Chapter 2
Lyapunov-Based Model Predictive Control

2.1 Introduction

MPC, also known as receding horizon control (RHC), is a popular control strategy for the design of high performance model-based process control systems because of its ability to handle multi-variable interactions, constraints on control (manipulated) inputs and system states, and optimization requirements in a systematic manner. MPC is an online optimization-based approach, which takes advantage of a system model to predict its future evolution starting from the current system state along a given prediction horizon. Using model predictions, a future control input trajectory is optimized by minimizing a typically quadratic cost function involving penalties on the system states and control actions. To obtain finite dimensional optimization problems, MPC optimizes over a family of piecewise constant trajectories with a fixed sampling time and a finite prediction horizon. Once the optimization problem is solved, only the first manipulated input value is implemented and the rest of the trajectory is discarded; this optimization procedure is then repeated in the next sampling step [25, 91]. This is the so-called receding horizon scheme. The success of MPC in industrial applications (e.g., [25, 89]) has motivated numerous research investigations into the stability, robustness and optimality of model predictive controllers [65]. One important issue arising from these works is the difficulty in characterizing, a priori, the set of initial conditions starting from where controller feasibility and closed-loop stability are guaranteed. This issue motivated research on LMPC designs [67, 68] (see also [42, 86]) which allow for an explicit characterization of the stability region of the closed-loop system and lead to a reduced computational complexity of the controller optimization problem. Despite this progress, the adoption of communication networks in the control loops and the use of heterogeneous measurements motivate the development of MPC schemes that take data losses (or asynchronous feedback) and time-varying delays explicitly into account. However, little attention has been given to these issues except for a few results on MPC of linear systems with delays (e.g., [36, 49]).

Motivated by the above considerations, in this chapter, we adopt the LMPC framework [67, 68] and introduce modifications on the LMPC design both in the

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optimization problem formulation and in the controller implementation to account for data losses and time-varying delays, respectively. The design of the LMPC is based on uniting receding horizon control with explicit Lyapunov-based nonlinear controller design techniques. In order to guarantee the closed-loop stability, in the design of the LMPCs, constraints based on Lyapunov functions are incorporated in the controller formulations. The theoretical results are illustrated through a chemical reactor example. The results of this chapter were first presented in [53, 72], and an application of the control methods to a continuous crystallizer can be found in [50].

2.2 Notation

Throughout this book, the operator $|\cdot|$ is used to denote the absolute value of a scalar and the operator $\|\cdot\|$ is used to denote Euclidean norm of a vector, while we use $\|\cdot\|_Q$ to denote the square of a weighted Euclidean norm, i.e., $\|x\|_Q = x^T Qx$ for all $x \in R^n$. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and satisfies $\alpha(0) = 0$. A function $\beta(r, s)$ is said to be a class $\mathscr{K}\mathscr{L}$ function if, for each fixed s, $\beta(r, s)$ belongs to class $\mathscr{K}$ function with respect to r and, for each fixed r, $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to 0$. The symbol Ω_r is used to denote the set $\Omega_r := \{x \in R^n : V(x) \le r\}$ where V is a scalar positive definite, continuous differentiable function and $V(0) = 0$, and the operator '/' denotes set subtraction, that is, $A/B := \{x \in R^n : x \in A, x \notin B\}$. The symbol $diag(v)$ denotes a square diagonal matrix whose diagonal elements are the elements of the vector v. The notation t_0 indicates the initial time instant. The set $\{t_{k\ge 0}\}$ denotes a sequence of synchronous time instants such that $t_k = t_0 + k\Delta$ and $t_{k+i} = t_k + i\Delta$ where Δ is a fixed time interval and i is an integer. Similarly, the set $\{t_{a\ge 0}\}$ denotes a sequence of asynchronous time instants such that the interval between two consecutive time instants is not fixed.

2.3 System Description

Consider nonlinear systems described by the following state-space model:

$$\dot{x}(t) = f\big(x(t), u(t), w(t)\big), \tag{2.1}$$

where $x(t) \in R^n$ denotes the vector of state variables, $u(t) \in R^m$ denotes the vector of control (manipulated) input variables, $w(t) \in R^w$ denotes the vector of disturbance variables and f is a locally Lipschitz vector function on $R^n \times R^m \times R^w$ such that $f(0, 0, 0) = 0$. This implies that the origin is an equilibrium point for the nominal system (i.e., system of Eq. 2.1 with $w(t) \equiv 0$ for all t) with $u = 0$.

The input vector is restricted to be in a nonempty convex set $U \subseteq R^m$ which is defined as follows:

$$U := \big\{u \in R^m : \|u\| \le u^{\max}\big\}, \tag{2.2}$$

where $u^{\max}$ is the magnitude of the input constraint.

The disturbance vector is bounded, that is, $w(t) \in W$ where:

$$W := \{w \in R^w : \|w\| \leq \theta, \theta > 0\} \tag{2.3}$$

with θ being a known positive real number. The vector of uncertain variables, $w(t)$, is introduced into the model in order to account for the occurrence of uncertainty in the values of the process parameters and the influence of disturbances in process control applications.

Remark 2.1 Note that the assumption that f is a locally Lipschitz vector function is a reasonable assumption for most of chemical process models.

2.4 Lyapunov-Based Control

We assume that there exists a feedback control law $u(t) = h(x(t))$ which satisfies the input constraint on u for all x inside a given stability region and renders the origin of the nominal closed-loop system asymptotically stable. This assumption is essentially equivalent to the assumption that the nominal system is stabilizable or that there exists a Lyapunov function for the nominal system or that the pair (A, B) in the case of linear systems is stabilizable. Using converse Lyapunov theorems [11, 40, 48, 64], this assumption implies that there exist functions $\alpha_i(\cdot)$, $i = 1, 2, 3, 4$ of class $\mathscr{K}$ and a continuously differentiable Lyapunov function $V(x)$ for the nominal closed-loop system, that satisfy the following inequalities:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{2.4}$$

$$\frac{\partial V(x)}{\partial x} f(x, h(x), 0) \leq -\alpha_3(\|x\|), \tag{2.5}$$

$$\left\| \frac{\partial V(x)}{\partial x} \right\| \leq \alpha_4(\|x\|), \tag{2.6}$$

$$h(x) \in U \tag{2.7}$$

for all $x \in O \subseteq R^n$ where O is an open neighborhood of the origin. We denote the region $\Omega_\rho \subseteq O$ as the stability region of the closed-loop system under the control $u = h(x)$. Note that explicit stabilizing control laws that provide explicitly defined regions of attraction for the closed-loop system have been developed using Lyapunov techniques for specific classes of nonlinear systems, particularly input-affine nonlinear systems; the reader may refer to [2, 11, 41, 97] for results in this area including results on the design of bounded Lyapunov-based controllers by taking explicitly into account constraints for broad classes of nonlinear systems [18, 19, 47].

By continuity, the local Lipschitz property assumed for the vector field $f(x, u, w)$, the fact that the manipulated input u is bounded in a convex set and the continuous differentiable property of the Lyapunov function V, there exists positive constants M, L_w, L_x and L'_x such that:

$$\left\| f(x,u,w)\right\| \leq M, \tag{2.8}$$

$$\left\| f(x,u,w) - f\left(x',u,0\right)\right\| \leq L_w\|w\| + L_x\left\|x - x'\right\|, \tag{2.9}$$

$$\left\| \frac{\partial V(x)}{\partial x} f(x,u,0) - \frac{\partial V(x')}{\partial x} f\left(x',u,0\right)\right\| \leq L'_x\left\|x - x'\right\| \tag{2.10}$$

for all $x, x' \in \Omega_\rho$, $u \in U$ and $w \in W$. These constants will be used in characterizing the stability properties of the system of Eq. 2.1 under LMPC designs.

Remark 2.2 Note that while there are currently no general methods for constructing Lyapunov functions for general nonlinear systems, for broad classes of nonlinear models arising in the context of chemical process control applications, quadratic Lyapunov functions are widely used and provide very good estimates of closed-loop stability regions.

Remark 2.3 Note that the inequalities of Eqs. 2.4–2.10 are derived from the basic assumptions (i.e., Lipschitz vector field and existence of a stabilizing Lyapunov-based controller). The various constants involved in the upper bounds are not assumed to be arbitrarily small.

2.5 Model Predictive Control

MPC is widely adopted in industry as an effective approach to deal with large multivariable constrained control problems. The main idea of MPC is to choose control actions by repeatedly solving an online constrained optimization problem, which aims at minimizing a performance index over a finite prediction horizon based on predictions obtained by a system model. In general, an MPC design is composed of three components:

1. A model of the system. This model is used to predict the future evolution of the system in open-loop and the efficiency of the calculated control actions of an MPC depends highly on the accuracy of the model.
2. A performance index over a finite horizon. This index will be minimized subject to constraints imposed by the system model, restrictions on control inputs and system state and other considerations at each sampling time to obtain a trajectory of future control inputs.
3. A receding horizon scheme. This scheme introduces the notion of feedback into the control law to compensate for disturbances and modeling errors.

Consider the control of the system of Eq. 2.1 and assume that the state measurements of the system of Eq. 2.1 are available at synchronous sampling time instants

$\{t_{k\geq 0}\}$, a standard MPC is formulated as follows [25]:

$$\min_{u\in S(\Delta)} \int_{t_k}^{t_{k+N}} \left[\|\tilde{x}(\tau)\|_{Q_c} + \|u(\tau)\|_{R_c} \right] d\tau + F\big(x(t_{k+N})\big), \tag{2.11}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f\big(\tilde{x}(t), u(t), 0\big), \tag{2.12}$$

$$u(t) \in U, \tag{2.13}$$

$$\tilde{x}(t_k) = x(t_k), \tag{2.14}$$

where $S(\Delta)$ is the family of piece-wise constant functions with sampling period Δ, N is the prediction horizon, Q_c and R_c are strictly positive definite symmetric weighting matrices, $\tilde{x}$ is the predicted trajectory of the nominal system due to control input u with initial state $x(t_k)$ at time t_k, and $F(\cdot)$ denotes the terminal penalty.

The optimal solution to the MPC optimization problem defined by Eqs. 2.11–2.14 is denoted as $u^*(t|t_k)$ which is defined for $t \in [t_k, t_{k+N})$. The first step value of $u^*(t|t_k)$ is applied to the closed-loop system for $t \in [t_k, t_{k+1})$. At the next sampling time t_{k+1}, when a new measurement of the system state $x(t_{k+1})$ is available, the control evaluation and implementation procedure is repeated. The manipulated input of the system of Eq. 2.1 under the control of the MPC of Eqs. 2.11–2.14 is defined as follows:

$$u(t) = u^*(t|t_k), \quad \forall t \in [t_k, t_{k+1}), \tag{2.15}$$

which is the standard receding horizon scheme.

In the MPC formulation of Eqs. 2.11–2.14, Eq. 2.11 defines a performance index or cost index that should be minimized. In addition to penalties on the state and control actions, the index may also include penalties on other considerations; for example, the rate of change of the inputs. Equation 2.12 is the nominal model of the system of Eq. 2.1 which is used in the MPC to predict the future evolution of the system. Equation 2.13 takes into account the constraint on the control input, and Eq. 2.14 provides the initial state for the MPC which is a measurement of the actual system state. Note that in the above MPC formulation, state constraints are not considered but can be readily taken into account.

It is well known that the MPC of Eqs. 2.11–2.14 is not necessarily stabilizing. To achieve closed-loop stability, different approaches have been proposed in the literature. One class of approaches is to use infinite prediction horizons or well-designed terminal penalty terms; please see [6, 65] for surveys of these approaches. Another class of approaches is to impose stability constraints in the MPC optimization problem [1, 4, 65]. There are also efforts focusing on getting explicit stabilizing MPC laws using offline computations [59]. However, the implicit nature of MPC control law makes it very difficult to explicitly characterize, a priori, the admissible initial conditions starting from where the MPC is guaranteed to be feasible and stabilizing. In practice, the initial conditions are usually chosen in an ad hoc fashion and tested through extensive closed-loop simulations.

2.6 Lyapunov-Based Model Predictive Control

In this section, we introduce the LMPC design proposed in [67, 68] which allows for an explicit characterization of the stability region and guarantees controller feasibility and closed-loop stability.

For the predictive control of the system of Eq. 2.1, the LMPC is designed based on an existing explicit control law $h(x)$ which is able to stabilize the closed-loop system and satisfies the conditions of Eqs. 2.4–2.7. The formulation of the LMPC is as follows:

$$\min_{u \in S(\Delta)} \int_{t_k}^{t_{k+N}} \left[\left\| \tilde{x}(\tau) \right\|_{Q_c} + \left\| u(\tau) \right\|_{R_c} \right] d\tau, \tag{2.16}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f\big(\tilde{x}(t), u(t), 0\big), \tag{2.17}$$

$$u(t) \in U, \tag{2.18}$$

$$\tilde{x}(t_k) = x(t_k), \tag{2.19}$$

$$\frac{\partial V(x(t_k))}{\partial x} f\big(x(t_k), u(t_k), 0\big) \le \frac{\partial V(x(t_k))}{\partial x} f\big(x(t_k), h\big(x(t_k)\big), 0\big), \tag{2.20}$$

where $V(x)$ is a Lyapunov function associated with the nonlinear control law $h(x)$. The optimal solution to this LMPC optimization problem is denoted as $u_l^*(t|t_k)$ which is defined for $t \in [t_k, t_{k+N})$. The manipulated input of the system of Eq. 2.1 under the control of the LMPC of Eqs. 2.16–2.20 is defined as follows:

$$u(t) = u_l^*(t|t_k), \quad \forall t \in [t_k, t_{k+1}), \tag{2.21}$$

which implies that this LMPC also adopts a standard receding horizon strategy.

In the LMPC defined by Eqs. 2.16–2.20, the constraint of Eq. 2.20 guarantees that the value of the time derivative of the Lyapunov function, $V(x)$, at time t_k is smaller than or equal to the value obtained if the nonlinear control law $u = h(x)$ is implemented in the closed-loop system in a sample-and-hold fashion. This is a constraint that allows one to prove (when state measurements are available every synchronous sampling time) that the LMPC inherits the stability and robustness properties of the nonlinear control law $h(x)$ when it is applied in a sample-and-hold fashion.

One of the main properties of the LMPC of Eqs. 2.16–2.20 is that it possesses the same stability region Ω_ρ as the nonlinear control law $h(x)$, which implies that the origin of the closed-loop system is guaranteed to be stable and the LMPC is guaranteed to be feasible for any initial state inside Ω_ρ when the sampling time Δ and the disturbance upper bound θ are sufficiently small. Note that the region Ω_ρ can be explicitly characterized; please refer to Sect. 2.4 for more discussion on this issue. The stability property of the LMPC is inherited from the nonlinear control law $h(x)$ when it is applied in a sample-and-hold fashion; please see [14, 79] for results on sampled-data systems. The feasibility property of the LMPC is also guaranteed by the nonlinear control law $h(x)$ since $u = h(x)$ is a feasible solution to the optimization problem of Eqs. 2.16–2.20. The main advantage of the LMPC approach with respect to the nonlinear control law $h(x)$ is that optimality considerations can be taken explicitly into account (as well as constraints on the inputs and the states [68]) in the computation of the control actions within an online optimization framework while improving the closed-loop performance of the system.

Remark 2.4 Since the closed-loop stability and feasibility of the LMPC of Eqs. 2.16–2.20 are guaranteed by the nonlinear control law $h(x)$, it is unnecessary to use a terminal penalty term in the cost index (see Eq. 2.16 and compare it with Eq. 2.11) and the length of the horizon N does not affect the stability of the closed-loop system but it affects the closed-loop performance.

2.7 LMPC with Asynchronous Feedback

In this section, we modify the LMPC introduced in the previous section to take into account data losses or asynchronous measurements, both in the optimization problem formulation and in the controller implementation. In this LMPC scheme, when feedback is lost, instead of setting the control actuator outputs to zero or to the last available values, the actuators implement the last optimal input trajectory evaluated by the controller (this requires that the actuators must store in memory the last optimal input trajectory received). The LMPC is designed based on a nonlinear control law which is able to stabilize the closed-loop system and inherits the stability and robustness properties in the presence of uncertainty and data losses of the nonlinear controller, while taking into account optimality considerations. Specifically, the LMPC scheme allows for an explicit characterization of the stability region, guarantees practical stability in the absence of data losses or asynchronous measurements, and guarantees that the stability region is an invariant set for the closed-loop system under data losses or asynchronous measurements if the maximum time in which the loop is open is shorter than a given constant that depends on the parameters of the system and the nonlinear control law that is used to formulate the optimization problem. A schematic diagram of the considered closed-loop system is shown in Fig. 2.1.

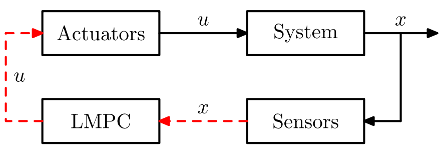


Fig. 2.1 LMPC design for systems subject to data losses. *Solid lines* denote point-to-point, wired communication links; *dashed lines* denote networked communication and/or asynchronous sampling/actuation

2.7.1 Modeling of Data Losses/Asynchronous Measurements

We assume that feedback of the state of the system of Eq. 2.1, $x(t)$, is available at asynchronous time instants t_a where $\{t_{a\geq 0}\}$ is a random increasing sequence of times; that is, the intervals between two consecutive instants are not fixed. The distribution of $\{t_{a\geq 0}\}$ characterizes the time the feedback loop is closed or the time needed to obtain a new state measurement. In general, if there exists the possibility of arbitrarily large periods of time in which feedback is not available, then it is not possible to provide guaranteed stability properties, because there exists a nonzero probability that the system operates in open-loop for a period of time large enough for the state to leave the stability region. In order to study the stability properties in a deterministic framework, we assume that there exists an upper bound T_m on the interval between two successive time instants in which the feedback loop is closed or new state measurements are available, that is:

$$\max_a \{t_{a+1} - t_a\} \leq T_m. \tag{2.22}$$

This assumption is reasonable from process control and networked control systems perspectives [69, 78, 110, 111] and allows us to study deterministic notions of stability. This model of feedback/measurements is of relevance to systems subject to asynchronous measurement samplings and to networked control systems, where the asynchronous property is introduced by data losses in the communication network connecting the sensors/actuators and the controllers.

2.7.2 LMPC Formulation with Asynchronous Feedback

When feedback is lost, most approaches set the control input to zero or to the last implemented value. Instead, in this LMPC for systems subject to data losses, when feedback is lost, we take advantage of the MPC scheme to update the input based on a prediction obtained using the system model. This is achieved using the following implementation strategy:

1. At a sampling time, t_a, when the feedback loop is closed (i.e., the current system state $x(t_a)$ is available for the controller and the controller can send information to the actuators), the LMPC evaluates the optimal future input trajectory $u(t)$ for $t \in [t_a, t_a + N\Delta)$.
2. The LMPC sends the entire optimal input trajectory (i.e., $u(t)\ \forall t \in [t_a, t_a + N\Delta)$) to the actuators.
3. The actuators implement the input trajectory until the feedback loop is closed again at the next sampling time t_{a+1}; that is, the actuators implement $u(t)$ in $t \in [t_a, t_{a+1})$.
4. When a new measurement is received ($a \leftarrow a + 1$), go to Step 1.

In this implementation strategy, when the state is not available, or the data sent from the controller to the actuators is lost, the actuators keep implementing the last received optimal trajectory. If data is lost for a period larger than the prediction horizon, the actuators set the inputs to the last implemented values or to fixed values. This strategy is a receding horizon scheme, which takes into account that data losses may occur. This strategy is motivated by the fact that when no feedback is available, a reasonable estimate of the future evolution of the system is given by the nominal trajectory. The LMPC design taking into account data losses/asynchronous measurements, therefore modifies the standard implementation scheme of switching off the actuators ($u = 0$) or setting the actuators to nominal values or to the last computed input values. The idea of using the model to predict the evolution of the system when no feedback is possible has also been used in the context of sampled-data linear systems, see [70, 71, 74, 75]. The actuators not only receive and implement given inputs, but must also be able to store future trajectories to implement them in case data losses occur. This means that to handle data losses, not only the control algorithms must be modified, but also the control actuator hardware that implements the control actions.

When data losses are present in the feedback loop, the existing LMPC schemes [42, 67, 68, 86] can not guarantee the closed-loop stability no matter whether the actuators keep the inputs at the last values, set the inputs to constant values, or keep on implementing the previously evaluated input trajectories. In particular, there is no guarantee that the LMPC optimization problems will be feasible for all time, i.e., that the state will remain inside the stability region for all time. In the LMPC design of Eqs. 2.16–2.20, the constraint of Eq. 2.20 only takes into account the first prediction step and does not restrict the behavior of the system after the first step. If no additional constraints are included in the optimization problem, no claims on the closed-loop behavior of the system can be made. For this reason, when data losses are taken into account, the constraints of the LMPC problem have to be modified. The LMPC that takes into account data losses in an explicit way is based on the following finite horizon constrained optimal control problem:

$$\min_{u \in S(\Delta)} \int_{t_a}^{t_a+N\Delta} \left[\left\| \tilde{x}(\tau) \right\|_{Q_c} + \left\| u(\tau) \right\|_{R_c} \right] d\tau, \tag{2.23}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f\big(\tilde{x}(t), u(t), 0\big), \tag{2.24}$$

$$\dot{\hat{x}}(t) = f\big(\hat{x}(t), h\big(\hat{x}(t_a + j\Delta)\big), 0\big), \quad \forall t \in \big[t_a + j\Delta, t_a + (j+1)\Delta\big), \tag{2.25}$$

$$u(t) \in U, \tag{2.26}$$

$$\tilde{x}(t_a) = \hat{x}(t_a) = x(t_a), \tag{2.27}$$

$$V\big(\tilde{x}(t)\big) \le V\big(\hat{x}(t)\big), \quad \forall t \in [t_a, t_a + N_R\Delta), \tag{2.28}$$

where $\hat{x}(t)$ is the trajectory of the nominal system under the nonlinear control law $u = h(\hat{x}(t))$ when it is implemented in a sample-and-hold fashion, $j = 0, 1, \ldots, N-1$, and N_R is the smallest integer satisfying $N_R\Delta \ge T_m$. This optimization problem does not depend on the uncertainty and assures that the LMPC inherits the properties of the nonlinear control law $h(x)$. To take full advantage of the use of the nominal model in the computation of the control action, the prediction horizon should be chosen in a way such that $N \geq N_R$.

The optimal solution to the LMPC optimization problem of Eqs. 2.23–2.28 is denoted as $u_a^*(t|t_a)$ which is defined for $t \in [t_a, t_a + N\Delta)$. The manipulated input of the system of Eq. 2.1 under the LMPC of Eqs. 2.23–2.28 is defined as follows:

$$u(t) = u_a^*(t|t_a), \quad \forall t \in [t_a, t_{a+1}), \tag{2.29}$$

where t_{a+1} is the next time instant in which the feedback loop will be closed again. This is a modified receding horizon scheme which takes advantage of the predicted input trajectory in the case of data losses.

In the design of the LMPC of Eqs. 2.23–2.28, the constraint of Eq. 2.25 is used to generate a system state trajectory under the nonlinear control law $u = h(x)$ implemented in a sample-and-hold fashion; this trajectory is used as a reference trajectory to construct the Lyapunov-based constraint of Eq. 2.28 which is required to be satisfied for a time period which covers the maximum possible open-loop operation time T_m. This Lyapunov-based constraint allows one to prove the closed-loop stability in the presence of data losses in the closed-loop system.

Remark 2.5 The LMPC of Eqs. 2.23–2.28 optimizes a cost function, subject to a set of constraints defined by the state trajectory corresponding to the nominal system in closed-loop. This allows us to formulate an LMPC problem that does not depend on the uncertainty and so it is of manageable computational complexity.

2.7.3 Stability Properties

The LMPC of Eqs. 2.23–2.28 computes the control input u applied to the system of Eq. 2.1 in a way such that in the closed-loop system, the value of the Lyapunov function at time instant t_a (i.e., $V(x(t_a))$) is a decreasing sequence of values with a lower bound. Following Lyapunov arguments, this property guarantees practical stability of the closed-loop system. This is achieved due to the constraint of Eq. 2.28. This property is summarized in Theorem 2.1 below. To state this theorem, we need the following propositions.

Proposition 2.1 *Consider the nominal sampled trajectory* $\hat{x}(t)$ *of the system of Eq.* 2.1 *in closed-loop for a controller* $h(x)$, *which satisfies the conditions of Eqs.* 2.4–2.7, *obtained by solving recursively*:

$$\dot{\hat{x}}(t) = f\big(\hat{x}(t), h\big(\hat{x}(t_k)\big), 0\big), \quad t \in [t_k, t_{k+1}), \tag{2.30}$$

where $t_k = t_0 + k\Delta$, $k = 0, 1, \ldots$. *Let* $\Delta, \varepsilon_s > 0$ *and* $\rho > \rho_s > 0$ *satisfy*:

$$-\alpha_3\big(\alpha_2^{-1}(\rho_s)\big) + L'_x M\Delta \leq -\varepsilon_s/\Delta. \tag{2.31}$$

Then if $\rho_{\min} < \rho$ *where*:

$$\rho_{\min} = \max\{V(\hat{x}(t+\Delta)) : V(\hat{x}(t)) \le \rho_s\} \tag{2.32}$$

and $\hat{x}(t_0) \in \Omega_\rho$, *the following inequality holds*:

$$V(\hat{x}(t)) \le V(\hat{x}(t_k)), \quad \forall t \in [t_k, t_{k+1}), \tag{2.33}$$

$$V(\hat{x}(t_k)) \le \max\{V(\hat{x}(t_0)) - k\varepsilon_s, \rho_{\min}\}. \tag{2.34}$$

Proof Following the definition of $\hat{x}(t)$, the time derivative of the Lyapunov function $V(x)$ along the trajectory $\hat{x}(t)$ of the system of Eq. 2.1 in $t \in [t_k, t_{k+1})$ is given by:

$$\dot{V}(\hat{x}(t)) = \frac{\partial V(\hat{x}(t))}{\partial x} f(\hat{x}(t), h(\hat{x}(t_k)), 0). \tag{2.35}$$

Adding and subtracting $\frac{\partial V(\hat{x}(t_k))}{\partial x} f(\hat{x}(t_k), h(\hat{x}(t_k)), 0)$ and taking into account Eq. 2.5, we obtain:

$$\begin{aligned}\dot{V}(\hat{x}(t)) \le -\alpha_3(\|\hat{x}(t_k)\|) + \frac{\partial V(\hat{x}(t))}{\partial x} f(\hat{x}(t), h(\hat{x}(t_k)), 0) \\ - \frac{\partial V(\hat{x}(t_k))}{\partial x} f(\hat{x}(t_k), h(\hat{x}(t_k)), 0).\end{aligned} \tag{2.36}$$

From the Lipschitz property of Eq. 2.10 and the above inequality of Eq. 2.36, we have that:

$$\dot{V}(\hat{x}(t)) \le -\alpha_3(\alpha_2^{-1}(\rho_s)) + L'_x \|\hat{x}(t) - \hat{x}(t_k)\| \tag{2.37}$$

for all $\hat{x}(t_k) \in \Omega_\rho / \Omega_{\rho_s}$. Taking into account the Lipschitz property of Eq. 2.8 and the continuity of $\hat{x}(t)$, the following bound can be written for all $t \in [t_k, t_{k+1})$:

$$\|\hat{x}(t) - \hat{x}(t_k)\| \le M\Delta. \tag{2.38}$$

Using the expression of Eq. 2.38, we obtain the following bound on the time derivative of the Lyapunov function for $t \in [t_k, t_{k+1})$, for all initial states $\hat{x}(t_k) \in \Omega_\rho / \Omega_{\rho_s}$:

$$\dot{V}(\hat{x}(t)) \le -\alpha_3(\alpha_2^{-1}(\rho_s)) + L'_x M\Delta. \tag{2.39}$$

If the condition of Eq. 2.31 is satisfied, then $\dot{V}(\hat{x}(t)) \le -\varepsilon_s/\Delta$. Integrating this bound on $t \in [t_k, t_{k+1})$ we obtain that the inequality of Eq. 2.33 holds. Using Eq. 2.33 recursively, it is proved that, if $x(t_0) \in \Omega_\rho / \Omega_{\rho_s}$, the state converges to Ω_{ρ_s} in a finite number of sampling times without leaving the stability region. Once the state converges to $\Omega_{\rho_s} \subseteq \Omega_{\rho_{\min}}$, it remains inside $\Omega_{\rho_{\min}}$ for all times. This statement holds because of the definition of $\rho_{\min}$ in Eq. 2.32. □

Proposition 2.1 ensures that if the nominal system under the control $u = h(x)$ implemented in a sample-and-hold fashion with state feedback every sampling time starts in the region Ω_ρ, then it is ultimately bounded in $\Omega_{\rho_{\min}}$. The following Proposition 2.2 provides an upper bound on the deviation of the system state trajectory obtained using the nominal model of Eq. 2.1, from the closed-loop state trajectory of the system of Eq. 2.1 under uncertainty (i.e., $w(t) \neq 0$) when the same control actions are applied.

Proposition 2.2 *Consider the systems*:

$$\dot{x}_a(t) = f\big(x_a(t), u(t), w(t)\big), \tag{2.40}$$

$$\dot{x}_b(t) = f\big(x_b(t), u(t), 0\big) \tag{2.41}$$

with initial states $x_a(t_0) = x_b(t_0) \in \Omega_\rho$. *There exists a class* $\mathcal{K}$ *function* $f_W(\cdot)$ *such that*:

$$\big\|x_a(t) - x_b(t)\big\| \leq f_W(t - t_0), \tag{2.42}$$

for all $x_a(t), x_b(t) \in \Omega_\rho$ *and all* $w(t) \in W$ *with*:

$$f_W(\tau) = \frac{L_w \theta}{L_x}\big(e^{L_x \tau} - 1\big). \tag{2.43}$$

Proof Define the error vector as $e(t) = x_a(t) - x_b(t)$. The time derivative of the error is given by:

$$\dot{e}(t) = f\big(x_a(t), u(t), w(t)\big) - f\big(x_b(t), u(t), 0\big). \tag{2.44}$$

From the Lipschitz property of Eq. 2.9, the following inequality holds:

$$\big\|\dot{e}(t)\big\| \leq L_w \big\|w(t)\big\| + L_x \big\|x_a(t) - x_b(t)\big\| \leq L_w \theta + L_x \big\|e(t)\big\| \tag{2.45}$$

for all $x_a(t), x_b(t) \in \Omega_\rho$ and $w(t) \in W$. Integrating $\|\dot{e}(t)\|$ with initial condition $e(t_0) = 0$ (recall that $x_a(t_0) = x_b(t_0)$), the following bound on the norm of the error vector is obtained:

$$\big\|e(t)\big\| \leq \frac{L_w \theta}{L_x}\big(e^{L_x (t - t_0)} - 1\big). \tag{2.46}$$

This implies that the inequality of Eq. 2.42 holds for:

$$f_W(\tau) = \frac{L_w \theta}{L_x}\big(e^{L_x \tau} - 1\big) \tag{2.47}$$

which proves this proposition. □

Proposition 2.3 below bounds the difference between the magnitudes of the Lyapunov function of two states in Ω_ρ.

Proposition 2.3 *Consider the Lyapunov function* $V(\cdot)$ *of the system of Eq.* 2.1. *There exists a quadratic function* $f_V(\cdot)$ *such that*:

$$V(x) \leq V\big(x'\big) + f_V\big(\|x - x'\|\big) \tag{2.48}$$

for all $x, x' \in \Omega_\rho$ *where*:

$$f_V(s) = \alpha_4\big(\alpha_1^{-1}(\rho)\big)s + M_v s^2 \tag{2.49}$$

with $M_v > 0$.

Proof Since the Lyapunov function $V(x)$ is continuous and bounded on compact sets, there exists a positive constant M_v such that a Taylor series expansion of V around x' yields:

$$V(x) \leq V\big(x'\big) + \frac{\partial V(x')}{\partial x}\|x - x'\| + M_v\|x - x'\|^2, \quad \forall x, x' \in \Omega_\rho. \tag{2.50}$$

Note that the term $M_v\|x - x'\|^2$ bounds the high order terms of the Taylor series of $V(x)$ for $x, x' \in \Omega_\rho$. Taking into account Eq. 2.6, the following bound for $V(x)$ is obtained:

$$V(x) \leq V\big(x'\big) + \alpha_4\big(\alpha_1^{-1}(\rho)\big)\|x - x'\| + M_v\|x - x'\|^2, \quad \forall x, x' \in \Omega_\rho, \tag{2.51}$$

which proves this proposition. □

In Theorem 2.1 below, we provide sufficient conditions under which the LMPC design of Eqs. 2.23–2.28 guarantees that the state of the closed-loop system of Eq. 2.1 is ultimately bounded in a region that contains the origin.

Theorem 2.1 *Consider the system of Eq.* 2.1 *in closed-loop, with the loop closing at asynchronous time instants* $\{t_{a\geq 0}\}$ *that satisfy the condition of Eq.* 2.22, *under the LMPC of Eqs.* 2.23–2.28 *based on a controller* $h(x)$ *that satisfies the conditions of Eqs.* 2.4–2.7. *Let* $\Delta, \varepsilon_s > 0$, $\rho > \rho_{\min} > 0$, $\rho > \rho_s > 0$ *and* $N \geq N_R \geq 1$ *satisfy the condition of Eq.* 2.31 *and the following inequality*:

$$-N_R\varepsilon_s + f_V\big(f_W(N_R\Delta)\big) < 0 \tag{2.52}$$

with $f_V(\cdot)$ *and* $f_W(\cdot)$ *defined in Eqs.* 2.49 *and* 2.43, *respectively, and* N_R *being the smallest integer satisfying* $N_R\Delta \geq T_m$. *If* $x(t_0) \in \Omega_\rho$, *then* $x(t)$ *is ultimately bounded in* $\Omega_{\rho_a} \subseteq \Omega_\rho$ *where*:

$$\rho_a = \rho_{\min} + f_V\big(f_W(N_R\Delta)\big) \tag{2.53}$$

with $\rho_{\min}$ *defined as in Eq.* 2.32.

Proof In order to prove that the closed-loop system is ultimately bounded in a region that contains the origin, we prove that $V(x(t_a))$ is a decreasing sequence of values with a lower bound. The proof is divided into two parts.

Part 1: In this part, we prove that the stability results stated in Theorem 2.1 hold in the case that $t_{a+1} - t_a = T_m$ for all a and $T_m = N_R \Delta$. This case corresponds to the worst possible situation in the sense that the LMPC needs to operate in open-loop for the maximum possible amount of time. In order to simplify the notation, we assume that all the notations used in this proof refer to the final solution of the LMPC of Eqs. 2.23–2.28 solved at time t_a. By Proposition 2.1 and the fact that $t_{a+1} = t_a + N_R \Delta$, the following inequality can be obtained:

$$V\big(\hat{x}(t_{a+1})\big) \leq \max\big\{V\big(\hat{x}(t_a)\big) - N_R \varepsilon_s, \rho_{\min}\big\}. \tag{2.54}$$

From the constraint of Eq. 2.28, the inequality of Eq. 2.54 and taking into account the fact that $\hat{x}(t_a) = \tilde{x}(t_a) = x(t_a)$, the following inequality can be written:

$$V\big(\tilde{x}(t_{a+1})\big) \leq \max\big\{V\big(x(t_a)\big) - N_R \varepsilon_s, \rho_{\min}\big\}. \tag{2.55}$$

When $x(t) \in \Omega_\rho$ for all times (this point will be proved below), we can apply Proposition 2.3 to obtain the following inequality:

$$V\big(x(t_{a+1})\big) \leq V\big(\tilde{x}(t_{a+1})\big) + f_V\big(\big\|\tilde{x}(t_{a+1}) - x(t_{a+1})\big\|\big). \tag{2.56}$$

Applying Proposition 2.2, we obtain the following upper bound on the deviation of $\tilde{x}(t)$ from $x(t)$:

$$\big\|x(t_{a+1}) - \tilde{x}(t_{a+1})\big\| \leq f_W(N_R \Delta). \tag{2.57}$$

From the inequalities of Eqs. 2.56 and 2.57, the following upper bound on $V(x(t_{a+1}))$ can be written:

$$V\big(x(t_{a+1})\big) \leq V\big(\tilde{x}(t_{a+1})\big) + f_V\big(f_W(N_R \Delta)\big). \tag{2.58}$$

Using the inequality of Eq. 2.55, we can rewrite the inequality of Eq. 2.58 as follows:

$$V\big(x(t_{a+1})\big) \leq \max\big\{V\big(x(t_a)\big) - N_R \varepsilon_s, \rho_{\min}\big\} + f_V\big(f_W(N_R \Delta)\big). \tag{2.59}$$

If the condition of Eq. 2.52 is satisfied, from the inequality of Eq. 2.59, we know that there exists $\varepsilon_w > 0$ such that the following inequality holds:

$$V\big(x(t_{a+1})\big) \leq \max\big\{V\big(x(t_a)\big) - \varepsilon_w, \rho_a\big\}, \tag{2.60}$$

which implies that if $x(t_a) \in \Omega_\rho / \Omega_{\rho_a}$, then $V(x(t_{a+1})) < V(x(t_a))$, and if $x(t_a) \in \Omega_{\rho_a}$, then $V(x(t_{a+1})) \leq \rho_a$.

Because $f_W(\cdot)$ and $f_V(\cdot)$ are strictly increasing functions of their arguments and $f_V(\cdot)$ is convex (see Propositions 2.2 and 2.3 for the expressions of $f_W(\cdot)$ and $f_V(\cdot)$), the inequality of Eq. 2.60 also implies that:

$$V\big(x(t)\big) \leq \max\big\{V\big(x(t_a)\big), \rho_a\big\}, \quad \forall t \in [t_a, t_{a+1}). \tag{2.61}$$

Using the inequality of Eq. 2.61 recursively, it can be proved that if $x(t_0) \in \Omega_\rho$, then the closed-loop trajectories of the system of Eq. 2.1 under the LMPC of Eqs. 2.23–2.28 stay in Ω_ρ for all times (i.e., $x(t) \in \Omega_\rho,\ \forall t$). Moreover, it can be proved that if $x(t_0) \in \Omega_\rho$, the closed-loop trajectories of the system of Eq. 2.1 satisfy:

$$\limsup_{t \to \infty} V\big(x(t)\big) \le \rho_a.$$

This proves that $x(t) \in \Omega_\rho$ for all times and $x(t)$ is ultimately bounded in Ω_{ρ_a} for the case when $t_{a+1} - t_a = T_m$ for all a and $T_m = N_R \Delta$.

Part 2: In this part, we extend the results proved in Part 1 to the general case, that is, $t_{a+1} - t_a \le T_m$ for all a and $T_m \le N_R \Delta$ which implies that $t_{a+1} - t_a \le N_R \Delta$. Because $f_W(\cdot)$ and $f_V(\cdot)$ are strictly increasing functions of their arguments and $f_V(\cdot)$ is convex, following similar steps as in Part 1, it can be shown that the inequality of Eq. 2.61 still holds. This proves that the stability results stated in Theorem 2.1 hold. □

Remark 2.6 Theorem 2.1 is important from an MPC point of view because if the maximum time without data losses is smaller than the maximum time that the system can operate in open-loop without leaving the stability region, the feasibility of the optimization problem for all times is guaranteed, since each time feedback is regained, the state is guaranteed to be inside the stability region, thereby yielding a feasible optimization problem.

Remark 2.7 In the LMPC of Eqs. 2.23–2.28, no state constraint has been considered but the presented approach can be extended to handle state constraints by restricting the closed-loop stability region further to satisfy the state constraints.

Remark 2.8 It is also important to remark that when there are data losses in the control system, standard MPC formulations do not provide guaranteed closed-loop stability results. For any MPC scheme, in order to obtain guaranteed closed-loop stability results, even in the case where initial feasibility of the optimization problem is given, the formulation of the optimization problem has to be modified accordingly to take into account data losses in an explicit way.

Remark 2.9 Although the proof of Theorem 2.2 is constructive, the constants obtained are conservative. This is the case with most of the results of the type presented in this book. In practice, the different constants are better estimated through closed-loop simulations. The various inequalities provided are more useful as guidelines on the interaction between the various parameters that define the system and the controller and may be used as guidelines to design the controller and the network.

2.7.4 Application to a Chemical Reactor

Consider a well mixed, nonisothermal continuously stirred tank reactor (CSTR) where three parallel irreversible elementary exothermic reactions take place of the form $A \rightarrow B$, $A \rightarrow C$ and $A \rightarrow D$. B is the desired product and C and D are byproducts. The feed to the reactor consists of pure A at flow rate F, temperature T_{A0} and molar concentration $C_{A0} + \Delta C_{A0}$ where ΔC_{A0} is an unknown time-varying uncertainty. Due to the nonisothermal nature of the reactor, a jacket is used to remove/provide heat to the reactor. Using first principles and standard modeling assumptions, the following mathematical model of the process is obtained [21]:

$$\frac{dT}{dt} = \frac{F}{V_r}(T_{A0} - T) - \sum_{i=1}^{3} \frac{\Delta H_i}{\sigma c_p} k_{i0} e^{\frac{-E_i}{RT}} C_A + \frac{Q}{\sigma c_p V_r}, \tag{2.62}$$

$$\frac{dC_A}{dt} = \frac{F}{V_r}(C_{A0} + \Delta C_{A0} - C_A) + \sum_{i=1}^{3} k_{i0} e^{\frac{-E_i}{RT}} C_A, \tag{2.63}$$

where C_A denotes the concentration of the reactant A, T denotes the temperature of the reactor, Q denotes the rate of heat input/removal, V_r denotes the volume of the reactor, $\Delta H_i, k_{i0}, E_i, i = 1, 2, 3$ denote the enthalpies, preexponential constants and activation energies of the three reactions, respectively, and c_p and σ denote the heat capacity and the density of the fluid in the reactor, respectively. The values of the process parameters are shown in Table 2.1.

For $Q_s = 0$ KJ/h (Q_s is the steady-state value of Q), the CSTR of Eqs. 2.62–2.63 has three steady-states (two locally asymptotically stable and one unstable). The control objective is to stabilize the system at the open-loop unstable steady state $T_s = 388$ K, $C_{As} = 3.59$ mol/l. The manipulated input is the rate of heat input Q. We consider a time-varying uncertainty in the concentration of the inflow $|\Delta C_{A0}| \leq 0.5$ kmol/m^3. The control system is subject to data losses in both the sensor-controller and the controller-actuator links.

To demonstrate the theoretical results, we first design the nonlinear control law $h(x)$ as a Lyapunov-based feedback law using the method presented in [97]. The

Table 2.1 Process parameters of the CSTR of Eqs. 2.62–2.63

F	4.998 [m^3/h]	k_{10}	3×10^6 [h^{-1}]
V_r	1 [m^3]	k_{20}	3×10^5 [h^{-1}]
R	8.314 [KJ/kmol K]	k_{30}	3×10^5 [h^{-1}]
T_{A0}	300 [K]	E_1	5×10^4 [KJ/kmol]
C_{A0}	4 [kmol/m^3]	E_2	7.53×10^4 [KJ/kmol]
ΔH_1	-5.0×10^4 [KJ/kmol]	E_3	7.53×10^4 [KJ/kmol]
ΔH_2	-5.2×10^4 [KJ/kmol]	σ	1000 [kg/m^3]
ΔH_3	-5.4×10^4 [KJ/kmol]	c_p	0.231 [KJ/kg K]

CSTR of Eqs. 2.62–2.63 belongs to the following class of nonlinear systems:

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t) + w\big(x(t)\big), \tag{2.64}$$

where $x^T = [T - T_s \; C_A - C_{As}]$ is the state, $u = Q - Q_s$ is the input and $w = \Delta C_{A0}$ is a time varying bounded disturbance with the upper bound $\theta = 0.5$ kmol/m^3. We consider the Lyapunov function $V(x) = x^T P x$ with:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 10^4 \end{bmatrix}. \tag{2.65}$$

The values of the weights have been chosen to account for the different range of numerical values for each state. The following feedback law [97] asymptotically stabilizes the open-loop unstable steady-state of the nominal process:

$$h(x) = \begin{cases} -\frac{L_f V + \sqrt{(L_f V)^2 + (L_g V)^4}}{L_g V} & \text{if } L_g V \neq 0, \\ 0 & \text{if } L_g V = 0, \end{cases} \tag{2.66}$$

where $L_f V = \frac{\partial V(x)}{\partial x} f(x)$ and $L_g V = \frac{\partial V(x)}{\partial x} g(x)$ denote the Lie derivatives of the scalar function V with respect to the vectors fields f and g in Eq. 2.64, respectively. This controller will be used in the design of the LMPC of Eqs. 2.16–2.20 and the LMPC of Eqs. 2.23–2.28. The stability region Ω_ρ is defined as $V(x) \leq 1000$, i.e., $\rho = 1000$.

First, we have to choose an appropriate sampling time and a maximum prediction horizon for the LMPC based on the properties of $h(x)$. The inequalities obtained in the main results of this section are conservative to be used to estimate an appropriate sampling time for a given uncertainty bound and the maximum time that the system can operate in open-loop without leaving the stability region. In order to obtain practical estimates, we resort to extensive off-line closed-loop simulations under the Lyapunov-based controller of Eq. 2.66. After trying different sampling times, we choose $\Delta = 0.05$ h. For this sampling time, the closed-loop system with $u = h(x)$ is practically stable and the performance is similar to the closed-loop system with continuous measurements. With this sampling time, the maximum time such that the system remains in Ω_ρ when controlled in open-loop with the nominal sampled input trajectory is 5Δ (i.e., $N_R = 5$). This value is also estimated using data from simulations.

We implement the LMPCs presented in the previous sections using a sampling time $\Delta = 0.05$ h and a prediction horizon $N = N_R = 5$. The cost function is defined by the weighting matrices $Q_c = P$ and $R_c = 10^{-6}$. The values of the weights have been tuned in a way such that the values of the control inputs are comparable to the ones computed by the Lyapunov-based controller (i.e., same order of magnitude of the input signal and convergence time of the closed-loop system when no uncertainty or data losses are taken into account).

We will first compare the LMPC of Eqs. 2.23–2.28 with the original LMPC of Eqs. 2.16–2.20. In this scheme, no data losses were taken into account. We implement the two LMPCs using the same strategy, that is, sending to the actuator the

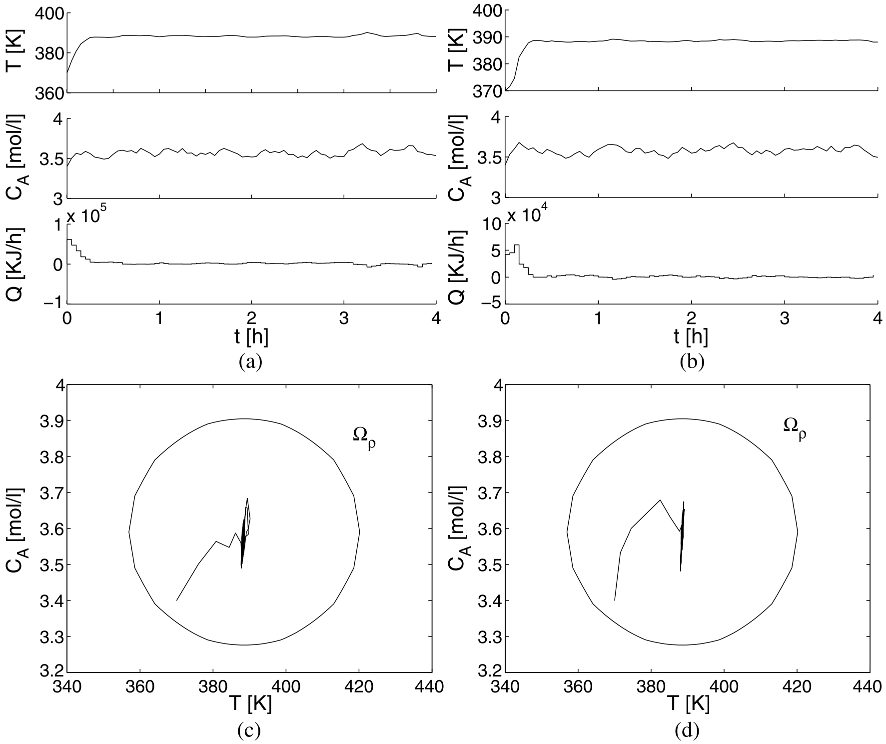


Fig. 2.2 (**a**, **c**) State and input trajectories of the CSTR of Eqs. 2.62–2.63 with the LMPC of Eqs. 2.23–2.28 with no data losses; (**b**, **d**) state and input trajectories of the CSTR of Eqs. 2.62–2.63 with the LMPC of Eqs. 2.16–2.20 with no data losses

whole optimal input trajectory, so in case data losses occur, the input is updated as in the modified receding horizon scheme. The same weights, sampling time and prediction horizon are used.

In Fig. 2.2, the trajectories of both LMPCs are shown assuming no data is lost, that is, the state $x(t_k)$ is available every sampling time. It can be seen that both closed-loop systems are practically stable. Note that regarding optimality, for a given state, the LMPC of Eqs. 2.16–2.20 (not necessarily the closed-loop trajectory) yields a lower cost than the LMPC of Eqs. 2.23–2.28, because the constraints that define the LMPC of Eqs. 2.16–2.20 are less restrictive (i.e., the Lyapunov-based constraint must hold only in the first sampling time whereas in the LMPC of Eqs. 2.23–2.28 it must hold along the whole prediction horizon).

When data losses occur, the LMPC of Eqs. 2.23–2.28 is more robust. The stability region is an invariant set for the closed-loop system if $T_m \leq N\Delta$. That is not the case with the LMPC of Eqs. 2.16–2.20. In Fig. 2.3, the trajectories of the closed-loop system under both LMPCs are shown for the worst case of data loss scenario with $T_m = 5\Delta$; that is, the system receives only one measurement of the actual state every 5 samples. These trajectories account for the worst-case effect

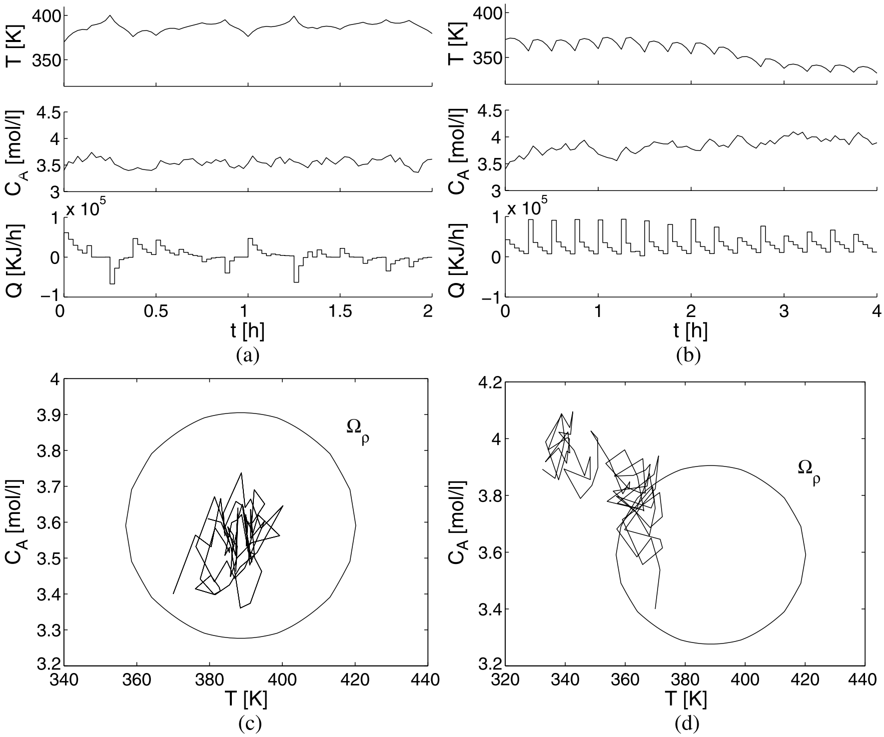


Fig. 2.3 (**a**, **c**) Worst case state and input trajectories of the CSTR of Eqs. 2.62–2.63 with the LMPC of Eqs. 2.23–2.28 with $T_m = 5\Delta$; (**b**, **d**) state and input trajectories of the CSTR of Eqs. 2.62–2.63 with the LMPC of Eqs. 2.16–2.20 with $T_m = 5\Delta$

of the data losses. The trajectories are shown in the state space along with the closed-loop stability region Ω_ρ. It can be seen that the trajectory under the LMPC of Eqs. 2.16–2.20 leaves the stability region, while the trajectory under the LMPC of Eqs. 2.23–2.28 remains inside. When data losses are taken into account, in order to inherit the stability properties of the Lyapunov-based controller of Eq. 2.66, the constraints must be modified to take into account data losses as in the LMPC of Eqs. 2.23–2.28.

We now compare the LMPC of Eqs. 2.23–2.28 with the Lyapunov-based controller of Eq. 2.66 applied in a sample-and-hold fashion following a "last available control" strategy, i.e., when data is lost, the actuator keeps implementing the last received input value. Note that, through extensive simulations, we have found that in this particular example, the strategy of setting the input to zero when data losses occur, yields worst results than the strategy of implementing the last available input. In Fig. 2.4, the worst case trajectories with $T_m = 2\Delta$ for both controllers are shown. It can be seen that, due to the instability of the open-loop steady state, for this small amount of losses, the Lyapunov-based controller is not able to stabilize the system.

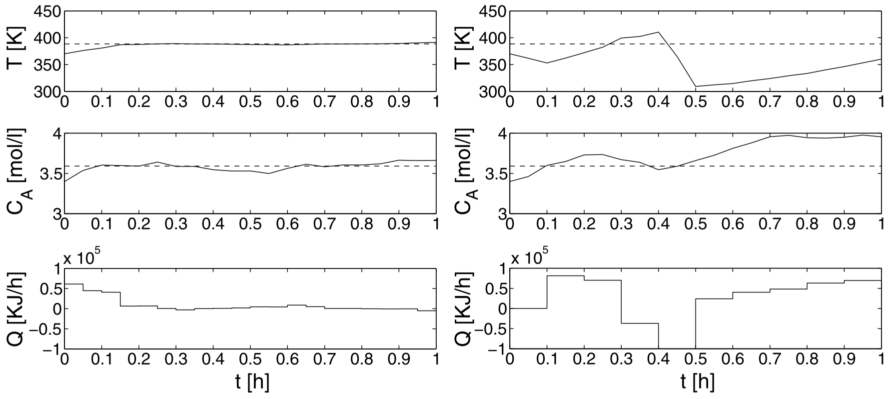


Fig. 2.4 Worst case state and input trajectories of the CSTR of Eqs. 2.62–2.63 with $T_m = 2\Delta$ in closed-loop with (**a**) the LMPC of Eqs. 2.23–2.28 and (**b**) the Lyapunov-based controller of Eq. 2.66

Table 2.2 Total performance costs along the closed-loop trajectories of the CSTR of Eqs. 2.62–2.63 under the Lyapunov-based controller of Eq. 2.66 and the LMPC of Eqs. 2.23–2.28

sim.	Lyapunov-based controller of Eq. 2.66	LMPC of Eqs. 2.23–2.28
1	0.1262×10^{12}	0.0396×10^{12}
2	0.3081×10^{12}	0.2723×10^{12}
3	0.0561×10^{12}	0.0076×10^{12}
4	0.9622×10^{11}	0.2884×10^{11}
5	3.8176×10^{11}	1.3052×10^{11}
6	0.9078×10^{11}	0.0950×10^{11}
7	0.4531×10^{12}	0.2678×10^{12}
8	0.6752×10^{11}	0.5689×10^{11}
9	1.0561×10^{11}	0.6776×10^{11}
10	0.5332×10^{12}	0.3459×10^{12}

This is due to the fact that this control scheme does not update the control actuator output using the model, as the LMPC of Eqs. 2.23–2.28 does.

We have also carried out another set of simulations to demonstrate that the LMPC of Eqs. 2.23–2.28, although inherits the same stability and robustness properties of the Lyapunov-based controller that it employs, it does outperform the Lyapunov-based controller of Eq. 2.66 from a performance index point of view. Table 2.2 shows the total cost computed for 10 different closed-loop simulations under the LMPC and the Lyapunov-based controller implemented in a sample-and-hold fashion, using the nominal model to predict the evolution of the system when data is lost. To carry out this comparison, we compute the total cost of each simulation based on

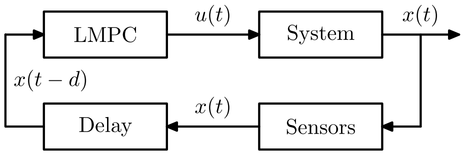


Fig. 2.5 LMPC design for systems subject to time-varying measurement delays

the performance index of the LMPC which has the form:

$$\int_{t_0}^{t_f} \left[\|x(\tau)\|_{Q_c} + \|u(\tau)\|_{R_c} \right] d\tau, \tag{2.67}$$

where $t_0 = 0$ is the initial time of the simulations and $t_f = 4$ h is the end of the simulation. For each pair of simulations (one for each controller), a different initial state inside the stability region, a different random uncertainty trajectory and a different data losses realization is chosen. As it can be seen in Table 2.2, the total cost under the LMPC of Eqs. 2.23–2.28 is lower than the corresponding total cost under the Lyapunov-based controller. This demonstrates that in this example, the LMPC shares the same robustness and stability properties and is more optimal than the Lyapunov-based controller, which is not designed taking into account any optimality consideration.

The simulations have been done in MATLAB® using *fmincon* and a Runge–Kutta solver with a fixed integration time of 0.001 h. To simulate the time-varying uncertainty, a different random value $w(t)$ has been applied at each integration step.

2.8 LMPC with Delayed Measurements

In this section, we deal with the design of LMPC for nonlinear systems subject to time-varying measurement delays in the feedback loop. In the LMPC design that will be presented, when measurement delays occur, the nominal model of the system is used together with the latest available measurement to estimate the current state, and the resulting estimate is used to evaluate the LMPC; at time instants where no measurements are available due to the delay, the actuator implements the last optimal input trajectory evaluated by the controller as discussed in the previous section. The LMPC accounting for delays is also designed based on a nonlinear control law which is able to stabilize the closed-loop system and inherits the stability and robustness properties in the presence of uncertainty and time-varying delays of the nonlinear control law, while taking into account optimality considerations. The closed-loop system considered in this section is shown in Fig. 2.5.

2.8.1 Modeling of Delayed Measurements

We assume that the state of the system of Eq. 2.1 is received by the controller at asynchronous time instants t_a where $\{t_{a\geq 0}\}$ is a random increasing sequence of times

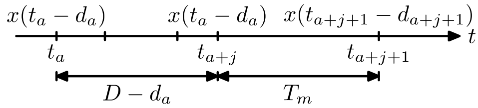


Fig. 2.6 A possible sequence of delayed measurements

and that there exists an upper bound T_m on the interval between two successive measurements as described in Eq. 2.22. We also assume that there are delays in the measurements received by the controller due to delays in the sampling process and data transmission. In order to model delays in measurements, another auxiliary variable d_a is introduced to indicate the delay corresponding to the measurement received at time t_a, that is, at time t_a, the measurement $x(t_a - d_a)$ is received. In general, if the sequence $\{d_{a \geq 0}\}$ is modeled using a random process, there exists the possibility of arbitrarily large delays. In this case, it is improper to use all the delayed measurements to estimate the current state and decide the control inputs, because when the delays are too large, they may introduce enough errors to destroy the stability of the closed-loop system. In order to study the stability properties in a deterministic framework, we assume that the delays associated with the measurements are smaller than an upper bound D, that is:

$$d_a \leq D. \tag{2.68}$$

The size of D is, in general, related to measurement sensor delays and data transmission network delays. We note that for chemical processes, the delay in the measurements received by a controller are mainly caused in the measurement sampling process. We also assume that the time instant when a measurement is sampled is recorded and transmitted together with the measurement. This assumption is practical for many process control applications and implies that the delay in a measurement received by the controller is calculable and can be assumed to be known.

Note that because the delays are time-varying, it is possible that at a time instant t_a, the controller may receive a measurement $x(t_a - d_a)$ which does not provide new information (i.e., $t_a - d_a \leq t_{a-1} - d_{a-1}$); that is, the controller has already received a measurement of the state after time $t_a - d_a$. We assume that each measurement is time-labeled, and hence the controller is able to discard a newly received measurement if $t_a - d_a < t_{a-1} - d_{a-1}$. Figure 2.6 shows part of a possible sequence of $\{t_{a \geq 0}\}$. At time t_a, the state measurement $x(t_a - d_a)$ is received. There exists a possibility that between t_a and t_{a+j}, with $t_{a+j} - t_a = D - d_a$ and j being an unknown integer, all the measurements received do not provide new information. Note that any measurements received after t_{a+j} provide new information because the maximum delay is D and the latest received measurement was $x(t_a - d_a)$. The maximum possible time interval between t_{a+j} and t_{a+j+1} is T_m. Therefore, the maximum amount of time in which the system might operate in open-loop following t_a is $D + T_m - d_a$. This upper bound will be used in the formulation of the LMPC design for systems subject to delayed measurements below.

Remark 2.10 The sequences $\{t_{a \geq 0}\}$ and $\{d_{a \geq 0}\}$ characterize the time needed to obtain a new measurement in the case of asynchronous measurements or the quality of the network link in the case of networked (wired or wireless) communications subject to data losses and time-varying delays. The model is general and can be used to model a wide class of systems subject to asynchronous, delayed measurements.

2.8.2 LMPC Formulation with Measurement Delays

A controller for a system subject to time-varying measurement delays must take into account two important issues. First, when a new measurement is received, this measurement may not correspond to the current state of the system. This implies that in this case, the controller has to make a decision using an estimate of the current state. Second, because the delays are time-varying, the controller may not receive new information every sampling time. This implies that in this case, the controller has to operate in open-loop using the last received measurements. To this end, when a delayed measurement is received the controller uses the nominal system model and the input trajectory that has been applied to the system to get an estimate of the current state and then an MPC optimization problem is solved in order to decide the optimal future input trajectory that will be applied until new measurements are received. This approach implies that the previous control input trajectory should be stored in the controller. The implementation strategy for the LMPC for systems subject to time-varying measurement delays is as follows:

1. When a measurement $x(t_a - d_a)$ is available at t_a, the LMPC checks whether the measurement provides new information. If $t_a - d_a > \max_{l<a} t_l - d_l$, go to Step 2. Else the measurement does not contain new information and is discarded, go to Step 5.
2. The LMPC estimates the current state of the system $\tilde{x}(t_a)$ and computes the optimal input trajectory of u based on $\tilde{x}(t_a)$ for $t \in [t_a, t_a + N\Delta)$.
3. The LMPC sends the entire optimal input trajectory to the actuators.
4. The actuators implement the input trajectory until a new measurement is received at time t_{a+1}.
5. When a new measurement is received ($a \leftarrow a + 1$), go to Step 1.

The LMPC that takes into account time-varying measurement delay in an explicit way is based on the following constrained optimal control problem:

$$\min_{u \in S(\Delta)} \int_{t_a}^{t_a+N\Delta} \left[\left\| \tilde{x}(\tau) \right\|_{Q_c} + \left\| u(\tau) \right\|_{R_c} \right] d\tau, \tag{2.69}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f\big(\tilde{x}(t), u(t), 0\big), \quad \forall t \in [t_a - d_a, t_a + N\Delta), \tag{2.70}$$

$$u(t) = u_d^*(t), \quad \forall t \in [t_a - d_a, t_a), \tag{2.71}$$

$$\tilde{x}(t_a - d_a) = x(t_a - d_a), \tag{2.72}$$

$$\dot{\hat{x}}(t) = f\big(\hat{x}(t), h\big(\hat{x}(t_a + j\Delta)\big), 0\big), \quad t \in \big[t_a + j\Delta, t_a + (j+1)\Delta\big), \tag{2.73}$$

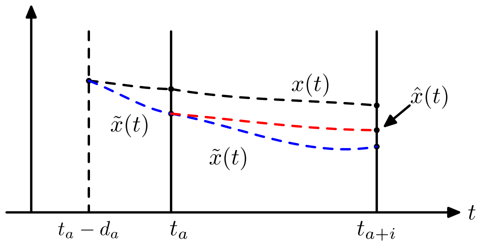


Fig. 2.7 A possible scenario of the measurements received by the LMPC of Eqs. 2.69–2.75 and the corresponding state trajectories defined in the LMPC

$$\hat{x}(t_a) = \tilde{x}(t_a), \tag{2.74}$$

$$V\big(\tilde{x}(t)\big) \le V\big(\hat{x}(t)\big), \quad \forall t \in [t_a, t_a + N_{D,a}\Delta), \tag{2.75}$$

where $u_d^*(t)$ indicates the actual control input trajectory that has been applied to the system, $x(t_a - d_a)$ is the delayed measurement that is received at t_a with delay size d_a, $\tilde{x}(t_a)$ is an estimate of the current system state, $j = 0, \dots, N-1$, and $N_{D,a}$ is the smallest integer satisfying $N_{D,a}\Delta \ge T_m + D - d_a$.

The optimal solution to the LMPC optimization problem of Eqs. 2.69–2.75 is denoted as $u_d^*(t|t_a)$ which is defined for $t \in [t_a, t_a + N\Delta)$. The manipulated input of the system of Eq. 2.1 under the control of the LMPC of Eqs. 2.23–2.28 is defined as follows:

$$u(t) = u_d^*(t|t_a), \quad \forall t \in [t_a, t_{a+i}), \tag{2.76}$$

for all t_a such that $t_a - d_a > \max_{l<a} t_l - d_l$ and for a given t_a, the variable i denotes the smallest integer that satisfies $t_{a+i} - d_{a+i} > t_a - d_a$.

In the LMPC design of Eqs. 2.69–2.75, if at a sampling time, a new measurement $x(t_a - d_a)$ is received, an estimate of the current state $\tilde{x}(t_a)$ is obtained using the nominal model of the system (the constraint of Eq. 2.70) and the control input trajectory applied to the system from $t_a - d_a$ to t_a (the constraint of Eq. 2.71) with the initial condition $\tilde{x}(t_a - d_a) = x(t_a - d_a)$ (the constraint of Eq. 2.72). The estimated state $\tilde{x}(t_a)$ is then used to obtain the optimal future control input trajectory. The LMPC of Eqs. 2.69–2.75 uses the nominal model to predict the future trajectory $\tilde{x}(t)$ for a given input trajectory $u(t) \in S(\Delta)$ with $t \in [t_a, t_a + N\Delta)$. A cost function is minimized (Eq. 2.69), while assuring that the value of the Lyapunov function along the predicted trajectory $\tilde{x}(t)$ satisfies a Lyapunov-based constraint (the constraint of Eq. 2.75) where $\hat{x}(t)$ is the state trajectory corresponding to the nominal system in closed-loop with the nonlinear control law $h(x)$ (the constraint of Eq. 2.73) with the initial condition $\hat{x}(t_a) = \tilde{x}(t_a)$ (the constraint of Eq. 2.74). Note that the length of the constraint $N_{D,a}$ depends on the current delay d_a so it may have different values at different time instants and has to be updated before solving the optimization problem of Eqs. 2.69–2.75. If the controller does not receive any new measurement at a sampling time, it keeps implementing the last evaluated optimal trajectory. This strategy is a receding horizon scheme, which takes time-varying measurement delays explicitly into account.

Figure 2.7 shows a possible scenario for a system of dimension 1. A delayed measurement $x(t_a - d_a)$ is received at time t_a and the next new measurement is not obtained until t_{a+i}. This implies that at time t_a we evaluate the LMPC of Eqs. 2.69–2.75 and we apply the optimal input $u_d^*(t|t_a)$ from t_a to t_{a+i}. The solid vertical lines are used to indicate sampling times in which a new measurement is obtained (that is, t_a and t_{a+i}) and the dashed vertical line is used to indicate the time corresponding to the measurement obtained in t_a (that is, $t_a - d_a$).

2.8.3 Stability Properties

In this subsection, we present the stability properties of the LMPC of Eqs. 2.69–2.75 for systems subject to time-varying measurement delays. Theorem 2.2 below provides sufficient conditions under which the LMPC of Eqs. 2.69–2.75 guarantees stability of the closed-loop system in the presence of time-varying measurement delays.

Theorem 2.2 *Consider the system of Eq.* 2.1 *in closed-loop, which closes at asynchronous time instants* $\{t_{a\geq 0}\}$ *that satisfy the condition of Eq.* 2.22, *under the LMPC of Eqs.* 2.69–2.75 *based on a controller* $h(x)$ *that satisfies the conditions of Eqs.* 2.4–2.7. *Let* $\Delta, \varepsilon_s > 0$, $\rho > \rho_{\min} > 0$, $\rho > \rho_s > 0$, $N \geq 1$ *and* $D \geq 0$ *satisfy the condition of Eq.* 2.31 *and the following inequality*:

$$-N_R \varepsilon_s + f_V\big(f_W(N_D \Delta)\big) + f_V\big(f_W(D)\big) < 0. \tag{2.77}$$

with $f_V(\cdot)$ *and* $f_W(\cdot)$ *defined in Eqs.* 2.49 *and* 2.43, *respectively,* N_D *being the smallest integer satisfying* $N_D \Delta \geq T_m + D$, *and* N_R *being the smallest integer satisfying* $N_R \Delta \geq T_m$. *If* $N \geq N_D$, $x(t_0) \in \Omega_\rho$ *and* $d_0 = 0$, *then* $x(t)$ *is ultimately bounded in* $\Omega_{\rho_d} \subseteq \Omega_\rho$ *where*:

$$\rho_d = \rho_{\min} + f_V\big(f_W(N_D \Delta)\big) + f_V\big(f_W(D)\big). \tag{2.78}$$

Proof In order to prove that the system of Eq. 2.1 in closed-loop with the LMPC of Eq. 2.69–2.75 is ultimately bounded in a region that contains the origin, we will prove that the Lyapunov function $V(x)$ is a decreasing function of time with a lower bound on its magnitude. We assume that the delayed measurement $x(t_a - d_a)$ is received at time t_a and that a new measurement is not obtained until t_{a+i}. The LMPC of Eq. 2.69–2.75 is solved at t_a and the optimal input trajectory $u_d^*(t|t_a)$ is applied from t_a to t_{a+i}.

Part 1: In this part, we prove that the stability results stated in Theorem 2.2 hold for $t_{a+i} - t_a = N_{D,a}\Delta$ and all $d_a \leq D$.

The trajectory $\hat{x}(t)$ corresponds to the nominal system in closed-loop with the nonlinear control law $u = h(\hat{x})$ implemented in a sample-and-hold fashion with initial condition $\tilde{x}(t_a)$; please see the constraint of Eqs. 2.73 and 2.74. By Proposition 2.1, the following inequality can be obtained:

$$V\big(\hat{x}(t_{a+i})\big) \leq \max\big\{V\big(\hat{x}(t_a)\big) - N_{D,a}\varepsilon_s, \rho_{\min}\big\}. \tag{2.79}$$

The constraint of Eq. 2.75 guarantees that:

$$V\big(\tilde{x}(t)\big) \leq V\big(\hat{x}(t)\big), \quad \forall t \in [t_a, t_a + N_{D,a}\Delta), \tag{2.80}$$

and the constraint of Eq. 2.74 guarantees that $V(\hat{x}(t_a)) = V(\tilde{x}(t_a))$. This implies that:

$$V\big(\tilde{x}(t_{a+i})\big) \leq \max\big\{V\big(\tilde{x}(t_a)\big) - N_{D,a}\varepsilon_s, \rho_{\min}\big\}. \tag{2.81}$$

When $x(t) \in \Omega_\rho$ for all times (this point will be proved below), we can apply Proposition 2.3 to obtain the following inequalities:

$$V\big(\tilde{x}(t_a)\big) \leq V\big(x(t_a)\big) + f_V\big(\big\|x(t_a) - \tilde{x}(t_a)\big\|\big), \tag{2.82}$$

$$V\big(x(t_{a+i})\big) \leq V\big(\tilde{x}(t_{a+i})\big) + f_V\big(\big\|x(t_{a+i}) - \tilde{x}(t_{a+i})\big\|\big). \tag{2.83}$$

Applying Proposition 2.2, we obtain the following upper bounds on the deviation of $\tilde{x}(t)$ from $x(t)$:

$$\big\|x(t_a) - \tilde{x}(t_a)\big\| \leq f_W(d_a), \tag{2.84}$$

$$\big\|x(t_{a+i}) - \tilde{x}(t_{a+i})\big\| \leq f_W(N_D\Delta). \tag{2.85}$$

Note that the constraints of Eqs. 2.70–2.72 and the implementation procedure allow us to apply Proposition 2.2 because it is guaranteed that the actual system state $x(t)$ and the state estimated using the nominal model $\tilde{x}(t)$ are obtained using the same input trajectory. Note also that we have taken into account that $N_D\Delta \geq T_m + D - d_a$ for all d_a. Using the inequalities of Eqs. 2.81–2.84, the following upper bound on $V(x(t_{k+j}))$ is obtained:

$$V\big(x(t_{a+i})\big) \leq \max\big\{V\big(x(t_a)\big) - N_{D,a}\varepsilon_s, \rho_{\min}\big\} + f_V\big(f_W(d_a)\big) + f_V\big(f_W(N_D\Delta)\big). \tag{2.86}$$

In order to prove that the Lyapunov function is decreasing between two consecutive new measurements, the following inequality must hold:

$$N_{D,a}\varepsilon_s > f_V\big(f_W(N_D\Delta)\big) + f_V\big(f_W(d_a)\big) \tag{2.87}$$

for all possible $0 \leq d_a \leq D$. Taking into account that $f_W(\cdot)$ and $f_V(\cdot)$ are strictly increasing functions of their arguments, that $N_{D,a}$ is a decreasing function of the delay d_a and that if $d_a = D$ then $N_{D,a} = N_R$, if the condition of Eq. 2.77 is satisfied, the condition of Eq. 2.87 holds for all possible d_a and there exists $\varepsilon_w > 0$ such that the following inequality holds:

$$V\big(x(t_{a+i})\big) \leq \max\big\{V\big(x(t_a)\big) - \varepsilon_w, \rho_d\big\}, \tag{2.88}$$

which implies that if $x(t_a) \in \Omega_\rho/\Omega_{\rho_d}$, then $V(x(t_{a+i})) < V(x(t_a))$, and if $x(t_a) \in \Omega_{\rho_d}$, then $V(x(t_{a+i})) \leq \rho_d$.

Because the upper bound on the difference between the Lyapunov function of the actual trajectory x and the nominal trajectory $\tilde{x}$ is a strictly increasing function of time, the inequality of Eq. 2.88 also implies that:

$$V\big(x(t)\big) \le \max\big\{V\big(x(t_a)\big), \rho_d\big\}, \quad \forall t \in [t_a, t_{a+i}). \tag{2.89}$$

Using the inequality of Eq. 2.89 recursively, it can be proved that if $x(t_0) \in \Omega_\rho$, then the closed-loop trajectories of the system of Eq. 2.1 under the LMPC of Eqs. 2.69–2.75 stay in Ω_ρ for all times (i.e., $x(t) \in \Omega_\rho, \forall t$). Moreover, using the inequality of Eq. 2.89 recursively, it can be proved that if $x(t_0) \in \Omega_\rho$, the closed-loop trajectories of the system of Eq. 2.1 under the LMPC of Eqs. 2.69–2.75 satisfy:

$$\limsup_{t\to\infty} V\big(x(t)\big) \le \rho_d. \tag{2.90}$$

This proves that $x(t) \in \Omega_\rho$ for all times and $x(t)$ is ultimately bounded in Ω_{ρ_d} for the case when $t_{a+i} - t_a = N_{D,a}\Delta$.

Part 2: In this part, we extend the results proved in Part 1 to the general case, that is, $t_{a+i} - t_a \le N_{D,a}\Delta$. Taking into account that $f_V(\cdot)$ and $f_W(\cdot)$ are strictly increasing functions of their arguments and $f_V(\cdot)$ is convex, following similar steps as in Part 1, it can be shown that the inequality of Eq. 2.87 holds for all possible $d_a \le D$ and $t_{a+i} - t_a \le N_{D,a}\Delta$. Using this inequality and following the same line of arguments as in the previous part, the stability results stated in Theorem 2.2 can be proved. □

Remark 2.11 When time-varying measurement delays are not present and new measurements of $x(t)$ are fed into the controller every synchronous sampling time, the LMPC of Eqs. 2.69–2.75 may be simplified to the LMPC of Eqs. 2.16–2.20. Comparing the LMPC of Eqs. 2.16–2.20 with the one of Eqs. 2.69–2.75, the difference is that the Lyapunov-based constraint of Eq. 2.20 has to hold only for one time step. This implies that even if the same implementation procedure is used, and the same optimization problem is solved (in order to estimate the current state), if the Lyapunov-based constraint is not changed, stability cannot be proved. This point will be illustrated in the example in Sect. 2.8.4.

Remark 2.12 In the LMPC of Eqs. 2.23–2.28 for systems with asynchronous feedback without delays, the Lyapunov-based constraint of Eq. 2.28 has to hold for a time period which is equal to or bigger than the maximum time without new measurement. This constraint makes the computed control action more conservative (and thus less optimal) because the controller may have to satisfy the Lyapunov-based constraint over unnecessarily large horizons. If the LMPC of Eqs. 2.23–2.28 is implemented for systems subject to time-varying delays, it will be, in general, less optimal than the LMPC of Eqs. 2.69–2.75. This point will also be illustrated in the example in Sect. 2.8.4.

2.8.4 Application to a Chemical Reactor

Consider the CSTR described by Eqs. 2.62–2.63 in Sect. 2.7.4. We assume that the manipulated input (the rate of heat input Q) is bounded by $|Q| \leq 10^5$ KJ/h and the time-varying uncertainty in the reactant concentration of the inflow is bounded by $|\Delta C_{A0}| \leq 0.2$ mol/l. The control system is subject to time-varying measurement delay in the measurements of the concentration of the reactant, C_A, and in the measurements of the temperature, T. Note that we do not consider the possible different sampling rates of temperature and concentration sensors in this example. Note also that the delay in the measurements could be regarded as the total time needed for online sensors to get a sample, analyze the sample and transmit the data to the controller. The same nonlinear controller of Eq. 2.66 with the same Lyapunov function $V(x)$ and weighting matrix P is used in the design of the LMPCs used in the simulations. The stability region Ω_ρ is defined as $V(x) \leq 700$, i.e., $\rho = 700$.

The sampling time of the LMPCs is chosen to be $\Delta = 0.025$ h, the maximum allowable measurement delay is $D = 6\Delta = 0.15$ h and the maximum interval between two consecutive measurements is $T_m = \Delta = 0.025$ h which implies that there is a measurement available every Δ but it may not contain new state information. The cost function is defined by the weighting matrices $Q_c = P$ and $R_c = 10^{-6}$.

We first compare the LMPC of Eqs. 2.69–2.75 with the LMPC of Eqs. 2.16–2.20 in the case where no time-varying measurement delays are present. For this simulation, we choose the prediction horizon of the two LMPCs N equal to 7 ($N \geq D + T_m$). We implement the LMPC of Eqs. 2.16–2.20 using the same approach employed in the implementation of the LMPC of Eqs. 2.69–2.75, that is, the current state is estimated using the nominal model when a delayed measurement is received and the last optimal input is applied when no new measurement is received. In Fig 2.8, the trajectories of the CSTR under both LMPCs are shown assuming no measurement delay is present, that is, the state $x(t_k)$ is available every sampling time. It can be seen that both closed-loop systems are practically stable and the trajectories remain in the stability region Ω_ρ.

In order to simulate the process in the presence of measurement delay, we use a random process to generate the delay sequence $\{d_{a\geq 0}\}$, and the time sequence $\{t_{a\geq 0}\}$ and corresponding delay sequence $\{d_{a\geq 0}\}$ in which the control system is subjected to is shown in Fig. 2.9. In this figure, we see the time-varying nature of the measurement delays and the largest delays are equal to the maximum allowable delay $D = 6\Delta = 0.15$ h. Note that when $d_{a+1} = d_a + \Delta$, the controller does not receive any new measurement.

When time-varying measurement delays are present, the LMPC of Eqs. 2.69–2.75 is more robust. The stability region is invariant for the closed-loop system if $D + T_m \leq N\Delta$. This is not the case with the LMPC of Eqs. 2.16–2.20. In Fig. 2.10, the trajectories of the closed-loop system under both controllers are shown in the presence of measurement delay with $D = 6\Delta = 0.15$ h. It can be seen that the

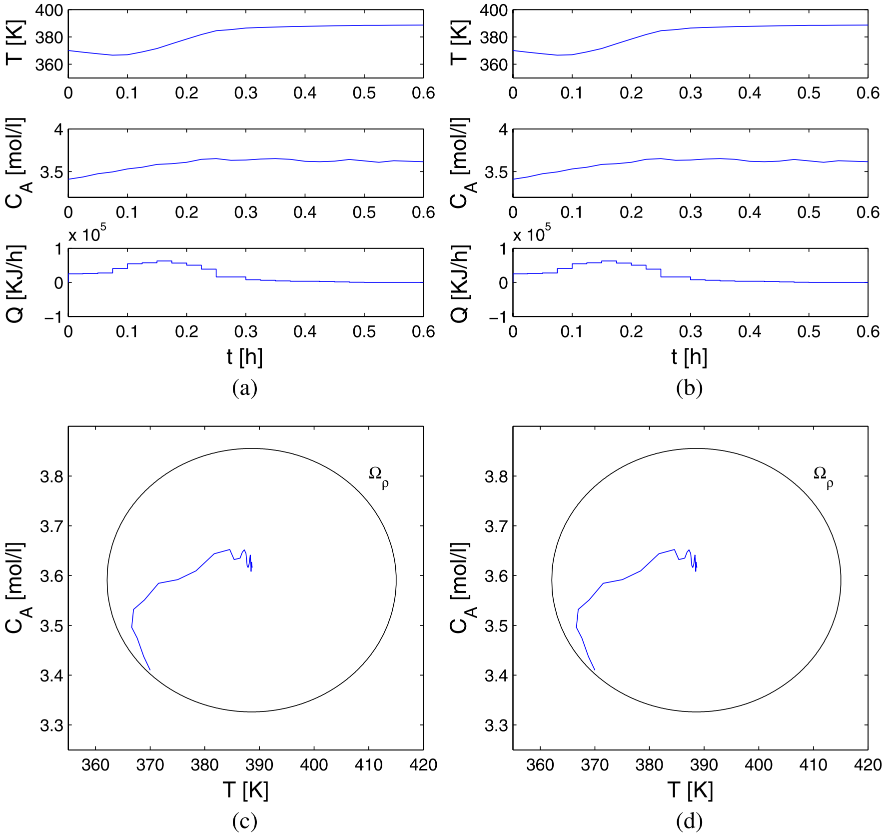
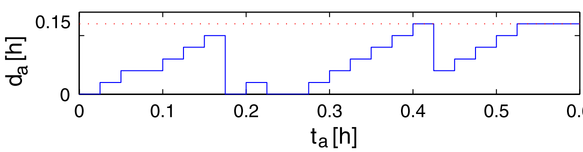


Fig. 2.8 (**a**, **c**) State and input trajectories of the CSTR of Eqs. 2.62–2.63 with the LMPC of Eqs. 2.69–2.75 when no measurement delay is present; (**b**, **d**) state and input trajectories of the CSTR of Eqs. 2.62–2.63 with the LMPC of Eqs. 2.16–2.20 when no measurement delay is present

Fig. 2.9 Time sequence $\{t_{a\geq 0}\}$ and corresponding delay sequence $\{d_{a\geq 0}\}$ used in the simulation shown in Fig. 2.10



LMPC of Eqs. 2.16–2.20 can not stabilize the system at the desired open-loop unstable steady-state and the trajectories leave the stability region, while the LMPC of Eqs. 2.69–2.75 keeps the trajectories inside the stability region. When measurement delay is present, in order to provide stability guarantees, the constraints must be modified to take into account the measurement delay as in the LMPC of Eqs. 2.69–2.75.

We have also carried out a set of simulations to compare the LMPC of Eqs. 2.69–2.75 with the LMPC of Eqs. 2.23–2.28 for nonlinear systems subject

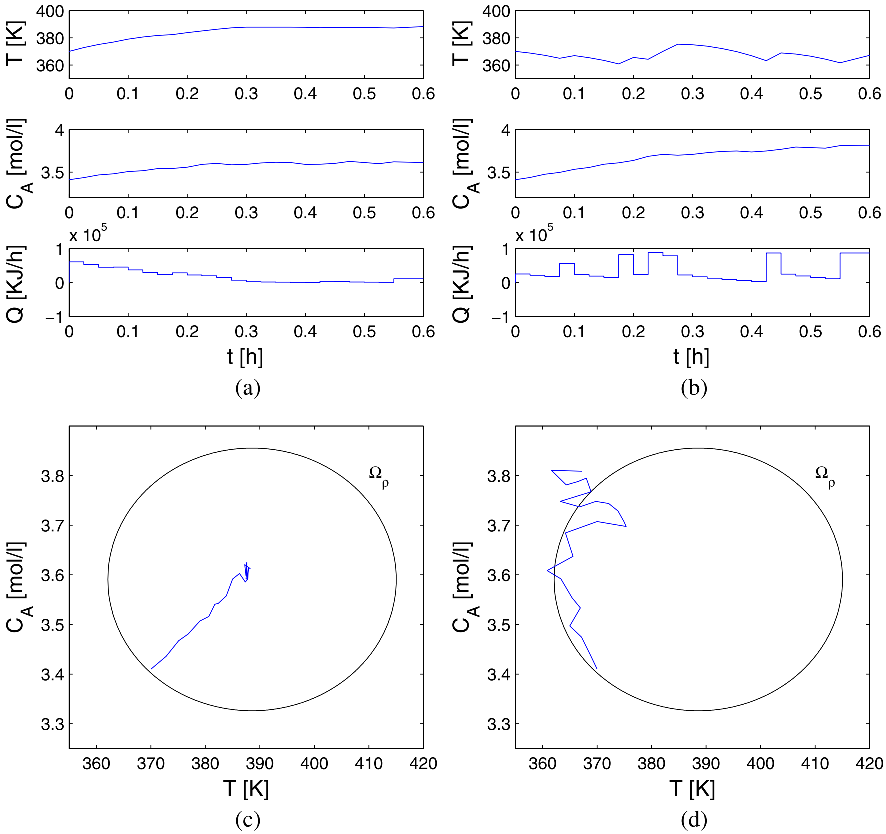


Fig. 2.10 (**a**, **c**) State and input trajectories of the CSTR of Eqs. 2.62–2.63 with the LMPC of Eqs. 2.69–2.75 when D is $6\varDelta$ and $T_m = \varDelta$; (**b**, **d**) state and input trajectories of the CSTR of Eqs. 2.62–2.63 with the LMPC of Eqs. 2.16–2.20 when D is $6\varDelta$ and $T_m = \varDelta$

to data losses from a performance index point of view. We also implement the LMPC of Eqs. 2.23–2.28 using the same approach employed in the implementation of the LMPC of Eqs. 2.69–2.75. Table 2.3 shows the total cost computed for 20 different closed-loop simulations under the LMPC of Eqs. 2.69–2.75 and the LMPC of Eqs. 2.23–2.28. To carry out this comparison, we have computed the total cost of each simulation based on the performance index of Eq. 2.67 with the initial simulation time $t_0 = 0$ and the final simulation time $t_f = 2$ h.

The prediction horizon in this set of simulations is $N = 10$. For each pair of simulations (one for each controller) a different initial state inside the stability region, a different uncertainty trajectory and a different random measurement delay sequence is chosen. As can be seen in Table 2.3, the LMPC of Eqs. 2.69–2.75 has a cost lower than the corresponding total cost under the LMPC designed for systems subject to data losses in 16 out of 20 simulations (see also Remark 2.12). This illustrates that the LMPC of Eqs. 2.69–2.75 is, in general, more optimal. This is because the LMPC

Table 2.3 Total performance costs along the closed-loop trajectories of the CSTR of Eqs. 2.62–2.63 under LMPC of Eqs. 2.69–2.75 and LMPC of Eqs. 2.23–2.28

sim.	LMPC of Eqs. 2.69–2.75	LMPC of Eqs. 2.23–2.28
1	1.8295×10^4	2.4428×10^4
2	4.2057×10^4	6.0522×10^4
3	3.2481×10^3	1.0428×10^4
4	7.4328×10^2	7.3961×10^2
5	1.4229×10^3	2.7798×10^5
6	4.9435×10^4	6.1596×10^4
7	3.2519×10^4	3.4319×10^4
8	2.7590×10^4	4.7075×10^4
9	9.4216×10^2	9.4866×10^2
10	5.4505×10^2	5.4322×10^2
11	1.9723×10^4	3.1282×10^4
12	2.7235×10^4	3.8772×10^4
13	1.8671×10^3	1.9200×10^3
14	3.7789×10^4	4.0050×10^4
15	2.1839×10^3	2.1392×10^3
16	4.2920×10^4	4.4594×10^4
17	1.5153×10^2	1.7190×10^2
18	4.9955×10^3	9.9094×10^3
19	3.2086×10^4	4.8838×10^4
20	1.5420×10^3	1.5197×10^3

designed for system subject to data losses requires the Lyapunov-based constraint of Eq. 2.28 to be satisfied along the whole possible maximum open-loop operation time (that is $t \in [t_a, t_a + N_R \Delta)$) which yields a more conservative controller from a performance point of view.

We have also carried out a set of simulations to study the dependence on the value of the maximum delay D of the set in which the trajectory of the process under the proposed LMPC scheme is ultimately bounded. In order to estimate the size of each set for a given D, we start the system very close to the equilibrium state and run it for a sufficient long time. In this set of simulations, we set $\Delta C_{A0} = 0.1$ kmol/m^3 and $N = 7$. The simulation time is 25 h. Figure 2.11 shows the location of the states, (C_A, T), at each sampling time and the estimated regions for $D = 2\Delta, 4\Delta, 6\Delta$. Three ellipses are used to estimate the boundaries of the sets, and they are chosen to be as small as possible but still include all the corresponding points indicating the states. From Fig. 2.11, we see that the size of these sets becomes larger as D increases. The results are expected because the size of the sets is not only dependent on the system and the controller, but it also depends on the maximum measurement delay. The longer the size of the delay, the further the system can move away from the steady-state which means a larger set (if the state is still in the stability region Ω_ρ). Note that all the sets for $D = 2\Delta, 4\Delta, 6\Delta$ are included

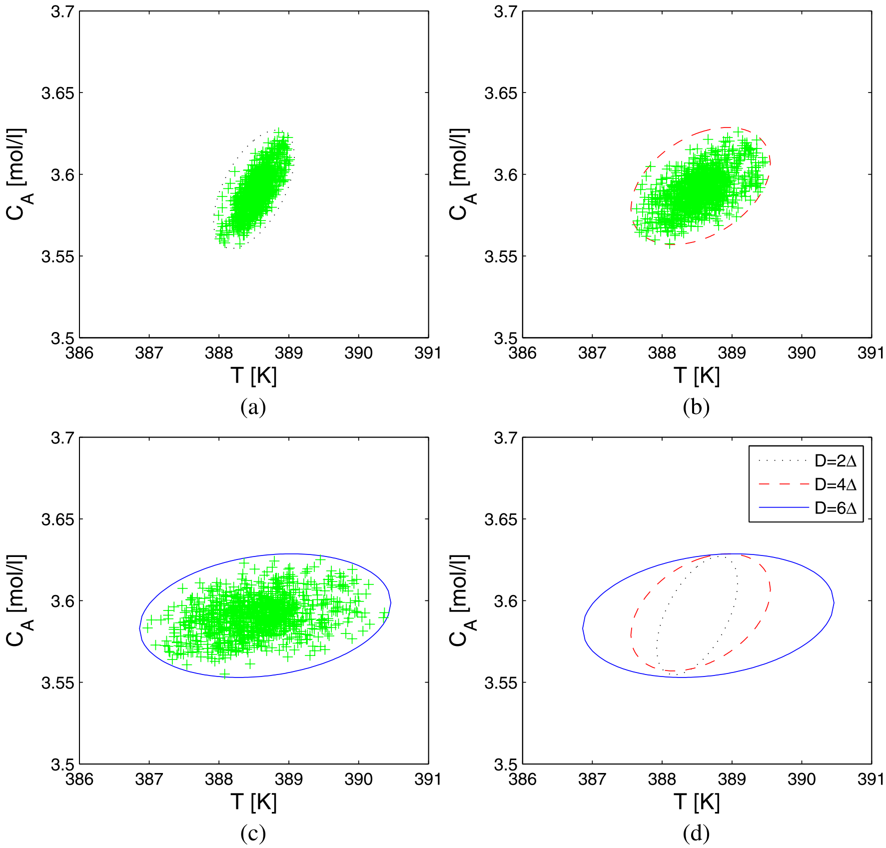


Fig. 2.11 (**a**) Estimate of the set in which the state trajectories of the CSTR of Eqs. 2.62–2.63 with the LMPC of Eqs. 2.69–2.75 are ultimately bounded when the maximum allowable measurement delay D is 2Δ; (**b**) estimate of the set in which the state trajectories of the CSTR of Eqs. 2.62–2.63 with the LMPC of Eqs. 2.69–2.75 are ultimately bounded when the maximum allowable measurement delay D is 4Δ; (**c**) estimate of the set in which the state trajectories of the CSTR of Eqs. 2.62–2.63 with the LMPC of Eqs. 2.69–2.75 are ultimately bounded when the maximum allowable measurement delay D is 6Δ; (**d**) comparison of the three sets

in the stability region of the closed loop system under the LMPC accounting for time-varying delays (Ω_ρ, $\rho = 700$).

2.9 Conclusions

In this chapter, LMPC designs were developed for the control of a broad class of nonlinear uncertain systems subject to data losses/asynchronous measurements and time-varying measurement delays. The main idea is that in order to provide guaranteed stability results in the presence of data losses or time-varying measurement delays, the constraints that define the LMPC optimization problems as well as the implementation procedures have to be modified to account for data losses/asynchronous measurements or time-varying measurement delays. The presented LMPCs possess an explicit characterization of the closed-loop system stability regions. The applications of the presented LMPCs were illustrated using a nonlinear CSTR example.

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