Preface

1. The beginning of a new century provides a good moment for looking back. Number theory has changed its appearance during the last hundred years. At the end of the 19th century it was regarded as a collection of dispersed results dealing with various old and newer problems, obtained by people who were mostly specializing in other subjects. After one hundred years number theory became a well-established part of mathematical sciences, having close relations to commutative algebra, homological algebra, algebraic geometry, function theory, real analysis, functional analysis, group theory and topology.

2. The aim of this book is to give a short survey of the development of the classical part of number theory between the proof of the Prime Number Theorem (PNT) and the proof of Fermat's Last Theorem (FLT), covering thus the twentieth century. Results obtained earlier or later will be also quoted, as far as they are connected with our main topics.

Actually it is now difficult to indicate the borders of number theory, as it tends to acquire grounds reserved earlier to analysis, algebra or geometry. It seems that A. Weil thought about limiting the possessions of number theory, when he wrote: "To the best of my understanding, analytic number theory is not number theory," [6630, p. 8] but nowadays it is fashionable to believe that number theory encompasses more and more of mathematical research.

The word "rational" in the title indicates that we shall concentrate on that part of number theory which deals with properties of integers and rational numbers, hence the theory of algebraic numbers will be excluded. This is motivated by the fact that its inclusion would enormously increase the size of the book, and, moreover, a large bibliography covering this part of number theory is available in my previous book [4543]. Nevertheless, some exceptions will be made, as we shall consider the class-number problems for quadratic and cyclotomic fields. The first of them coincides with the class-number problem for binary quadratic forms, and the second is intimately connected with the earlier approach to Fermat's Last Theorem. We shall also comment on the generalization of the Waring problem to algebraic number fields and describe the creation of class-field theory because of its influence on the reciprocity laws.

The history of the theory of modular forms which played a decisive role in the proof of Fermat's Last Theorem, and which underwent great progress in the last century, deserves a book of its own. Therefore we shall describe only those parts of its development which had a direct influence on number theory proper. This applies also to other branches of mathematics providing tools for arithmetical research. In particular we will not touch the more advanced topics in Diophantine geometry.

In consecutive chapters we shall present the main achievements of the relevant period, accompanied by comments about the development occurring in the next periods. An exception will be made for Fermat's Last Theorem, to which the last chapter is devoted. Our exposition will be concise, sometimes imitating the style of the celebrated Dickson's *History of the Theory of Numbers* [1545], although there is neither the possibility nor need to comment on all number-theoretical production. We have tried to list all the main achievements, quote many important papers, but restrain from including technical details in order to make the text available to non-specialists also. More attention will be paid to earlier work, in the hope that this will help to save it from falling into oblivion.

3. The first chapter contains a very short summary of the development of number theory in the 19th century, starting with Gauss's book *Disquisitiones Arithmeticae* [2208], and ending with the proof of the Prime Number Theorem by Hadamard and de la Vallée-Poussin and Hilbert's talk at the 1900 Congress of Mathematicians in Paris. The second chapter begins with a survey of some famous old problems (perfect numbers, Mersenne and Fermat primes, primality, ...), and then brings the story of our subject at the begin of the century (solution of the Waring problem, Brun's sieve, theorem of Thue, ...). In the next chapter the development up to 1930 will be covered (the inventing of the circle method by Hardy and Ramanujan, progress in the theory of Diophantine equations starting with Siegel's thesis, Mordell's finite basis theorem in the theory of elliptic curves). The most important events in the thirties, covered in Chap. 4, were Vinogradov's proof of the ternary Goldbach conjecture for large numbers, the solution of Hilbert's problem about transcendence of numbers α^β (with algebraic $\alpha \neq 0, 1$ and algebraic irrational β) obtained by Gelfond and Schneider, and the revival of the theory of modular forms by Hecke. The next two chapters report on later development, including the creation of the large sieve, and Chen's theorem on the binary Goldbach problem. The last chapter is devoted to Fermat's Last Theorem.

Information about results obtained after the period considered in each particular chapter is set in a smaller font.

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