*:

$$\begin{aligned} \mathbb{A} &\xrightarrow{f} k^n \\ Q &\longmapsto (q_1, \dots, q_n), \end{aligned}$$

where $\overrightarrow{PQ} = q_1 e_1 + \cdots + q_n e_n$.

It is clear that f is a bijective map. To see that f is an affinity we compute $\tilde{f}_P$. Let $v \in E$ and let $Q \in \mathbb{A}$ be the unique point such that $v = \overrightarrow{PQ}$. Then

$$\tilde{f}_P(v) = \tilde{f}_P(\overrightarrow{PQ}) = \overrightarrow{f(P)f(Q)} = (q_1, \dots, q_n),$$

since $f(Q) = (q_1, \dots, q_n)$ and $f(P) = (0, \dots, 0)$.

That is, $\tilde{f}_P$ sends the vector of E with components $(q_1, \dots, q_n)$ in the basis $(e_1, \dots, e_n)$ to the vector $(q_1, \dots, q_n) \in k^n$.

Hence,

$$\tilde{f}_P(u+v) = \tilde{f}_P(u) + \tilde{f}_P(v), \quad u, v \in E,$$

$$\tilde{f}_P(\lambda u) = \lambda \tilde{f}_P(u), \quad u \in E, \lambda \in k,$$

since the components of $u+v$ are the components of u plus the components of v, and the components of λu are the components of u multiplied by λ.

Hence, $\tilde{f}_P$ is linear and f is an affinity.

2.4 The Affine Group

We shall see that the set of all bijective affinities from an affine space into itself has the structure of a group. The group operation is, of course, composition of affinities. We begin with some slightly more general results.

Proposition 2.9

Let $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ and $g : \mathbb{A}_2 \longrightarrow \mathbb{A}_3$ be affinities. Then their composition $g \circ f : \mathbb{A}_1 \longrightarrow \mathbb{A}_3$ is an affinity with associated linear map $\tilde{g} \circ \tilde{f}$.

Proof

For each $P \in \mathbb{A}_1$ and each $v \in E_1$ we have

$$(g \circ f)(P+v) = g(f(P) + \tilde{f}(v)) = (g \circ f)(P) + \tilde{g} \circ \tilde{f}(v).$$

By Proposition 2.2, $g \circ f$ is an affinity with

$$\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}.$$

□

Proposition 2.10

Let $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ be a bijective affinity. Then its inverse $f^{-1} : \mathbb{A}_2 \longrightarrow \mathbb{A}_1$ is an affinity with associated linear map $\tilde{f}^{-1}$.

Proof

For each $P \in \mathbb{A}_1$ and each $v \in E_1$ we have

$$f^{-1}(P+v) = f^{-1}(P) + \tilde{f}^{-1}(v),$$

as can be seen directly by applying f to both sides of this equality. By Proposition 2.2, f^{-1} is an affinity with

$$\widetilde{f^{-1}} = \tilde{f}^{-1}.$$

□

Theorem 2.11 (Affine Group)

The set of all bijective affinities from an affine space $\mathbb{A}$ into itself is a group with respect to composition of maps, called the *affine group* or *group of affinities*, and is denoted $\mathbb{G}A$.

Proof

This is an immediate consequence of the above Propositions 2.9 and 2.10. Note that composition of affinities is associative and that the unit element is the identity. □

The natural action of the affine group $\mathbb{G}A$ on the space $\mathbb{A}$ itself is given by

$$\begin{aligned} \mathbb{G}A \times \mathbb{A} &\longrightarrow \mathbb{A} \\ f, P &\longmapsto f(P). \end{aligned}$$

Thus, as noted in Observation 2.7, the group of affinities of the straight line acts simply transitively over ordered pairs of points (given two ordered pairs of points, each pair formed by different points, there exists a unique affinity taking one pair onto the other), the group of affinities of the plane acts simply transitively over ordered triples of points (given two ordered triples of points, each triple formed by different non-collinear points, there exists a unique affinity taking one triple onto the other), etc. Recall the concept of a *simply transitive action* on points (not on pairs, triples, etc.) on page 2.

Now, following F. Klein, we can say that *Affine Geometry is the study of the properties of the figures of* $\mathbb{A}$ *which are invariant under the action of the affine group* $\mathbb{G}A$.

Observation 2.12

When there is a bijective affinity between two affine spaces we say that these affine spaces are isomorphic. Recall that the concept of affinity is only meaningful when the vector spaces associated to the corresponding affine spaces are modeled on the same field k. Two affine spaces are isomorphic if and only if they have the same dimension. Concretely we have:

Proposition 2.13

Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be affine spaces over the k-vector spaces E_1 and E_2, respectively. Then $\mathbb{A}_1$ and $\mathbb{A}_2$ are isomorphic if and only if they have the same dimension.

Proof

It is clear, by Proposition 2.6, that if the spaces are isomorphic then they have the same dimension.

Conversely, if the spaces have the same dimension, we take an isomorphism $\phi : E_1 \longrightarrow E_2$ between the associated vector spaces, which have, by definition, the same dimension. By Proposition 2.4, there exists an affinity $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ with $\tilde{f} = \phi$, which, by Proposition 2.6, is bijective. □

2.5 Affinities and Linear Varieties

Proposition 2.14

Affinities take linear varieties to linear varieties. Concretely, let $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ be an affinity, $L_1 = P + [F]$ a linear variety of $\mathbb{A}_1$, and $L_2 = Q + [G]$ a linear variety of $\mathbb{A}_2$. Then

$$f(P + [F]) = f(P) + [\tilde{f}(F)],$$
$$f^{-1}(Q + [G]) = P + [\tilde{f}^{-1}(G)], \quad \text{if } P \in f^{-1}(Q + [G]).$$

Proof

Since $f(P + v) = f(P) + \tilde{f}(v)$ for all $P \in \mathbb{A}_1$ and $v \in E_1$, the first equality is evident.

To prove the second equality we observe that P is any point such that $\overrightarrow{Qf(P)} \in G$. Note that in some cases this point P does not exist; in these cases we have

$$f^{-1}(Q + [G]) = \emptyset.$$

Note also that a point $X \in \mathbb{A}_1$ belongs to $f^{-1}(Q + [G])$ if and only if $\overrightarrow{Qf(X)} \in G$.

Finally, a point $X \in \mathbb{A}_1$ belongs to $P + [\tilde{f}^{-1}(G)]$ if and only if $\tilde{f}(\overrightarrow{PX}) \in G$. Since

$$\tilde{f}(PX) = \overrightarrow{f(P)f(X)} = \overrightarrow{f(P)Q} + \overrightarrow{Qf(X)},$$

we have $X \in f^{-1}(Q + [G])$ if and only if $X \in P + \tilde{f}^{-1}(G)$. □

Corollary 2.15

Injective affinities take linear varieties to linear varieties of the same dimension. In particular, they take straight lines to straight lines.

Proof

Injectivity ensures $\dim \tilde{f}(F) = \dim F$. □

The Fundamental Theorem of Affine Geometry, which we shall meet later (Theorem 2.46, page 81), deals with the converse of this proposition. The question is: is a map taking straight lines to straight lines necessarily an affinity?

Proposition 2.16

Injective affinities preserve the simple ratio.

Proof

Let $A, B, C \in \mathbb{A}_1$ be three different collinear points such that $\overrightarrow{AB} = \lambda \overrightarrow{AC}$. Let $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ be an injective affinity. By the previous corollary, the three points $f(A), f(B), f(C) \in \mathbb{A}_2$, which are distinct, are collinear. Applying $\tilde{f}$ to the above equality one obtains

$$\tilde{f}(\overrightarrow{AB}) = \overrightarrow{f(A)f(B)} = \tilde{f}(\lambda \overrightarrow{AC}) = \lambda \overrightarrow{f(A)f(C)},$$

and hence

$$(f(A), f(B), f(C)) = \lambda = (A, B, C).$$

□

2.6 Equations of Affinities

Let $f : \mathbb{A}_1 \longrightarrow A_2$ be an affinity, $\mathcal{R}_1 = \{P_1; (e_1, \ldots, e_n)\}$ an affine frame in $\mathbb{A}_1$ and $\mathcal{R}_2 = \{P_2; (v_1, \ldots, v_m)\}$ an affine frame in $\mathbb{A}_2$.

The aim of this section is to relate the coordinates $(x_1, \ldots, x_n)$ of a point $X \in \mathbb{A}_1$ to the coordinates $(y_1, \ldots, y_m)$ of the point $f(X) \in \mathbb{A}_2$.

For this we write

$$\overrightarrow{P_2 f(P_1)} = \sum_{j=1}^{m} a_j v_j,$$

$$\overrightarrow{P_1 X} = \sum_{i=1}^{n} x_i e_i,$$

$$\overrightarrow{P_2 f(X)} = \sum_{j=1}^{m} y_j v_j.$$

That is,

$$f(P_1) = (a_1, \ldots, a_m),$$

$$X = (x_1, \ldots, x_n),$$

$$f(X) = (y_1, \ldots, y_m).$$

We also write

$$\tilde{f}(e_i) = \sum_{j=1}^{m} a_{ji} v_j.$$

That is, $A = (a_{ij})$ is the matrix of $\tilde{f}$ with respect to the bases $\mathcal{B}_1 = (e_1, \ldots, e_n)$ and $\mathcal{B}_2 = (v_1, \ldots, v_m)$, which is usually denoted by $M(\tilde{f}, \mathcal{B}_1, \mathcal{B}_2)$.

Then

$$\begin{aligned} \overrightarrow{P_2 f(X)} &= \overrightarrow{P_2 f(P_1)} + \overrightarrow{f(P_1) f(X)} \\ &= \overrightarrow{P_2 f(P_1)} + \tilde{f}(\overrightarrow{P_1 X}), \end{aligned}$$

that is,

$$\sum_{j=1}^{m} y_j v_j = \sum_{j=1}^{m} a_j v_j + \sum_{i=1}^{n} x_i \sum_{j=1}^{m} a_{ji} v_j.$$

Equating coefficients we get

$$y_j = a_j + \sum_{i=1}^{n} x_i a_{ji}, \quad j = 1, \ldots, m.$$

Matricially

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} + \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

or, in a more compact form,

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \\ 1 \end{pmatrix} = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & a_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & a_m \\ \hline 0 & \dots & 0 & 1 \end{array}\right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix}.$$

Or, in an even more abbreviated form,

$$\boxed{\begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}} \tag{2.1}$$

or

$$\boxed{y = Ax + a}$$

with

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \qquad A = (a_{ij}), \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad a = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}.$$

We shall use the notation

$$M(f, \mathcal{R}_1, \mathcal{R}_2) = \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$$

to indicate the matrix of f in the affine frames $\mathcal{R}_1$ and $\mathcal{R}_2$. When $\mathcal{R}_1 = \mathcal{R}_2$ (and thus $\mathbb{A}_1 = \mathbb{A}_2$) we shall simply write $M(f, \mathcal{R}_1)$.

Sometimes, when these affine frames are implicitly given, we shall simply write

$$M(f) = \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}.$$

Now it is very easy to construct examples of affinities: we simply give the two matrices A and a.

Observation 2.17

Given the affine frames $\mathcal{R}_1 = \{P_1; \mathcal{B}_1\}$ of $\mathbb{A}_1$, $\mathcal{R}_2 = \{P_2; \mathcal{B}_2\}$ of $\mathbb{A}_2$, and a matrix

$$M = \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix},$$

there exists a unique affinity $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ such that

$$M(f, \mathcal{R}_1, \mathcal{R}_2) = \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}.$$

In fact, we define f giving its associated linear map $\tilde{f}$ by the condition

$$M(\tilde{f}, \mathcal{B}_1, \mathcal{B}_2) = A$$

and its value on a point by the condition $f(P_1) = a^{\mathsf{T}}$. That is, $f(P_1)$ is the point with coordinates $(a_1, \dots, a_n)$ in $\mathcal{R}_2$. From Proposition 2.4, such an affinity exists and is unique.

Observation 2.18

We have seen that, once we fix affine frames, affinities are given by affine equations of the form $y = Ax + a$. If we change these affine frames, the equations will change, but there will still be an affine relationship between the coordinates of a point and those of its transformed image.

For instance, affinities from k^n to k^m are the maps $f : k^n \longrightarrow k^m$, with $f(x_1, \dots, x_n) = (y_1, \dots, y_m)$, such that

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix},$$

with $A \in \mathcal{M}_{m \times n}(k)$. It suffices to identify the components $(x_1, \dots, x_n)$ of a point with its coordinates in the canonical affine frame. Here, recall, $\mathcal{M}_{m \times n}(k)$ is the set of matrices with m rows and n columns with elements in the field k.

Observation 2.19

Given two affine frames $\mathcal{R}$ and $\mathcal{R}'$ in an affine space $\mathbb{A}$, we have

$$M(\mathrm{id}, \mathcal{R}', \mathcal{R}) = M(\mathcal{R}', \mathcal{R}),$$

where $M(\mathcal{R}', \mathcal{R})$ is the matrix of the change of coordinates introduced on page 19.

Equation (2.1), page 58, tells us that the relationship between the coordinates x of a point and the coordinates y of the image of this point by an affinity is given by only one matrix, as is the case for linear maps between vector spaces. So we can manipulate affinities "as if" they are linear maps.

In particular, by arguments similar to those given in the composition of linear maps (see [8], page 300), and by the change of basis formula (see [8], page 302), we obtain the next two propositions.

Proposition 2.20

The matrix of the composition of affinities is the product of the matrices of these affinities. Concretely, if $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ and $g : \mathbb{A}_2 \longrightarrow \mathbb{A}_3$ are affinities and we fix affine frames $\mathcal{R}_i$ on $\mathbb{A}_i$, $i = 1, 2, 3$, then

$$M(g \circ f, \mathcal{R}_1, \mathcal{R}_3) = M(g, \mathcal{R}_2, \mathcal{R}_3) \cdot M(f, \mathcal{R}_1, \mathcal{R}_2).$$

Proof

If

$$\begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}$$

are the equations of f, and

$$\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} B & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1 \end{pmatrix}$$

are the equations of g, then

$$\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} B & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}$$

are the equations of $g \circ f$. Since the product of matrices is associative, we have the result. □

We summarize this result by simply writing

$$\boxed{M(g \circ f) = M(g) \cdot M(f)}$$

Corollary 2.21

The matrix of the inverse of an affinity is the inverse of the matrix of this affinity. Concretely, if $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ is a bijective affinity and we fix affine frames $\mathcal{R}_i$ on $\mathbb{A}_i$, $i = 1, 2$, then

$$M(f^{-1}, \mathcal{R}_2, \mathcal{R}_1) = M(f, \mathcal{R}_1, \mathcal{R}_2)^{-1}.$$

Proof

We apply the above proposition with $g = f^{-1}$ and $\mathcal{R}_1 = \mathcal{R}_3$. □

We summarize this result by simply writing

$$\boxed{M(f^{-1}) = M(f)^{-1}}$$

Proposition 2.22

Let $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ be an affinity, $\mathcal{R}_1, \mathcal{R}'_1$ affine frames in $\mathbb{A}_1$, and $\mathcal{R}_2, \mathcal{R}'_2$ affine frames in $\mathbb{A}_2$. Then

$$M(f, \mathcal{R}_1, \mathcal{R}_2) = M(\mathrm{id}_2, \mathcal{R}_2, \mathcal{R}'_2)^{-1} M(f, \mathcal{R}'_1, \mathcal{R}'_2) M(\mathrm{id}_1, \mathcal{R}_1, \mathcal{R}'_1).$$

Proof

This is a consequence of Proposition 2.20 and the equality

$$f \circ \mathrm{id}_1 = \mathrm{id}_2 \circ f,$$

where id_i denotes the identity map of $\mathbb{A}_i$, $i = 1, 2$. □

In the particular case in which $\mathbb{A}_1 = \mathbb{A}_2$, we can take $\mathcal{R}_1 = \mathcal{R}_2$ and $\mathcal{R}'_1 = \mathcal{R}'_2$ and we have

$$\boxed{M(f, \mathcal{R}_1, \mathcal{R}_1) = C^{-1} M(f, \mathcal{R}'_1, \mathcal{R}'_1) C}$$

where $C = M(\mathrm{id}, \mathcal{R}_1, \mathcal{R}'_1) = M(\mathcal{R}_1, \mathcal{R}'_1)$ is the matrix of the change of coordinates.

We have already stated that we will write this equation as

$$\boxed{M(f, \mathcal{R}_1) = C^{-1} M(f, \mathcal{R}'_1) C} \tag{2.2}$$

Example 2.23

Find, in the affine plane $\mathbb{R}^2$, the equations of an affinity sending the triangle $\triangle ABC$ onto the triangle $\triangle A'B'C'$, with $A=(1,1)$, $B=(3,2)$, $C=(4,4)$, $A'=(-1,0)$, $B'=(-5,2)$, $C'=(7,4)$.

Solution

It follows, from Theorem 2.5, that such an affinity exists and is unique if we assume that the points A,B,C are mapped, respectively, to the points A',B',C'. We can also consider, for example, the case where A,B,C are mapped, respectively, to A',C',B', or to any other permutation of A',B',C'. Here we only study the first case.

Let us consider the affine frames

$$\mathcal{R}=\{A;(\overrightarrow{AB},\overrightarrow{AC})\}=\{(1,1);((2,1),(3,3))\},$$
$$\mathcal{R}'=\{A';(\overrightarrow{A'B'},\overrightarrow{A'C'})\}=\{(-1,0);((-4,2),(8,4))\}.$$

By definition of the matrix associated to an affinity, we have

$$M(f,\mathcal{R},\mathcal{R}')=I_2,$$

where f is the affinity we are looking for.

We shall use the notation I_n for the $n\times n$ identity matrix, i.e. the diagonal matrix with 1s on the diagonal and zeros elsewhere.

By Proposition 2.22 we have

$$\begin{aligned}M(f,\mathcal{C},\mathcal{C})&=M(\mathrm{id},\mathcal{C},\mathcal{R}')^{-1}M(f,\mathcal{R},\mathcal{R}')M(\mathrm{id},\mathcal{C},\mathcal{R})\\&=M(\mathcal{R}',\mathcal{C})M(\mathcal{R},\mathcal{C})^{-1},\end{aligned}$$

where $\mathcal{C}$ is the canonical affine frame.

These two matrices are easily computable. Concretely we have

$$M(\mathcal{R}',\mathcal{C})=\begin{pmatrix}-4&8&-1\\2&4&0\\0&0&1\end{pmatrix}\quad\text{and}\quad M(\mathcal{R},\mathcal{C})=\begin{pmatrix}2&3&1\\1&3&1\\0&0&1\end{pmatrix}$$

and the affinity we are looking for is

$$\begin{cases}x'=-\dfrac{20}{3}x+\dfrac{28}{3}y-\dfrac{11}{3},\\[2mm] y'=\dfrac{2}{3}x+\dfrac{2}{3}y-\dfrac{4}{3}.\end{cases}$$

□

2.7 Invariant Varieties

In this section we only consider affinities from an affine space $\mathbb{A}$ into itself.

A point $P \in \mathbb{A}$ is a *fixed point* of an affinity $f : \mathbb{A} \longrightarrow \mathbb{A}$ if and only if $f(P) = P$.

A linear variety $L = P + [F] \subset \mathbb{A}$ is *invariant* under an affinity $f : \mathbb{A} \longrightarrow \mathbb{A}$ if and only if $f(L) \subset L$.

In particular, fixed points are invariant linear varieties of dimension zero.

Proposition 2.24

A linear variety $L = P + [F]$ of the affine space $\mathbb{A}$ is invariant under an affinity $f : \mathbb{A} \to \mathbb{A}$ if and only if
(1) $\overrightarrow{Pf(P)} \in F$; and
(2) $\tilde{f}(F) \subset F$.

Proof

By Proposition 2.14 we have

$$f(L) = f(P) + [\tilde{f}(F)] = P + \overrightarrow{Pf(P)} + [\tilde{f}(F)],$$

and hence $f(L) \subset L$ if and only if $\overrightarrow{Pf(P)} \in F$ and $\tilde{f}(F) \subset F$. □

In the next corollary we use the notion of an eigenvector, which is reviewed on page 333 of the Appendix.

Corollary 2.25

A straight line $L = P + \langle v \rangle$ of an affine space $\mathbb{A}$ is invariant under an affinity $f : \mathbb{A} \to \mathbb{A}$ if and only if there exist $\lambda, \mu \in k$ such that
(1) $\overrightarrow{Pf(P)} = \lambda v$; and
(2) $\tilde{f}(v) = \mu v$ (that is, v *is an eigenvector* of $\tilde{f}$).

Proof

This is the above proposition in dimension 1. □

Notice that the condition that the direction vector of a straight line L is an eigenvector of $\tilde{f}$ is a necessary but not sufficient condition for L to be invariant under f. For instance, the straight line $y=1$ of $\mathbb{R}^2$ is not invariant under the affinity

$$\begin{cases} x' = x, \\ y' = y+1, \end{cases}$$

although its direction vector is an eigenvector of $\tilde{f}$, since $\tilde{f} = \mathrm{id}$. However, the straight line $x=0$ is invariant, since its direction vector is $v=(0,1)$ and, taking $P=(0,0)$, we have $\overrightarrow{Pf(P)} = v$.

Proposition 2.26

If f is a bijective affinity and $L = P + [F]$ is an invariant linear variety, then $f(L) = L$.

Proof

In this case $\tilde{f}$ is also bijective, and hence $\tilde{f}(F) = F$. Thus,

$$f(L) = f(P) + [\tilde{f}(F)] = P + \overrightarrow{Pf(P)} + [F] = P + [F] = L.$$

□

Note that we can have $f(L) = L$, but $f(P) \neq P$ for all $P \in L$. For instance, the translation T_u (see Section 2.8) leaves invariant any straight line with direction vector u, but it does not have any fixed points.

Proposition 2.27

The set $\mathrm{Fix}(f)$ of fixed points of an affinity f is either a linear variety directed by $\ker(\tilde{f} - \mathrm{id})$ or it is the empty set.

Proof

Let us assume that there exists a $P \in \mathbb{A}$ such that $f(P) = P$. Then, for each $u \in E$, $P+u$ is a fixed point of f if and only if

$$P + u = f(P+u) = f(P) + \tilde{f}(u) = P + \tilde{f}(u),$$

that is, if and only if $\tilde{f}(u) = u$, or, equivalently, $u \in \ker(\tilde{f} - \mathrm{id})$.

Hence,

$$\mathrm{Fix}(f) = P + [\ker(\tilde{f} - \mathrm{id})].$$

□

The study of fixed points will be of great interest in what follows, so we begin with the following result.

Proposition 2.28

An affinity $f : \mathbb{A} \longrightarrow \mathbb{A}$ has a unique fixed point if and only if the associated linear map $\tilde{f} : E \longrightarrow E$ does not have eigenvalue 1.

Proof

Let us assume that P is the unique fixed point. Since $\mathrm{Fix}(f) = P + [\ker(\tilde{f} - \mathrm{id})]$, it follows from the above proposition that $\ker(\tilde{f} - \mathrm{id}) = \{\vec{0}\}$, and hence there is no eigenvector with eigenvalue 1.

Conversely, let us assume that $\tilde{f}$ does not have eigenvalue 1.

This means

$$\ker(\tilde{f} - \mathrm{id}) = \{\vec{0}\}$$

and hence, by the above proposition, it only remains to prove that $\mathrm{Fix}(f) \neq \emptyset$.

For this, we look for a point Q and a vector v such that $f(Q + v) = Q + v$. That is, such that $f(Q) + \tilde{f}(v) = Q + v$. Equivalently, we are looking for a point Q and a vector v such that

$$\overrightarrow{Qf(Q)} = -(\tilde{f} - \mathrm{id})(v). \tag{2.3}$$

But, since the kernel of $(\tilde{f} - \mathrm{id})$ is zero, $(\tilde{f} - \mathrm{id})$ is invertible. Hence, given any point Q, there exists a (unique) vector v satisfying (2.3), and so the point $P = Q + v$, with v thus constructed, is fixed. □

Using the equations of an affinity, we can give a very simple proof of Proposition 2.28.

Proposition 2.29 (Equations of fixed points)

Let us assume that the equations of an affinity f in some frame $\mathcal{R} = \{P; \mathcal{B}\}$ of an affine space $\mathbb{A}$ of dimension n is $x' = Ax + a$. Then f has a unique fixed point if and only if $\det(A - I_n) \neq 0$.

Proof

The point with coordinates x is a fixed point if and only if

$$x = Ax + a, \tag{2.4}$$

that is,

$$(A - I_n)x = -a.$$

But this system has a unique solution if and only if $\det(A - I_n) \neq 0$. □

Note that the condition $\det(A - I_n) \neq 0$ is equivalent to the condition that 1 is not an eigenvalue of A. But A is the matrix of $\tilde{f}$ in $\mathcal{B}$, and hence f has a unique fixed point if and only if $\tilde{f}$ does not have eigenvalue 1.

Equation (2.4) is the *equation of fixed points* of f.

2.8 Examples of Affinities

2.8.1 Translations

Definition 2.30

A *translation* is an affinity $f : \mathbb{A} \longrightarrow \mathbb{A}$ such that $\tilde{f} = \mathrm{id}$.

Proposition 2.31

An affinity f is a translation if and only if there is a $u \in E$ such that

$$f(Q) = Q + u \quad \text{for all } Q \in \mathbb{A}.$$

Proof

Since

$$\overrightarrow{Pf(P)} = \overrightarrow{PQ} + \overrightarrow{Qf(Q)} + \overrightarrow{f(Q)f(P)} = \overrightarrow{PQ} + \overrightarrow{Qf(Q)} + \tilde{f}(\overrightarrow{QP}),$$

every affinity satisfies the fundamental relation

$$\boxed{\overrightarrow{Pf(P)} + (\tilde{f} - \mathrm{id})\overrightarrow{PQ} = \overrightarrow{QfQ}}$$

Hence, if $\tilde{f} = \mathrm{id}$, we have $\overrightarrow{Pf(P)} = \overrightarrow{QfQ}$ for all $P, Q \in \mathbb{A}$. Let $u = \overrightarrow{Pf(P)}$. Then, for all $Q \in \mathbb{A}$, we have

$$f(Q) = Q + \overrightarrow{Qf(Q)} = Q + \overrightarrow{Pf(P)} = Q + u.$$

Conversely, if there is a $u \in E$ such that $u = \overrightarrow{Pf(P)} = \overrightarrow{Qf(Q)}$, for all $P, Q \in \mathbb{A}$, the *fundamental relation* directly implies $\tilde{f} = \mathrm{id}$. □

For this reason we shall denote translations by T_u and we will say that T_u is *the translation by vector* u, where u is called the *translation vector*.

Equations of Translations Let $\mathcal{R} = \{P; \mathcal{B}\}$ be an arbitrary affine frame and assume that the translation vector u has components $u = (u_1, \ldots, u_n)$ in $\mathcal{B}$. In particular, $T_u(P) = P + u = (u_1, \ldots, u_n)$. Then, by Section 2.6, we have

$$M(T_u, \mathcal{R}) = \begin{pmatrix} I_n & u \\ 0 & 1 \end{pmatrix},$$

or, equivalently,

$$\begin{cases} x'_1 = x_1 + u_1, \\ \quad\vdots \\ x'_n = x_n + u_n. \end{cases}$$

If $u \neq 0$, we can complete the translation vector to a basis $\mathcal{B} = (u, e_2, \ldots, e_n)$ of E, and then, in the affine frame $\mathcal{R}' = \{P; \mathcal{B}\}$, with $P \in \mathbb{A}$ arbitrary, we have

$$\begin{cases} x'_1 = x_1 + 1, \\ x'_2 = x_2, \\ \quad\vdots \\ x'_n = x_n. \end{cases}$$

Observation 2.32

The set of all translations of an affine space $\mathbb{A}$ is a group with respect to composition of maps. The identity element is translation by the vector $\vec{0}$. The group properties follow from the equalities $T_u \circ T_v = T_{u+v}$ (the composition of translations is a translation) and $T_u^{-1} = T_{-u}$ (the inverse of a translation is a translation). This group, denoted by $\mathbb{T}$, is a subgroup of the group of affinities $\mathbb{G}A$. Let

$$\begin{aligned} \Phi : \mathbb{G}A &\longrightarrow \text{End E} \\ f &\mapsto \tilde{f} \end{aligned}$$

be the map sending each affinity f to its associated endomorphism $\tilde{f}$.

By definition of translation we have

$$\mathbb{T} = \ker \Phi.$$

In particular, $\mathbb{T}$ is a normal subgroup of $\mathbb{G}A$, and, from the Isomorphism Theorem (see [8], page 284) and because Φ is surjective, we have

$$\mathbb{G}A/\mathbb{T} \cong \text{End E}.$$

2.8.2 Homotheties

Definition 2.33

A *homothety* is an affinity $f : \mathbb{A} \longrightarrow \mathbb{A}$ such that $\tilde{f} = \lambda\,\text{id}$, $\lambda \neq 0, 1$. λ is called the *similitude ratio* of the homothety.

Proposition 2.34

Homotheties have a unique fixed point.

Proof

Let us fix $P \in \mathbb{A}$ and assume that $Q \in \mathbb{A}$ is a fixed point. Then,

$$\begin{aligned} P + \overrightarrow{PQ} &= Q \\ &= f(Q) \\ &= f(P + \overrightarrow{PQ}) \\ &= f(P) + \lambda\overrightarrow{PQ} \\ &= P + \overrightarrow{Pf(P)} + \lambda\overrightarrow{PQ}. \end{aligned}$$

Hence, Q is a fixed point if and only if

$$\overrightarrow{Pf(P)} = (1 - \lambda)\overrightarrow{PQ},$$

that is, if and only if

$$Q = P + \frac{1}{1-\lambda}\overrightarrow{Pf(P)}. \tag{2.5}$$

Moreover, it is clear that Q is the unique fixed point. To see this let us assume that there are two different fixed points Q, Q'. Then

$$\tilde{f}(\overrightarrow{QQ'}) = \overrightarrow{f(Q)f(Q')} = \overrightarrow{QQ'},$$

implying $\tilde{f} = \mathrm{id}$, a contradiction since $\tilde{f} = \lambda\,\mathrm{id}$ with $\lambda \neq 1$. □

Note that, in particular, we have proved that for all $P, R \in \mathbb{A}$ we have

$$P + \frac{1}{1-\lambda}\overrightarrow{Pf(P)} = R + \frac{1}{1-\lambda}\overrightarrow{Rf(R)}.$$

Since homotheties are determined by the fixed point, called the *center* of the homothety, and by the similitude ratio λ, we shall denote by $h_{P,\lambda}$ the homothety with center P and similitude ratio λ.

Equations of homotheties Let $\mathcal{R} = \{P; (e_1, \ldots, e_n)\}$ be an affine frame with origin the unique fixed point P of the homothety and with $(e_1, \ldots, e_n)$ an arbitrary basis of E. It follows from Section 2.6 that

$$M(h_{P,\lambda}, \mathcal{R}) = \begin{pmatrix} \lambda I_n & 0 \\ 0 & 1 \end{pmatrix},$$

or, equivalently,

$$\begin{cases} x_1' = \lambda x_1, \\ \quad\vdots \\ x_n' = \lambda x_n. \end{cases}$$

Observation 2.35

The set of all homotheties of an affine space $\mathbb{A}$ is *not* a group with respect to composition of maps. The identity translation is not a homothety.

Even if we add the identity to the set of all homotheties, we still don't have a group, since *the composition of homotheties with different centers and inverse similitude ratios is a translation.*

Indeed, if we denote by $h_{X,\nu}$ the homothety with center $X \in \mathbb{A}$ and similitude ratio $\nu \in k$, it follows from Proposition 2.9 that

$$h_{P,\lambda} \circ h_{Q,\mu} = \begin{cases} T_{QQ'} & \lambda\mu = 1, \\ h_{R,\lambda\mu} & \lambda\mu \neq 1, \end{cases}$$

where $Q' = h_{P,\lambda}(Q)$ and

$$R = Q + \frac{1}{1-\lambda\mu}\overrightarrow{Qh_{P,\lambda}(Q)}.$$

We can arrive at the same conclusion using coordinates. Indeed, if we take an affine frame $\mathcal{R}$ with origin at Q, by Proposition 2.20 we have

$$\begin{aligned} M(h_{P,\lambda} \circ h_{Q,\mu}, \mathcal{R}) &= M(h_{P,\lambda}, \mathcal{R}) \circ M(h_{Q,\mu}, \mathcal{R}) \\ &= \begin{pmatrix} \lambda I_n & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu I_n & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda\mu I_n & a \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

which gives the above result, that is, $h_{P,\lambda} \circ h_{Q,\mu}$ is a homothety if $\lambda\mu \neq 1$, or a translation if $\lambda\mu = 1$. Here $n = \dim \mathbb{A}$.

Affinities such that $\tilde{f} = \lambda\,\mathrm{id}$ (homotheties and translations) are called *dilations*, and they constitute a group (see the definition of dilation given in the introduction, page viii).

2.8.3 Symmetries

Definition 2.36

A *symmetry* is an affinity $f : \mathbb{A} \longrightarrow \mathbb{A}$ such that $f^2 = \mathrm{id}$.

Note first that, for all points $P \in \mathbb{A}$, the point

$$Q = P + \frac{1}{2}\overrightarrow{Pf(P)}$$

is fixed.

Hence, the linear variety $\mathrm{Fix}(f)$ is non-empty. Let us assume $\dim \mathrm{Fix}(f) = r$. Recall $\mathrm{Fix}(f) = Q + [\ker(\tilde{f} - \mathrm{id})]$, where Q is any fixed point.

Since

$$u = \frac{1}{2}(u + \tilde{f}(u)) + \frac{1}{2}(u - \tilde{f}(u)), \quad \text{for all } u \in E,$$

we have

$$E = \ker(\tilde{f} - \mathrm{id}) \oplus \ker(\tilde{f} + \mathrm{id}),$$

since $\frac{1}{2}(u + \tilde{f}(u)) \in \ker(\tilde{f} - \mathrm{id})$, $\frac{1}{2}(u - \tilde{f}(u)) \in \ker(\tilde{f} + \mathrm{id})$ and, obviously, the intersection of these subspaces is the zero vector.

This is the decomposition given by the annihilating polynomial, see [8], page 361. Note that $f^2 = \mathrm{id}$ implies $\tilde{f}^2 = \mathrm{id}$, and hence $(\tilde{f} - \mathrm{id}) \circ (\tilde{f} + \mathrm{id}) = 0$. That is, the polynomial $(x-1)(x+1)$ is a multiple of the minimal polynomial.

Every point $P \in \mathbb{A}$ can be written as (see Figure 2.1)

$$P = Q + \overrightarrow{QP} = Q + u + v, \quad u \in \ker(\tilde{f} - \mathrm{id}),\ v \in \ker(\tilde{f} + \mathrm{id}),$$

where Q is the above fixed point.

Thus,

$$f(P) = f(Q + u + v) = Q + u - v = P - 2v.$$

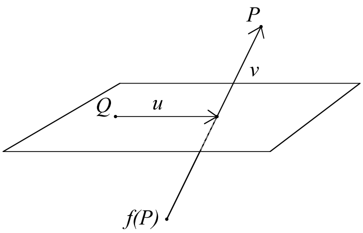

Figure 2.1. Symmetry

Note that if $r = 0$, the linear variety $\mathrm{Fix}(f)$ is a point. In this case f is said to be a *central symmetry*. In particular, $\ker(\tilde{f} - \mathrm{id}) = \{\vec{0}\}$ and $P = Q + v$ with $v \in \ker(\tilde{f} + \mathrm{id})$. The image of a point is given by $f(P) = P - 2v = P - 2\overrightarrow{QP}$.

If $r = 1$, $\mathrm{Fix}(f)$ is a straight line, and we say that f is an *axial symmetry*; if $r = 2$, $\mathrm{Fix}(f)$ is a plane, and we say that f is a *mirror symmetry*.

Equations of symmetries Let $\mathcal{R} = \{P; (e_1, \dots, e_n)\}$ be an affine frame with origin a fixed point P of the given symmetry, and let $(e_1, \dots, e_n)$ be a basis with $e_i \in \ker(\tilde{f} - \mathrm{id})$, $i = 1, \dots, r$, and $e_i \in \ker(\tilde{f} + \mathrm{id})$, $i = r+1, \dots, n$. Then, from Section 2.6, we have

$$M(f, \mathcal{R}) = \left(\begin{array}{ccc|c} 1 & & & 0 \\ & \ddots & & \vdots \\ & & -1 & 0 \\ \hline 0 & \dots & 0 & 1 \end{array}\right),$$

or, equivalently,

$$\begin{cases} x'_1 = x_1, \\ \quad\vdots \\ x'_r = x_r, \\ x'_{r+1} = -x_{r+1}, \\ \quad\vdots \\ x'_n = -x_n. \end{cases}$$

Observation 2.37

Each decomposition of the vector space E as a direct sum $E = F \oplus G$, together with a point P, gives rise to a symmetry $s_L : \mathbb{A} \longrightarrow \mathbb{A}$ defined by

$$s_L(P+v) = P + v_1 - v_2, \quad \text{for all } v = v_1 + v_2 \in E, \text{ with } v_1 \in F, v_2 \in G,$$

and called the *symmetry with respect to* $L = P + [F]$ *in the direction* G. Note that $s_F^2 = \mathrm{id}$.

If $\dim L = 0$, we have a *central symmetry*; if $\dim L = 1$, we have an *axial symmetry*; if $\dim L = 2$, we have a *mirror symmetry*.

2.8.4 Projections

Definition 2.38

A *projection* is an affinity $f : \mathbb{A} \longrightarrow \mathbb{A}$ such that $f^2 = f$.

In this case it is clear that $\mathrm{Fix}(f) = \Im(f)$. Hence, the linear variety $\mathrm{Fix}(f)$ is non-empty. Let us assume $\dim \mathrm{Fix}(f) = r$ and set $\mathrm{Fix}(f) = Q + [\ker(\tilde{f} - \mathrm{id})]$, where Q is any fixed point.

Since

$$u = \tilde{f}(u) + (u - \tilde{f}(u)), \quad \text{for all } u \in E,$$

we have

$$E = \ker(\tilde{f} - \mathrm{id}) \oplus \ker(\tilde{f}),$$

because $\tilde{f}(u) \in \ker(\tilde{f} - \mathrm{id})$, $(u - \tilde{f}(u)) \in \ker(\tilde{f})$ and, obviously, the intersection of these subspaces is the zero vector.

This is the decomposition induced by the annihilating polynomial of f, $x(x-1)$, see [8], page 361. In fact, $f^2 = f$ implies $\tilde{f}^2 = \tilde{f}$, and hence, $(\tilde{f} - \mathrm{id}) \circ \tilde{f} = 0$. That is, the polynomial $x(x-1)$ is a multiple of the minimal polynomial.

Every point $P \in \mathbb{A}$ can be written in a unique way as (see Figure 2.2)

$$P = Q + \overrightarrow{QP} = Q + u + v, \quad u \in \ker(\tilde{f} - \mathrm{id}),\ v \in \ker(\tilde{f}),$$

where Q is the above fixed point.

Thus

$$f(P) = f(Q + u + v) = Q + u = P - v.$$

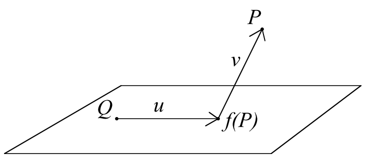

Figure 2.2. Projection

Note that if $r = 0$, the linear variety $\mathrm{Fix}(f)$ reduces to the point P. Then $f(X) = P$ for all $X \in \mathbb{A}$. If $r = \dim E$, then $f = \mathrm{id}$.

Equations of projections Let $\mathcal{R} = \{P; (e_1, \ldots, e_n)\}$ be an affine frame with origin a fixed point P of the symmetry, and let $(e_1, \ldots, e_n)$ be a basis with $e_i \in \ker(\tilde{f} - \mathrm{id})$, $i = 1, \ldots, r$ and $e_i \in \ker(\tilde{f})$, $i = r+1, \ldots, n$. Then, by Section 2.6, we have

$$M(f, \mathcal{R}) = \left(\begin{array}{ccc|c} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 0 & 0 \\ \hline 0 & \ldots & 0 & 1 \end{array}\right),$$

or, equivalently,

$$\begin{cases} x'_1 = x_1, \\ \quad\vdots \\ x'_r = x_r, \\ x'_{r+1} = 0, \\ \quad\vdots \\ x'_n = 0. \end{cases}$$

Observation 2.39

Each decomposition of the vector space E as a direct sum $E = F \oplus G$, together with a point P, gives rise to a projection $p_L : \mathbb{A} \longrightarrow \mathbb{A}$ defined by

$$p_L(P+v) = P + v_1, \text{ for all } v = v_1 + v_2 \in E, \text{ with } v_1 \in F, v_2 \in G,$$

called the *projection on* $L = P + [F]$ *in the direction* G.

2.9 Characterization of Affinities of the Line

Theorem 2.40

Let $\mathbb{A}_1$, $\mathbb{A}_2$ be two affine spaces over the k-vector spaces E_1, E_2, respectively, of dimension 1, and with $k \neq \mathbb{Z}/2\mathbb{Z}$. Let $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ be a map preserving the simple ratio. Then f is an affinity.

Proof

Fix $P \in \mathbb{A}_1$ and let us study $\tilde{f}_P$. First we want to prove that $\tilde{f}_P(u+v) = \tilde{f}_P(u) + \tilde{f}_P(v)$ for all $u, v \in E_1$. Since this formula is true for $u = \vec{0}$ or $v = \vec{0}$, we can assume given two vectors $u, v \in E_1$ different from zero. We know that there are points (unique) Q, R, S, with $P \neq Q, P \neq R, Q \neq S$ and $R \neq S$, such that

$$u = \overrightarrow{PQ},$$
$$v = \overrightarrow{PR},$$
$$u + v = \overrightarrow{PS}.$$

Note that

$$\overrightarrow{QS} = \overrightarrow{QP} + \overrightarrow{PS} = -u + u + v = \overrightarrow{PR}.$$

Analogously

$$\overrightarrow{RS} = \overrightarrow{RP} + \overrightarrow{PQ} + \overrightarrow{QS} = -v + u + v = \overrightarrow{PQ}.$$

If the three points P, Q, R are distinct (that is, $u \neq v$) we can compute their simple ratio. Put $(P, Q, R) = \lambda$, that is, $\overrightarrow{PQ} = \lambda \overrightarrow{PR}$. Since the simple ratio is preserved, the three points $P' = f(P)$, $Q' = f(Q)$ and $R' = f(R)$ are also distinct and

$$(P', Q', R') = \lambda,$$

that is,

$$\overrightarrow{P'Q'} = \lambda \overrightarrow{P'R'}. \tag{2.6}$$

Note, on the other hand, that if the three points P, Q, S are distinct (that is, $u \neq -v$) we can compute their simple ratio, obtaining

$$(Q, P, S) = -\lambda,$$

since

$$\overrightarrow{QP} = -\lambda \overrightarrow{PR} = -\lambda \overrightarrow{QS}.$$

Since the simple ratio is preserved, the three points P', Q' and $S' = f(S)$ are also distinct and

$$(Q', P', S') = -\lambda,$$

that is,

$$\overrightarrow{Q'P'} = -\lambda \overrightarrow{Q'S'}. \tag{2.7}$$

From (2.6) and (2.7) we directly deduce that $\overrightarrow{P'R'} = \overrightarrow{Q'S'}$.

Thus,

$$\begin{aligned}
\tilde{f}_P(u+v) &= \tilde{f}_P(\overrightarrow{PS}) \\
&= \overrightarrow{P'S'} \\
&= \overrightarrow{P'Q'} + \overrightarrow{Q'S'} \\
&= \overrightarrow{P'Q'} + \overrightarrow{P'R'} \\
&= \tilde{f}_P(u) + \tilde{f}_P(v).
\end{aligned}$$

This proves that $\tilde{f}_P$ preserves vector addition (with the hypothesis that these vectors are neither equal nor opposite). It remains to prove that $\tilde{f}_P$ preserves scalar multiplication. Since the formula that we want to prove, $\tilde{f}_P(\lambda v) = \lambda \tilde{f}_P(v)$, is clearly true for $\lambda = 0$, $\lambda = 1$ or $v = \vec{0}$, we can assume from now on that $\lambda \in k$ and $v \in E_1$, with $\lambda \neq 0$, $\lambda \neq 1$ and $v \neq 0$.

We know that there exists a unique point $Q \in \mathbb{A}_1$ such that $v = \overrightarrow{PQ}$, and a unique point $T \in \mathbb{A}_1$ such that $\lambda v = \overrightarrow{PT}$.

Then it is clear that the points P, Q, T are distinct and that

$$(P, T, Q) = \lambda,$$

hence

$$(P', T', Q') = \lambda,$$

where $P' = f(P)$, $T' = f(T)$, $Q' = f(Q)$.

Thus,

$$\tilde{f}_P(\lambda v) = \tilde{f}_P(\overrightarrow{PT}) = \overrightarrow{P'T'} = \lambda \overrightarrow{P'Q'} = \lambda \tilde{f}_P(v).$$

Finally we remark that, since $\tilde{f}_P$ preserves scalar multiplication, the formula $\tilde{f}_P(u+v) = \tilde{f}_P(u) + \tilde{f}_P(v)$ is also true for $u = \pm v$. This completes the proof. □

For a slightly different proof of this theorem, see Exercise 2.11 of this chapter, page 88.

Note that when $k = \mathbb{Z}/2\mathbb{Z}$ the straight lines have only two points and the hypothesis on the simple ratio doesn't make sense. For this reason, there are maps between affine spaces over $\mathbb{Z}/2\mathbb{Z}$ which are not affinities. For instance, considering the field $k = \mathbb{Z}/2\mathbb{Z}$ as an affine space, the map $f : \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ given by $f(0) = f(1) = 1$ is not an affinity because $f(1+1) \neq f(1) + f(1)$.

2.10 The Fundamental Theorem of Affine Geometry

First let us recall a definition from linear algebra.

Definition 2.41

Let E_1, E_2 be two k-vector spaces. A map $\tilde{f} : E_1 \longrightarrow E_2$ is called *semi-linear* if there exists an *automorphism* σ of the field k such that

$$\tilde{f}(u+v) = \tilde{f}(u) + \tilde{f}(v), \quad \text{for all } u, v \in E_1,$$

$$\tilde{f}(\lambda u) = \sigma(\lambda)\tilde{f}(u), \quad \text{for all } \lambda \in k, u \in E_1.$$

Recall that an *automorphism* of the field k is a bijective map $\sigma : k \longrightarrow k$ such that $\sigma(a+b) = \sigma(a) + \sigma(b)$, $\sigma(ab) = \sigma(a)\sigma(b)$, for each $a, b \in k$, and $\sigma(1) \neq 0$. This implies $\sigma(0) = 0$ and $\sigma(1) = 1$.

Let us return our attention to affine spaces.

Definition 2.42

A map $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ between two affine spaces is called a *semi-linear affine transformation* if the map $\tilde{f}_P : E_1 \longrightarrow E_2$, induced by f and by a point $P \in \mathbb{A}_1$ on the corresponding k-vector spaces is semi-linear.

In this case we also say that f is a *semi-affinity*.

Recall that, using only the fact that $\tilde{f}_P$ preserves vector addition, one can prove $\tilde{f}_P = \tilde{f}_Q$ for all $Q \in \mathbb{A}$. Hence, it is natural to denote simply by $\tilde{f}$ the semi-linear map associated with the semi-affinity f.

Equivalently, we have the following.

Proposition 2.43

A map $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ is a semi-affinity if and only if there is a semi-linear map $\tilde{f} : E_1 \longrightarrow E_2$ such that

$$f(P+u) = f(P) + \tilde{f}(u), \quad \text{for all } P \in \mathbb{A} \text{ and } u \in E_1.$$

Proof

Compare the proof of Proposition 2.2. □

If such a map $\tilde{f}$ exists, it is unique.

The Fundamental Theorem of Affine Geometry, which we are going to prove in this section, states that *if a bijective map f takes collinear points to collinear points then it is a semi-affinity.*

Let us first show that the bijective map f maps collinear points to collinear points if and only if it maps straight lines to straight lines.

It could be the case that the image of a straight line is only a proper part of a straight line. For instance, it is easy to construct a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(\mathbb{R}) \subset (0,1)$. Such a map sends collinear points to collinear points, but it does not send straight lines to straight lines. However, this f is not bijective. Does there exist a bijective map $f : \mathbb{A} \longrightarrow \mathbb{A}$ sending collinear points to collinear points such that the image $f(L)$ of a straight line L is properly contained in a straight line L'?

Before answering this question, we carefully study the special case of the plane. Readers who wish to proceed directly to the general case may prefer to skip to Proposition 2.45.

Proposition 2.44

Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two affine planes over the k-vector spaces E_1, E_2, respectively. Assume $k \neq \mathbb{Z}/2\mathbb{Z}$.

Then every bijective map $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ sending collinear points to collinear points bijectively sends straight lines onto straight lines.

Proof

Let L be a straight line in $\mathbb{A}_1$ and P_1, P_2 be distinct points of L. Let L' be the straight line determined by the distinct points $P_1' = f(P_1)$, $P_2' = f(P_2)$.

Since f maps collinear points to collinear points, it is clear that

$$f(L) \subset L'.$$

It remains to prove the opposite inclusion.

Let $Z' \in L'$. There is a $Z \in \mathbb{A}_1$ such that $f(Z) = Z'$. We want to prove that $Z \in L$. Let us assume that $Z \notin L$, see Figure 2.3.

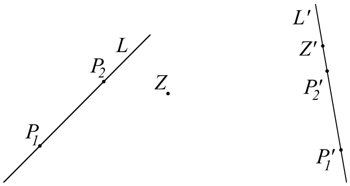

Figure 2.3. Collinear points to collinear points

Take an arbitrary point $X \in \mathbb{A}_1$. If the straight line determined by the points X, Z cuts L in a point Y (see the drawing on the left of Figure 2.4), then $Y' = f(Y) \in L'$; since $Z' \in L'$, and $X' = f(X)$ belongs to the straight line determined by Y' and Z', we must have $X' \in L'$.

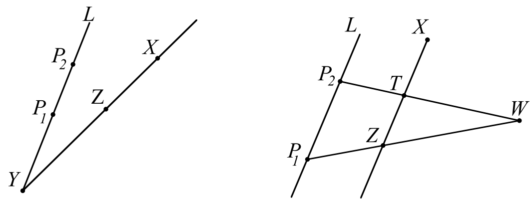

Figure 2.4. Collinear points to collinear points

If the straight line determined by the points X, Z does not cut L (and, therefore, the two lines are parallel, since they lie in a plane, see the drawing on the right of Figure 2.4), we take the point $T = Z + t\overrightarrow{P_1P_2}$ with $t \neq 0, 1$ (here we use the assumption that $k \neq \mathbb{Z}/2\mathbb{Z}$). Then the straight lines P_2T and P_1Z meet in a point W. Since W is collinear with P_1 and Z, we have $W' = f(W) \in L'$.

Since T is collinear with P_2 and W, we have $T' = f(T) \in L'$. Finally, since X is collinear with T and Z, we have $X' \in L'$.

Therefore, in all cases, given $X \in \mathbb{A}_1$ we have $X' \in L'$, that is, we have $f(\mathbb{A}_1) \subset L'$, contradicting the bijectivity of f. Hence, $Z \in L$, and this completes the proof. □

Let us turn to the general case.

Proposition 2.45

Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be affine spaces of dimension $n \geq 2$ over the k-vector spaces E_1, E_2, respectively. Assume $k \neq \mathbb{Z}/2\mathbb{Z}$.

Then every bijective map $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ sending collinear points to collinear points also bijectively maps linear varieties of dimension r onto linear varieties of dimension r, $r = 1, \ldots, n$. In particular, f bijectively maps straight lines onto straight lines.

Proof

The central idea of the proof is to show that f bijectively maps hyperplanes onto hyperplanes. In fact, once we have done this, we can restrict f to a hyperplane. The hypotheses of the theorem will then continue to hold, but in dimension $n-1$; repeating the argument we eventually prove that f bijectively maps straight lines onto straight lines.

Let L be a hyperplane in $\mathbb{A}_1$. Let $P_1, \ldots, P_n$ be points generating this hyperplane, that is, such that

$$L = P_1 + \langle \overrightarrow{P_1P_2}, \ldots, \overrightarrow{P_1P_n} \rangle, \qquad \dim L = n-1.$$

Let L' be the linear variety generated by the distinct points $P_1' = f(P_1), \ldots, P_n' = f(P_n)$.

Since f takes collinear points to collinear points we have $f(L) \subset L'$. To see this, we define

$$L_i = P_1 + [\langle e_2, \ldots, e_i \rangle], \quad e_i = \overrightarrow{P_1P_i},\ i = 2, \ldots, n,$$

and, since $L = L_n$, we use induction on i. If $i = 2$, since L_1 is the straight line determined by the points P_1 and P_2, it is clear that $f(L_1) \subset L'$.

Let us assume $f(L_k) \subset L'$.

In order to prove that $f(L_{k+1}) \subset L'$ let us take an arbitrary point $X \in L_{k+1}$. Note that X can be written as

$$X = X_0 + xe_{k+1}, \quad X_0 \in L_k.$$

Let $Y = P_1 + \epsilon e_{k+1}$ be a point with $\epsilon \neq 0, x$ (we are assuming that the field has more than two elements). Since Y belongs to the straight line determined by the points P_1 and P_{k+1}, it is clear that $Y' = f(Y) \in L'$.

A short calculation shows that the point

$$T = X + t\overrightarrow{XY}, \quad \text{with } t = \frac{x}{x - \epsilon},$$

belongs to L_k. Hence, $T' = f(T) \in L'$.

Since the points X, Y, T are collinear, the points $X' = f(X)$, Y', T' are also collinear, and hence $X' \in L'$. This proves that $f(L_{k+1}) \subset L'$ and, by induction, that $f(L) \subset L'$.

In order to ensure that *the image of a hyperplane is a hyperplane*, it remains to prove that $f(L) = L'$, and that $\dim L' = n - 1$.

Let us take $Z' \in L'$. We must show that $Z' \in f(L)$. We know that there exists a $Z \in \mathbb{A}_1$ such that $f(Z) = Z'$, but it is not clear a priori that $Z \in L$. Let us assume $Z \notin L$.

Take an arbitrary point $X \in \mathbb{A}_1$. If the straight line determined by the points X, Z cuts L in a point Y, then $Y' = f(Y) \in L'$; since also $Z' \in L'$, and $X' = f(X)$ belongs to the straight line determined by Y' and Z', we must have $X' \in L'$.

If the straight line determined by the points X, Z does not cut L, we have

$$\overrightarrow{ZX} \in \langle \overrightarrow{P_1P_2}, \dots, \overrightarrow{P_1P_n} \rangle.$$

Let us take a point $T = Z + t\overrightarrow{ZX}$ with $t \neq 0, 1$ (here we use the assumption that $k \neq \mathbb{Z}/2\mathbb{Z}$).

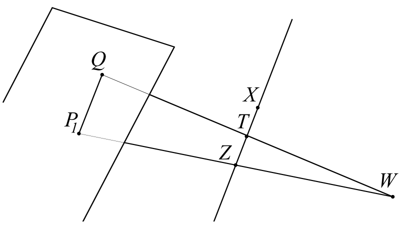

Figure 2.5. The image of a hyperplane is a hyperplane

Let $Q = P_1 + \overrightarrow{ZX} \in L$. Then the straight lines P_1Z and QT meet in a point W, see Figure 2.5. Since W is collinear with P_1 and Z, we have $W' =$

$f(W) \in L'$. Since T is collinear with Q and W, we have $T' = f(T) \in L'$. Finally, since X is collinear with T and Z, we have $X' \in L'$.

Thus, in all cases, given $X \in \mathbb{A}_1$ we have $X' \in L'$, that is, we have $f(\mathbb{A}_1) \subset L'$, contradicting the bijectivity of f. Hence, $Z \in L$, and this proves that $f(L) = L'$.

Finally, note that given a point $P' \notin L'$ it is easy to see, by an argument similar to that used above, that each point of $\mathbb{A}_2$ belongs either to a straight line connecting P' with a point of L' or to the linear variety through P' with the same direction as L. But this implies, see Exercise 1.14 of Chapter 1, page 40, that $\dim L' = n - 1$, and this completes the proof. □

Theorem 2.46 (Fundamental theorem of affine geometry)

Let $\mathbb{A}_1$, $\mathbb{A}_2$ be two affine spaces over the k-vector spaces E_1, E_2 respectively, of the same dimension n, with $n \geq 2$. Suppose that $k \neq \mathbb{Z}/2\mathbb{Z}$. Let $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ be a bijective map sending collinear points to collinear points.

Then f is a semi-affinity.

Proof

First part: f takes parallel straight lines to parallel straight lines. If $n = 2$, this is evident because f is injective.

Let r_1, r_2 be two distinct parallel straight lines of $\mathbb{A}_1$, with $\dim A_1 = n > 2$. Let us take two distinct points A, B on r_1 and a point C on r_2. Let $D = C + \overrightarrow{AB} \in r_2$.

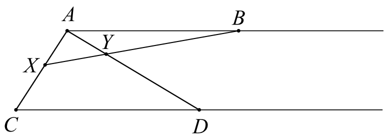

Figure 2.6. Construction of the point X

Take $Y \in AD$ distinct from A and D (here we use $k \neq \mathbb{Z}/2\mathbb{Z}$).

Denote by r_3 the straight line AC, by r_4 the straight line YB and by r_5 the straight line AD.

The straight lines r_3 and r_4 meet in a point X distinct from A, see Figure 2.6. This point X exists because all straight lines of the above diagram are in a plane (concretely in the plane $\Sigma : A + \langle \overrightarrow{AB}, \overrightarrow{AC} \rangle$) and the direction vectors of r_3 and r_4, $\overrightarrow{AC}$ and $\overrightarrow{BY}$, are linearly independent (since $\overrightarrow{AC} = \overrightarrow{BD}$).[1]

[1] If the characteristic of the field is different from 2, we can take $Y = A + \frac{1}{2}\overrightarrow{AD}$, and then $X = C$.

Since f is injective and maps straight lines to straight lines, $f(r_1)$ and $f(r_2)$ are non-intersecting straight lines. It still remains to prove that they are in the same plane.

The straight lines $f(r_1)$, $f(r_5)$ meet in $f(A)$ and, therefore, they determine a plane Π. Since $f(Y)$ and $f(B)$ belong to this plane, we have $f(r_4) \subset \Pi$ and in particular $f(X) \in \Pi$. But this implies $f(r_3) \subset \Pi$, and hence $f(C) \in \Pi$. Since also $f(D) \in \Pi$, we have $f(r_2) \subset \Pi$ and $f(r_1)$, $f(r_2)$ are coplanar. Since they do not meet, $f(r_1)$ and $f(r_2)$ are parallel, and this completes the first part of the proof.

Second part: $\tilde{f}_P$ is additive. Let us fix a point $P \in \mathbb{A}_1$. We must show that the map $\tilde{f}_P : E_1 \longrightarrow E_2$ given by

$$\tilde{f}_P(\overrightarrow{PX}) = \overrightarrow{f(P)f(X)}, \quad \text{for all } X \in \mathbb{A}_1,$$

preserves vector addition.

Let $u = \overrightarrow{PQ}$, $v = \overrightarrow{PR}$ be linearly independent vectors and let $S = Q + \overrightarrow{PR}$. We have $u + v = \overrightarrow{PQ} + \overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PS}$.

By the first part of the proof, and by Proposition 1.17, we know that the points $f(P)$, $f(Q), f(R), f(S)$ are the vertices of a parallelogram, that is, $\overrightarrow{f(P)f(R)} = \overrightarrow{f(Q)f(S)}$. Thus,

$$\begin{aligned} \tilde{f}_P(u) + \tilde{f}_P(v) &= \overrightarrow{f(P)f(Q)} + \overrightarrow{f(P)f(R)} \\ &= \overrightarrow{f(P)f(Q)} + \overrightarrow{f(Q)f(S)} \\ &= \overrightarrow{f(P)f(S)} \\ &= \tilde{f}_P(u+v). \end{aligned}$$

Hence, $\tilde{f}_P$ preserves the addition of linearly independent vectors.

If $w = \lambda u$, we have

$$\begin{aligned} \tilde{f}_P(u+w) + \tilde{f}_P(v) &= \tilde{f}_P(u+w+v) \\ &= \tilde{f}_P(u) + \tilde{f}_P(w+v) \\ &= \tilde{f}_P(u) + \tilde{f}_P(w) + \tilde{f}_P(v). \end{aligned}$$

Hence, $\tilde{f}_P$ preserves the addition of vectors, linearly independent or not; that is,

$$\tilde{f}(u+v) = \tilde{f}(u) + \tilde{f}(v), \quad \text{for all } u, v \in E_1.$$

Third part: the behavior of $\tilde{f}_P$ with scalars. Since $\tilde{f}_P$ is additive we have $\tilde{f}_P = \tilde{f}_Q$, for all $P, Q \in \mathbb{A}_1$, and so from now on we shall write $\tilde{f}_P = \tilde{f}$.

Fix $u \in E_1$, $u \neq 0$. Let $\lambda \in k$, $\lambda \neq 0$. There are unique points Q and R such that $u = \overrightarrow{PQ}$ and $\lambda u = \overrightarrow{PR}$.

Since P, Q, R are collinear, the points $f(P), f(Q), f(R)$ are also collinear, and hence there exists a $\mu_u(\lambda) \in k$ such that

$$\tilde{f}(\lambda u) = \overrightarrow{f(P)f(R)} = \mu_u(\lambda)\overrightarrow{f(P)f(Q)} = \mu_u(\lambda)\tilde{f}(u). \tag{2.8}$$

Moreover, since f is bijective, $\tilde{f}(u) \neq 0$ and, therefore, the scalar $\mu_u(\lambda)$ is unique. Thus we have a map

$$\begin{aligned} \mu_u : k &\longrightarrow k \\ \lambda &\mapsto \mu_u(\lambda). \end{aligned}$$

It is also easy to see that, since f is bijective, μ_u is also bijective.

Note that this map μ_u depends, a priori, on the chosen vector $u \in E_1$. Also note that $\mu_u(0) = 0$ and $\mu_u(1) = 1$.

By definition of semi-affinity, we must prove that $\tilde{f}$ is semi-linear, that is, that the scalars are transformed via an automorphism of the field. Hence, by (2.8), we must verify the following two properties:

(a) $\mu_u = \mu_v$ for all $v \in E_1$; and

(b) μ_u is an automorphism of k.

Proof of (a) *for every $v \in E_1$ linearly independent with u.* Let $v \in E_1$ be linearly independent with u. Set $u = \overrightarrow{PQ}$, $v = \overrightarrow{PR}$, $\lambda u = \overrightarrow{PQ'}$ and $\lambda v = \overrightarrow{PR'}$.

From the converse of the corollary of Thales' theorem, see Exercise 1.44 of Chapter 1, page 45, the straight lines RQ and $R'Q'$ are parallel. From the first part of the present proof, page 81, the straight lines $f(R)f(Q)$ and $f(R')f(Q')$ are also parallel, see Figure 2.7.

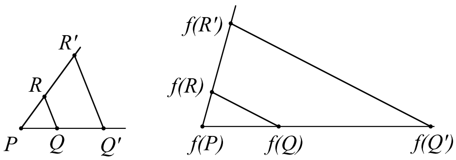

Figure 2.7. Thales' theorem

By Thales' theorem, if the scalar $\tau \in k$ is such that

$$\overrightarrow{f(P)f(Q')} = \tau\overrightarrow{f(P)f(Q)},$$

then

$$\overrightarrow{f(P)f(R')} = \tau\overrightarrow{f(P)f(R)}.$$

Equivalently, if

$$\tilde{f}(\lambda u) = \tau\tilde{f}(u), \tag{2.9}$$

then

$$\tilde{f}(\lambda v) = \tau \tilde{f}(v). \tag{2.10}$$

From (2.9) we deduce $\tau = \mu_u(\lambda)$ and from (2.10) we deduce $\tau = \mu_v(\lambda)$; hence, $\mu_u = \mu_v$ for all $v \in E_1$ linearly independent with u.

Proof of (b). The equality $\mu_u = \mu_v$ for all $v \in E_1$ linearly independent with u enables us to prove that μ_u is an automorphism of k.

Indeed, since f is bijective, there exists a $v \in E_1$ linearly independent with u such that $\tilde{f}(u) \neq 0$. If $\lambda' \in k$, $\lambda' \neq 0$, then setting $\mu = \mu_u$, we have

$$\mu(\lambda\lambda')\tilde{f}(v) = \tilde{f}(\lambda\lambda' v) = \mu(\lambda)\tilde{f}(\lambda' v) = \mu(\lambda)\mu(\lambda')\tilde{f}(v),$$

since, as $\lambda' v$ is linearly independent with u, $\mu_u = \mu_{\lambda' v}$.

Thus,

$$\mu(\lambda\lambda') = \mu(\lambda)\mu(\lambda'), \quad \lambda, \lambda' \in k \backslash \{0\}.$$

Since $\mu(0) = 0$, this equality holds for all λ, $\lambda' \in k$.

Since $\tilde{f}((\lambda + \lambda')u) = \tilde{f}(\lambda u) + \tilde{f}(\lambda' u)$, we have $\mu(\lambda + \lambda') = \mu(\lambda) + \mu(\lambda')$ and clearly $\mu(1) = 1$. Hence $\mu : k \longrightarrow k$ is an automorphism of k.

End of the proof of (a). It remains only to prove that

$$\mu_u = \mu_v \quad \text{when } v = \nu u,\ \nu \in k.$$

But we have

$$\tilde{f}(\lambda v) = \tilde{f}(\lambda \nu u) = \mu_u(\lambda\nu)\tilde{f}(u) = \mu_u(\lambda)\mu_u(\nu)\tilde{f}(u),$$

and also

$$\tilde{f}(\lambda v) = \mu_v(\lambda)\tilde{f}(\nu u) = \mu_v(\lambda)\mu_u(\nu)\tilde{f}(u).$$

Comparing these two equalities we obtain the result. □

Corollary 2.47

Let $\mathbb{A}_1, \mathbb{A}_2$ be two affine spaces over the k-vector spaces E_1, E_2, respectively, of the same dimension n, with $n \geq 2$. Let us assume that $k \neq \mathbb{Z}/2\mathbb{Z}$. Let $f : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ be a bijective map that takes straight lines to straight lines and preserves the simple ratio.

Then f is an affinity.

Proof

Every element λ of the field can be written as the simple ratio $\lambda = (A, B, C)$ of three points.

Since f is a semi-affinity we have

$$\tilde{f}(\overrightarrow{AB}) = \tilde{f}(\lambda\overrightarrow{AC}) = \sigma(\lambda)\tilde{f}(\overrightarrow{AC}).$$

Hence,

$$\lambda = (A, B, C) = (f(A), f(B), f(C)) = \sigma(\lambda),$$

where σ is the automorphism of the field associated to f. Thus, $\sigma(\lambda) = \lambda$ for all λ, that is, σ is the identity, and hence f is an affinity. □

Example 2.48

Find a bijective map of an affine space $\mathbb{A}$ into itself taking straight lines onto straight lines and such that it is not a semi-affinity.

Solution

By the Fundamental Theorem, we must look for an affine space over a $\mathbb{Z}/2\mathbb{Z}$-vector space. We take $\mathbb{A} = E = (\mathbb{Z}/2\mathbb{Z})^3$ and

$$\begin{aligned} f : \mathbb{A} &\longrightarrow \mathbb{A} \\ (0,0,0) &\mapsto (0,0,0) \\ (1,0,0) &\mapsto (1,0,1) \\ (1,1,0) &\mapsto (1,1,1) \\ (0,1,0) &\mapsto (0,1,1) \\ (0,0,1) &\mapsto (0,0,1) \\ (1,0,1) &\mapsto (1,0,0) \\ (1,1,1) &\mapsto (1,1,0) \\ (0,1,1) &\mapsto (0,1,0) \end{aligned}$$

f is bijective and takes straight lines onto straight lines, but it is not a semi-affinity. Indeed, take $P = (0,0,0)$, $Q = (1,0,0)$, $R = (0,1,0)$ and denote $\tilde{f}_P$ by $\tilde{f}$. Notice that $\overrightarrow{PQ} = (1,0,0)$ and $\overrightarrow{PR} = (0,1,0)$. We have

$$\tilde{f}(\overrightarrow{PQ} + \overrightarrow{PR}) = \tilde{f}(1,1,0) = (1,1,1).$$

But,

$$\begin{aligned}\tilde{f}(\overrightarrow{PQ}) + \tilde{f}(\overrightarrow{PR}) &= \tilde{f}(1,0,0) + \tilde{f}(0,1,0) \\ &= (1,0,1) + (0,1,1) \\ &= (1,1,0) \neq (1,1,1).\end{aligned}$$

□

2.10.1 A Note on Automorphisms of Fields

It is useful to know that the fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{Z}/p\mathbb{Z}$ have a unique automorphism: the identity. Hence, when we work over one of these fields (with $p \neq 2$), the Fundamental Theorem of Affine Geometry states that *every bijective map taking straight lines to straight lines is an affinity.*

The only automorphism of $\mathbb{Z}/p\mathbb{Z}$ *is the identity.* Let σ be an automorphism of $\mathbb{Z}/p\mathbb{Z}$. Since $\sigma(1) = 1$, the other elements are also fixed, because $\mathbb{Z}/p\mathbb{Z}$ is additively generated by 1. That is, $\sigma(m) = m\sigma(1) = m$, and hence $\sigma = \mathrm{id}$.

The only automorphism of $\mathbb{Q}$ *is the identity.* Let σ be an automorphism of $\mathbb{Q}$. Since $\sigma(1) = 1$, all integers are fixed, because $\mathbb{Z}$ is additively generated by 1. Moreover, $\sigma(n \cdot \frac{1}{n}) = \sigma(n) \cdot \sigma(\frac{1}{n}) = 1$; hence, $\sigma(\frac{1}{n}) = \frac{1}{\sigma(n)} = \frac{1}{n}$.

Thus,

$$\sigma\left(\frac{m}{n}\right) = m \cdot \sigma\left(\frac{1}{n}\right) = \frac{m}{n},$$

and so $\sigma = \mathrm{id}$.

The only automorphism of $\mathbb{R}$ *is the identity.* Let σ be an automorphism of $\mathbb{R}$. We know, by the above argument, that σ is the identity on $\mathbb{Q}$.

Let $a \in \mathbb{R}$. Let us assume $a > 0$. Then $a = b^2$, with $b \in \mathbb{R}$. Thus, $\sigma(a) = \sigma(b)^2$, and hence $\sigma(a) > 0$. Since σ is bijective, we have $a > 0$ if and only if $\sigma(a) > 0$.

Let us assume that there is a real number c such that $\sigma(c) - c > 0$ (a very similar argument can be applied if $\sigma(c) - c < 0$). Choose $r \in \mathbb{Q}$ such that $\sigma(c) < r < c$. Then

$$\sigma(c - r) = \sigma(c) - \sigma(r) = \sigma(c) - r < 0,$$

a contradiction, since $c - r > 0$. Hence, $\sigma(c) = c$ for all $c \in \mathbb{R}$, and $\sigma = \mathrm{id}$.

See, for instance, [30], page 49.

The field $\mathbb{C}$, however, has infinitely many automorphisms[2] and the fields of p^r elements have exactly r automorphisms; see for instance [20], page 184.

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[2] It seems impossible (or very difficult) to explicitly describe these automorphisms (except, of course, for the identity and conjugation). Nevertheless, given any permutation P of a family of complex numbers S, algebraically independent over the

EXERCISES

2.1. Verify if the following maps are affinities or not:

$$\begin{array}{ccc} \mathbb{C} \longrightarrow \mathbb{C} & \mathbb{R}^2 \longrightarrow \mathbb{R}^2 & \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \\ z \mapsto \overline{z} & (x,y) \mapsto (x,-y) & (x,y,z) \mapsto (1+x,\sqrt{\pi}) \end{array}$$

In the first case consider $\mathbb{C}$ as an affine space over $\mathbb{C}$ and also over $\mathbb{R}$.

2.2. Consider the affinity $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given in the respective canonical affine frames by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Find the equations of f in the affine frames $\mathcal{R}_1$ and $\mathcal{R}_2$ given by

$$\mathcal{R}_1 = \{(1,0,0);((2,-1,0),(0,2,-1),(-1,0,2))\},$$

$$\mathcal{R}_2 = \{(2,1);((1,1),(1,-1))\}.$$

2.3. Find, in the affine plane $\mathbb{R}^2$, the equations of the projection on the straight line r: $x+y=1$ in the direction of the straight line s: $x-y=2$.

2.4. Find, in the affine space $\mathbb{R}^3$, the equations of the symmetry with respect to $L=(1,1,0)+[F]$ in the direction G in the following cases:
1. $F=\langle(1,2,3),(0,0,1)\rangle$, $G=\langle(3,1,1)\rangle$.
2. $F=\langle(3,1,1)\rangle$, $G=\langle(1,2,3),(0,0,1)\rangle$.
3. $F=\{\vec{0}\}$.
4. $G=\{\vec{0}\}$.

2.5. Find, in the affine space $\mathbb{R}^3$, the equations of the projection on $L=(1,1,0)+[F]$ in the direction G in the same four cases considered in the previous exercise.

2.6. Consider, in an affine space of dimension 3, the linear varieties given in some affine frame $\mathcal{R}$ by

$$L=\{(x,y,x) : x+y+z=3, x-4y+2z=1\},$$

$$M=\{(x,y,x) : 2x-3y-z=1\}.$$

Find the equations of the symmetry with respect to L in the direction of M and the equations of the symmetry with respect to M in the direction of L.

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field of algebraic numbers $\bar{Q}$, and any automorphism σ of $\bar{Q}$, there exists an automorphism of $\mathbb{C}$ that restricted to S acts as P and restricted to $\bar{Q}$ acts as σ. See, for instance, [20], page 270, Exercise 1.

2.7. Let $f:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be an affinity of the affine space $\mathbb{R}^3$ such that the points of the plane Π: $x+z=1$ are fixed and such that $f(0,0,0) = (0,-3,0)$.
(a) Find the matrix of f in the canonical affine frame $\mathcal{C}$ of the affine space $\mathbb{R}^3$.
(b) Prove that on the planes parallel to Π, f acts as a translation, and find the translation vector.

2.8. Let $\mathcal{R} = \{P;(e_1,e_2)\}$ be an affine frame of a given affine plane. Find the equations of the axial symmetries, and their compositions, with respect to the straight lines given in $\mathcal{R}$ by the equations $2x+3y=2$ and $x+3y=2$, in the directions e_1+e_2 and e_1 respectively.

2.9. Prove that dilations of an affine plane satisfy the Axioms 4 and 5 of Affine Geometry given in the introduction, page viii.

2.10. Prove that the image under an affinity of the barycenter of r points is equal to the barycenter of the images of these points. The same is true for the barycenter with weights. Is a map preserving the barycenter with weights necessarily an affinity?

2.11. Prove Theorem 2.40 in the following three steps:
(1) Reduce the problem to the case $A_1 = A_2 = k$.
(2) Further reduce the problem to the case $f(0)=0$ (composing with a translation).
(3) Observe that $f(b) = a\cdot b$ where $f(1)=a$.
Thus, f is multiplication by a, which is linear. This exercise was suggested by F. Cedó.

2.12. Let f be an affinity of an affine plane $\mathbb{A}$, given in the affine frame $\mathcal{R} = \{P;(e_1,e_2)\}$ by the equations $f(x,y) = (x-y+1, y+2)$. Find the equations of f in the affine frame $\mathcal{R}' = \{(3,2);(e_1-e_2, 2e_1+e_2)\}$.

2.13. Consider the affinity of the affine space $\mathbb{R}^3$ given by

$$T(x,y,z) = (1+x+2y+z, 2+y-z, -1+x+9z).$$

Is it bijective? Find the associated endomorphism. Find the image of the straight line $x=a$, $y=2-a$, $z=-1$ and the image of the plane $2x+y-z=1$. Find the preimage of these linear varieties. Find the fixed points. Find also the equations of T in the affine frame $\{(1,0,4);((1,1,0),(2,0,1),(8,0,7))\}$.

2.14. Find the affinity f of the affine space $\mathbb{R}^3$ leaving the plane Π: $x+y=1$ invariant, acting on this plane as a translation by the vector $v=(0,0,1)$, and such that $f(1,1,0)=(0,0,0)$.

2.15. Let f be the affinity of an affine space of dimension 3 given in an affine frame $\mathcal{R}$ by

$$\begin{cases} x' = 1 - \frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z, \\ y' = \frac{2}{3}x - \frac{1}{3}y + \frac{2}{3}z, \\ z' = -1 + \frac{2}{3}x + \frac{2}{3}y - \frac{1}{3}z. \end{cases}$$

Find, with respect to $\mathcal{R}$:

(a) The fixed points and the invariant straight lines.

(b) The image of the straight line

$$\frac{x}{1} = \frac{y-2}{2} = \frac{z+1}{3}.$$

(c) The preimage of the plane $x' - y' - z' = 2$.

2.16. (a) Construct all the affinities of the affine plane $\mathbb{R}^2$ which leave invariant the figure formed by a point P and a straight line r with $P \notin r$.

(b) Give, in the affine plane $\mathbb{R}^2$, the expression of all affinities f such that $f(P) = P$ and $f(r) = r$ where $P = (1,0)$ and r: $y - x = 0$.

2.17. Prove that, in an affine plane, given two intersecting straight lines r, s and a point P not belonging to them, and given another analogous configuration, that is, two intersecting straight lines r', s' and a point P' not belonging to them, there exists a unique affinity f such that $f(r) = r'$, $f(s) = s'$ and $f(P) = P'$. Find this affinity, in the affine plane $\mathbb{R}^2$, where

$$r\colon x - y = 2, \qquad s\colon x - 2y = -1, \quad P = (0,0);$$
$$r'\colon x = 4, \qquad s'\colon x - y = 1, \quad P' = (1,2).$$

2.18. (a) Find, with respect to the canonical affine frame of the affine plane $\mathbb{R}^2$, the equations of a homothety of center $(2,3)$ and similitude ratio -1.

(b) Is it true that the image of a straight line that does not contain the center of a homothety is another straight line parallel to it?

(c) Find a homothety which, when composed with the homothety of part a), yields a translation, and give the translation vector.

2.19. Let $\mathbb{A}$ be an affine plane. Find the affinities of $\mathbb{A}$ fixing one vertex of a triangle and permuting the other two. Find the affinities of $\mathbb{A}$ leaving invariant a given straight line.

2.20. Let $\mathbb{A}$ be an affine space of dimension 3. Find the affinities of $\mathbb{A}$ with one straight line of fixed points r and leaving invariant another straight line parallel to r.

2.21. Consider the map $f:\mathbb{R}^4 \to \mathbb{R}^4$ given by

$$f(x,y,z,t) = (1+x+y, -1+y+z, 2+z+t, 1+\alpha x+t), \quad \alpha \in \mathbb{R}.$$

Find the value of α for which f is not bijective. Let Π: $ax+by+cz+dt=1$ be a hyperplane in the affine space $\mathbb{R}^4$. Find, for the previous value of α, the condition that must be fulfilled by a,b,c,d in order that $f(\Pi)$ be a linear variety of dimension smaller than 3.

2.22. Study the affinities of the affine space $\mathbb{R}^2$ preserving the hyperbola $xy=1$.

2.23. Let f be an affine map. Prove:
(a) If f^2 has a fixed point, then f also has a fixed point.
(b) If there exists an $n \in \mathbb{N}$ such that f^n has a fixed point, then f also has a fixed point.

2.24. Prove that given two triangles T_1 and T_2 of an affine plane $\mathbb{A}$, there exists a bijective affinity $f:\mathbb{A}\to\mathbb{A}$ such that $f(T_1)=T_2$. How many maps with this property are there? Is the previous statement true if we replace triangles by quadrilaterals, parallelograms or triples of points?

2.25. Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles whose sides are respectively parallel. Prove that there exists a translation or a homothety mapping one of them onto the other.

2.26. (a) Find, with respect to the canonical affine frame of the affine space $\mathbb{R}^2$, the equations of an affinity mapping the points $A=(0,0)$, $B=(1,0)$, $C=(0,1)$, respectively onto the points B, C, A. What is the image under this affinity of the barycenter of the triangle $\triangle ABC$?
(b) Find, with respect to the canonical affine frame of the affine space $\mathbb{R}^2$, the equations of an affinity mapping the points $A=(a_1,a_2)$, $B=(b_1,b_2)$, $C=(c_1,c_2)$, respectively onto the points B,C,A. What is the image under this affinity of the barycenter of the triangle $\triangle ABC$?

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