Chapter 2
Connections

2.1 The Structure of a Tangent Bundle to a Vector Bundle

Let $\pi : \Theta \to M$ be a vector bundle with standard fiber $\mathbb{R}^d$, $\dim M = n$. Denote by Θ_m the fiber at $m \in M$ and by $(m, \vartheta) = \vartheta_m$ the points of this fiber. Consider a chart $\mathcal{U}_\alpha$ on a manifold M with local coordinates $(q^1, \dots, q^n)$ and a trivialization $\mathcal{F}_\alpha$ of the bundle over that chart. Let $e_1, \dots, e_d$ be the standard basis in $\mathbb{R}^d$. Since $\mathcal{F}_\alpha(\pi^{-1}\mathcal{U}_\alpha) = \mathcal{U}_\alpha \times \mathbb{R}^d$, this basis generates a basis in every fiber Θ_m, $m \in \mathcal{U}_\alpha$. We obtain a smooth field of bases that will also be denoted by $e_1, \dots, e_d$. Thus every cross-section ϑ of the bundle Θ can be represented in terms of coordinates with respect to these bases in the form $\vartheta = \vartheta^i e_i$, $i = 1, \dots, d$. In $\mathcal{U}_\alpha \times \mathbb{R}^d$ the set of vectors $\mathcal{U}_\alpha \times \{X_0\}$ for some $X_0 \in \mathbb{R}^d$ corresponds to the vectors ϑ from Θ_m, $m \in \mathcal{U}_\alpha$ that have the same coordinates with respect to $e_1, \dots, e_d$ as X_0. Another trivialization of the bundle over $\mathcal{U}_\alpha$ would generate another set of vectors equivalent to X_0 that is different from the former.

In the vector bundle the set $\mathcal{F}_\alpha(\pi^{-1}\mathcal{U}_\alpha) = \mathcal{U}_\alpha \times \mathbb{R}^d$ can be considered as a chart on the total space Θ. Denote by ϑ^i the coordinates in the fibers Θ_m, $m \in \mathcal{U}_\alpha$, whose coordinate axes are spanned by the basis vectors e_i in the fibers. We obtain the coordinate system $(q^1, ..., q^n, \vartheta^1, ..., \vartheta^d)$ in the chart $\mathcal{U}_\alpha \times \mathbb{R}^d$ on Θ. By a general scheme (see Section 1.1) this system generates a basis $\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}, \frac{\partial}{\partial \vartheta^1}, \dots, \frac{\partial}{\partial \vartheta^d}$ in the tangent space $T_{(m,\vartheta)}\Theta$ to the total space Θ of the bundle at every point (m, ϑ), $m \in \mathcal{U}_\alpha$.

The symbols $\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}$ for the first "half" of the basis vectors coincide with the symbols for the basis vectors $\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}$ in the tangent space $T_m M$, $m \in \mathcal{U}_\alpha$, to M generated by coordinates $(q^1, \dots, q^n)$. This is natural since the former vectors are tangent to a submanifold $\mathcal{U}_\alpha \times \{V\}$ in $\mathcal{F}_\alpha(\pi^{-1}\mathcal{U}_\alpha)$ where the coordinates $(q^1, \dots, q^n)$ are generated by the projection of the same coordinates from $\mathcal{U}_\alpha$, and that projection is an isomorphism. On the

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other hand, by construction the vectors $\frac{\partial}{\partial \vartheta^1}, \dots, \frac{\partial}{\partial \vartheta^d}$ are tangent to the fiber Θ_m, i.e., they belong to the tangent space $T_\vartheta \Theta_m$.

Definition 2.1. The tangent space $T_\vartheta \Theta_m$ to the fiber of a bundle Θ at m is called the *vertical subspace* in the tangent space $T_{(m,\vartheta)}\Theta$ to the total space of the bundle and is denoted by $\mathsf{V}_{(m,\vartheta)}$. The vectors of $\mathsf{V}_{(m,\vartheta)}$ are said to be *vertical*.

For every vector $Y_{(m,\vartheta)}$ at a point $(m,\vartheta) \in \Theta$ we can find its coordinate decomposition with respect to the basis mentioned above: $Y = Y^i \frac{\partial}{\partial q^i} + \dot{Y}^j \frac{\partial}{\partial \vartheta^j}$, $i = 1, \dots, n$, $j = 1, \dots, d$. By introducing vectors $Y_1 = Y^i \frac{\partial}{\partial q^i} \in T_m M$ and $Y_2 = \dot{Y}^j \frac{\partial}{\partial \vartheta^j} \in T_\vartheta \Theta_m$ the vectors $Y_{(m,\vartheta)} \in T_{(m,V)}\Theta$ are represented as quadruples (m, ϑ, Y_1, Y_2). This notation is compatible with that of Convention 1.3 for tangent vectors as points of the tangent bundle: here (m,ϑ) is a point of the manifold Θ and (Y_1, Y_2) is a tangent vector to Θ.

Let us find the formula of transformation of Y_1 and Y_2 under standard changes of coordinates of the form $(\varphi_{\beta\alpha}, g_{\beta\alpha}(m))$ (see Definition 1.32) on the total space Θ. Recall that, since Θ is a vector bundle, $g_{\beta\alpha}(m)$ is a linear operator in $\mathbb{R}^d$ and so it is equal to its derivative. On the other hand the derivative in $m \in M$ of the linear operator $g_{\beta\alpha}(m)$, depending on m, is a bilinear operator. Denote it by $g'_{\beta\alpha}(m)(\cdot,\cdot)$. The first argument of this operator is a vector from the fiber and the second argument is a vector tangent to the base M. In particular, the derivative $g_{\beta\alpha}(m)$ in m at the point $(m,\vartheta) \in \Theta$ takes the form $g'_{\beta\alpha}(m)(\vartheta,\cdot)$. Since $\varphi_{\beta\alpha}$ does not depend on the points of fiber, the derivative of $\varphi_{\beta\alpha}$ in $\mathbb{R}^d$ equals zero. Taking this into account it is easy to see that the derivative of the change of coordinates $(\varphi_{\beta\alpha}, g_{\beta\alpha}(m))$ at the point $(m,\vartheta) \in \Theta$ is represented in the form

$$(\varphi_{\beta\alpha}, g_{\beta\alpha}(m))' = \begin{pmatrix} \varphi'_{\beta\alpha} & 0 \\ g'_{\beta\alpha}(m)(\vartheta,\cdot) & g_{\beta\alpha}(m) \end{pmatrix}.$$

This means that under the above-mentioned changes of coordinates the column (Y_1, Y_2) transforms by the formula

$$\begin{aligned}(Y_1^\beta, Y_2^\beta)_{(m^\beta,\vartheta^\beta)} &= \begin{pmatrix} \varphi'_{\beta\alpha} & 0 \\ g'_{\beta\alpha}(m^\alpha)(\vartheta^\alpha,\cdot) & g_{\beta\alpha}(m^\alpha) \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \end{pmatrix} \\ &= \begin{pmatrix} \varphi'_{\beta\alpha} Y_1^\alpha \\ g'_{\beta\alpha}(m^\alpha)(\vartheta^\alpha, Y_1^\alpha) + g_{\beta\alpha}(m^\alpha)(Y_2^\alpha) \end{pmatrix}. \end{aligned} \tag{2.1}$$

In terms of quadruples formula (2.1) takes the form

$$\begin{aligned}&(m^\beta, \vartheta^\beta, Y_1^\beta, Y_2^\beta) \\ &= (\varphi_{\beta\alpha} m^\alpha, g_{\beta\alpha}(m^\alpha)\vartheta^\alpha, \varphi'_{\beta\alpha} Y_1^\alpha, g'_{\beta\alpha}(m^\alpha)(\vartheta^\alpha, Y_1^\alpha) + g_{\beta\alpha}(m^\alpha)(Y_2^\alpha)).\end{aligned} \tag{2.2}$$

By the definition of the projection π we have $\pi(q^1, \dots, q^n, \vartheta^1, \dots, \vartheta^d) = (q^1, \dots, q^n)$. Hence, the Jacobi matrix (presentation of the differential $d_{(m,\vartheta)}\pi$ in the given coordinate system) takes the form $(I \quad 0)$ where I and 0 are the unit $n \times n$ matrix and zero $k \times n$ matrix, respectively. As a consequence we obtain the formula for $T\pi : T\Theta \to TM$ in the form

$$T\pi(m, \vartheta, Y_1, Y_2) = (m, Y_1) \tag{2.3}$$

(by definition the tangent mapping $T\pi$ acts as π on the points (m, ϑ) and as $\mathrm{d}_{(m,\vartheta)}\pi$ on the vectors (Y_1, Y_2)). Recall that by construction Y_1 on the left-hand side of (2.3) belongs to $T_{(m,\vartheta)}\Theta$ while Y_1 on the right-hand side of (2.3) belongs to T_mM but both vectors have the same coordinates with respect to $\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}$ (the first "half"of the basis in $T_{(m,\vartheta)}\Theta$ and the entire basis in T_mM, respectively, are isomorphic to each other, see above). We do not distinguish between these two vectors or the frames in this notation.

Formula (2.3) means that

$$T\pi\left(Y^i \frac{\partial}{\partial q^i} + \dot{Y}^j \frac{\partial}{\partial \vartheta^j}\right) = Y^i \frac{\partial}{\partial q^i}. \tag{2.4}$$

Remark 2.2. As on every manifold, there is a natural projection of $T\Theta$ onto Θ. Denote it by $\pi_1 : T\Theta \to \Theta$. In coordinates it is represented in the form

$$\pi_1(m, \vartheta, Y_1, Y_2) = (m, \vartheta). \tag{2.5}$$

We emphasize the difference between (2.3) and (2.5).

Let Y be a cross-section of the bundle Θ. Over the chart $\mathcal{U}_\alpha$ we have the decomposition $Y = Y^i e_i$ (see above). Recall that we consider the cross-section Y as a mapping $Y : M \to \Theta$ such that $\pi Y = \mathrm{id}$ (see Definition 1.38). Consider also its tangent mapping $TY : TM \to T\Theta$. Since Y has the form

$$Y(q^1, \dots, q^n) = (q^1, \dots, q^n, Y^1(q^1, \dots, q^n), \dots, Y^k(q^1, \dots, q^n)),$$

its Jacobi matrix takes the form

$$\mathrm{d}_m Y = \begin{pmatrix} I \\ \left(\frac{\partial Y^i}{\partial q^j}\right) \end{pmatrix},$$

where I is the unit $n \times n$ matrix and $\left(\frac{\partial Y^i}{\partial q^j}\right)$ is the $n \times d$ Jacobi matrix of Y. Thus for $(m, X) \in TM$ we obtain

$$TY(m, X) = \left(m, Y, X, \left(\frac{\partial Y^i}{\partial q^j}\right) X\right) \tag{2.6}$$

(recall that TY acts as Y on points m and as $\mathrm{d}_m X$ on X).

On the vector bundle Θ the so-called *action of the real line* is given as follows. For every $a \in \mathbb{R}$ defined $a : \Theta \to \Theta$ (we denote the number and the corresponding mapping by the same symbol a) by:

$$a(m, \vartheta) = (m, a\vartheta), \tag{2.7}$$

where $(m, \vartheta) \in \Theta$, i.e. the action consists of multiplying all vectors from all fibers of Θ by a. Thus $a(q^1, \dots, q^n, \vartheta^1, \dots, \vartheta^d) = (q^1, \dots, q^n, a\vartheta^1, \dots, a\vartheta^d)$ and evidently $\mathrm{d}_{(m,\vartheta)}a = \begin{pmatrix} I & 0 \\ 0 & aI \end{pmatrix}$, where I and 0 are the unit and zero matrices, respectively, of corresponding dimensions. Hence

$$Ta(m, \vartheta, Y_1, Y_2) = (m, a\vartheta, Y_1, aY_2). \tag{2.8}$$

Below we shall often use the constructions described in this section on tangent and cotangent bundles. For these cases we have to define the previous formulae and notation more precisely.

Since the fibers of a tangent bundle are tangent spaces, they already have the standard frames $\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}$. For the case of a tangent bundle we most often use such frames and the trivialization generated by them in a tangent bundle over charts (the construction of this trivialization is described in Section 1.1). Sometimes we shall also use alternative trivializations but those cases will be mentioned explicitly.

In this case the notation $\dot{q}^i$ for coordinates in fibers is compatible with the interpretation of a tangent vector as a velocity of some curve. We replace ϑ^i by this notation. Thus the frames in tangent spaces to TM have the form $\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}, \frac{\partial}{\partial \dot{q}^1}, \dots, \frac{\partial}{\partial \dot{q}^n}$.

For an analogous trivialization in a cotangent bundle we use the basis $\mathrm{d}q^1, \dots, \mathrm{d}q^n$ and coordinates in fibers with respect to those frames are denoted by p_i (here we take into account the interpretation of cotangent vectors as momenta). Respectively, the frame in a cotangent space to T^*M takes the form $\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}$.

The frame in a cotangent space to TM is denoted by $\mathrm{d}q^1, \dots, \mathrm{d}q^n, \mathrm{d}\dot{q}^1, \dots, \mathrm{d}\dot{q}^n$ and in a cotangent space to T^*M by $\mathrm{d}q^1, \dots, \mathrm{d}q^n, \mathrm{d}p_1, \dots, \mathrm{d}p_n$.

We consider in detail the case of tangent bundles. The constructions on cotangent bundles are analogous.

Definition 2.3. The tangent bundle to a tangent bundle TM is called the *second tangent bundle* to the manifold M and is denoted by TTM or T^2M.

The vectors of the second tangent bundle, i.e., tangent vectors to TM, are described as quadruples of the form (m, X, Y_1, Y_2) where X and Y_1 belong to T_mM while Y_2 is a vector tangent to T_mM.

The Jacobi matrix of the natural projection $\pi : TM \to M$ has the form $(I, 0)$ where I is the unit matrix and 0 is the zero matrix, both $n \times n$. Thus $T\pi\left(Y^i \frac{\partial}{\partial q^i} + \tilde{Y}^i \frac{\partial}{\partial \dot{q}^i}\right) = Y^i \frac{\partial}{\partial q^i}$.

The action of the real line on TM is defined as a particular case of the general definition. For $a \in \mathbb{R}$ the Jacobi matrix of the corresponding mapping $a : TM \to TM$ has the same form as above.

The transformation rule for quadruples describing vectors of the second tangent bundle under changes of coordinates also has to be specified. First of all on TM the transformation $g_{\beta\alpha}(m)$ in fibers takes the form $g_{\beta\alpha}(m) = \varphi'_{\beta\alpha}$. Hence $g'_{\beta\alpha}(m)(\cdot,\cdot) = \varphi''_{\beta\alpha}(\cdot,\cdot)$ where $''$ denotes the second derivative of the change of coordinates $\varphi_{\beta\alpha}$. Since the fiber of TM at m is the tangent space T_mM, both arguments in $\varphi''_{\beta\alpha}(\cdot,\cdot)$ have the same nature: they are vectors tangent to M. This is why we replace the symbol ϑ in the notation for an element in the fiber of Θ by the symbol X of a tangent vector to M.

Thus formula (2.1) is transformed into

$$(Y_1^\beta, Y_2^\beta)_{(m^\beta, X^\beta)} = \begin{pmatrix} \varphi'_{\beta\alpha} & 0 \\ \varphi''_{\beta\alpha}(X^\alpha, \cdot) & \varphi'_{\beta\alpha} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \end{pmatrix} = \begin{pmatrix} \varphi'_{\beta\alpha} Y_1^\alpha \\ \varphi''_{\beta\alpha}(m^\alpha)(X^\alpha, Y_1^\alpha) + \varphi'_{\beta\alpha}(m^\alpha)(Y_2^\alpha) \end{pmatrix}. \tag{2.9}$$

So, by formula (2.9) the transformation of quadruples as vectors tangent to TTM under the change of coordinates $\varphi_{\beta\alpha}$ on M has the form

$$(m^\beta, X^\beta, Y_1^\beta, Y_2^\beta) = (\varphi_{\beta\alpha} m^\alpha, \varphi'_{\beta\alpha} X^\alpha, \varphi'_{\beta\alpha} Y_1^\alpha, \varphi''_{\beta\alpha}(X^\alpha, Y_1^\alpha) + \varphi'_{\beta\alpha}(Y_2^\alpha)). \tag{2.10}$$

We return to the general case.

Recall that by Definition 2.1 the space $T_\vartheta \Theta_m$ is called the vertical subspace in $T_{(m,\vartheta)}\Theta$ and is denoted by $\mathsf{V}_{(m,\vartheta)}$. The vectors belonging to $\mathsf{V}_{(m,\vartheta)}$ are said to be vertical.

As a direct consequence of the construction we obtain the following:

Proposition 2.4 *The space $\mathsf{V}_{(m,\vartheta)}$ does not depend on the choice of the chart $\mathfrak{U}_\alpha$, its coordinate system $(q^1, \dots, q^n)$ in a neighborhood of $m \in M$, or on the choice of the trivialization of $\pi^{-1}\mathfrak{U}_\alpha$.*

Indeed, the fiber Θ_m and hence the tangent space $T_\vartheta \Theta_m = V_{(m,\vartheta)}$ are determined without use of any coordinate system. The system $(q^1, \dots, q^n)$ is involved only in representing $\mathsf{V}_{(m,\vartheta)}$ as the linear span of $\frac{\partial}{\partial \vartheta^1}, \dots, \frac{\partial}{\partial \vartheta^d}$. Notice that these vectors do depend on the trivialization of $\pi^{-1}\mathfrak{U}_\alpha$.

Recall that by formula (1.2) we introduced the linear isomorphism $\mathbf{p}$ of a tangent space to a vector space onto the vector space. Thus here $\mathbf{p} : V_{(m,\vartheta)} \to \Theta_m$ is well-defined and takes the coordinate representation

$$\mathbf{p}\left(\frac{\partial}{\partial \vartheta^i}\right) = e_i. \tag{2.11}$$

Denote by $\mathsf{H}^E_{(m,\vartheta)}$ the linear span of the vectors $\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}$ in $T_{(m,\vartheta)}\Theta$. By construction, $\mathsf{H}^E_{(m,\vartheta)}$ is the tangent space to the submanifold $\mathcal{U}_\alpha \times \vartheta$ in $\pi^{-1}\mathcal{U}_\alpha$ with respect to the given trivialization of $\pi^{-1}\mathcal{U}_\alpha$. Immediately from the definition we get $T_{(m,\vartheta)}\Theta = \mathsf{H}^E_{(m,\vartheta)} \oplus \mathsf{V}_{(m,\vartheta)}$ where $\oplus$ is the direct sum. Notice that for the vector $Y_{(m,\vartheta)} = (m, \vartheta, Y_1, Y_2) \in T_{(m,\vartheta)}\Theta$ by definition $Y_1 \in \mathsf{H}^E_{(m,\vartheta)}$ and $Y_2 \in \mathsf{V}_{(m,\vartheta)}$ so that $Y_{(m,\vartheta)} = Y_1 \oplus Y_2$.

Proposition 2.5 *The subspace* $\mathsf{H}^E_{(m,\vartheta)}$ *depends on the choice of trivialization of* $\pi^{-1}\mathcal{U}_\alpha$ *and hence on the chart* $\mathcal{U}_\alpha$*.*

Proof. Indeed, consider another chart $\mathcal{U}_\beta$ with non-empty intersection $\mathcal{U}_{\alpha\beta}$ with $\mathcal{U}_\alpha$. Let a trivialization of $\pi^{-1}\mathcal{U}_\beta$ be given so that the standard basis in $\mathbb{R}^d$ generates another field of bases $e'_1, \ldots, e'_d$ different from the field $e_1, \ldots, e_d$ generated by the above trivialization of $\pi^{-1}\mathcal{U}_\alpha$. The layers in $\mathcal{U}_\alpha \times \mathbb{R}^d$ of the vectors $\mathcal{U}_\alpha \times \vartheta$ and in $\mathcal{U}_\beta \times \mathbb{R}^d$ of the vectors $\mathcal{U}_\beta \times \vartheta$ for a specified vector $\vartheta \in \mathbb{R}^d$ are different since the former is generated by those ϑ''s from $\Theta_{m'}$, $m' \in \mathcal{U}_\alpha$ whose coordinates with respect to $e_1, \ldots, e_d$ are the same as the coordinates of ϑ and the latter by those whose coordinates with respect to $e'_1 \ldots, e'_d$ are the same as those of ϑ. Since the layers going through (m, ϑ) are different, their tangent spaces at (m, ϑ) are also different. □

Remark 2.6. From the definitions it immediately follows that the quadruple for a vector from $\mathsf{H}^E_{(m,\vartheta)}$ takes the form $(m, \vartheta, Y_1, 0)$ and, for a vector from $\mathsf{V}_{(m,\vartheta)}$, the form $(m, \vartheta, 0, Y_2)$.

Proposition 2.7 $T\pi$ *sends any* $\mathsf{H}^E_{(m,\vartheta)}$ *isomorphically onto* T_mM *and* $\mathsf{V}_{(m,\vartheta)}$ *is the kernel of* $T\pi$ *at any* $T_{(m,\vartheta)}\Theta$*.*

Indeed, a vector from $\mathsf{H}^E_{(m,\vartheta)}$ takes the form $(m, \vartheta, Y_1, 0)$ and from $\mathsf{V}^E_{(m,\vartheta)}$ the form $(m, \vartheta, 0, Y_2)$ (see Remark 2.6). So, by (2.3) $T\pi(m, \vartheta, Y_1, 0) = (m, Y_1)$ and $T\pi(m, \vartheta, 0, Y_2) = (m, 0)$.

2.2 Connections on Vector Bundles

Connection and connector

Definition 2.8. Let Θ be a vector bundle and suppose that in every tangent space $T_{(m,\vartheta)}\Theta$ a subspace $\mathsf{H}_{(m,\vartheta)}$, complementary to $\mathsf{V}_{(m,\vartheta)}$ (i.e. $T_{(m,\vartheta)}\Theta = \mathsf{H}_{(m,\vartheta)} \oplus \mathsf{V}_{(m,\vartheta)}$ at any $(m, \vartheta) \in \Theta$), is specified such that the total family of subspaces $\mathsf{H} = \{\mathsf{H}_{(m,\vartheta)} | (m, \vartheta) \in \Theta\}$ satisfies the following two properties:

(i) the space $\mathsf{H}_{(m,\vartheta)}$ depends smoothly on $(m, \vartheta) \in \Theta$ (in the sense described below);

(ii) the family H is invariant with respect to the action of the real line on Θ, i.e., $Ta\mathsf{H}_{(m,\vartheta)} = \mathsf{H}_{(m,a\vartheta)}$ for every $a \in \mathbb{R}$ and $(m, \vartheta) \in \Theta$.

Then H is said to be a *connection* on Θ.

The subspaces $\mathsf{H}_{(m,\vartheta)}$ of a connection H are called *horizontal*, as are the vectors of $T_{(m,\vartheta)}\Theta$ belonging to $\mathsf{H}_{(m,\vartheta)}$.

The precise meaning of the statement that $\mathsf{H}_{(m,\vartheta)}$ is smooth in (m,ϑ) is as follows. In a neighborhood of any point $(m,\vartheta) \in \Theta$ there are n smooth linearly independent vector fields such that, for any (m',ϑ') in the neighborhood, the subspace $\mathsf{H}_{(m',\vartheta')}$ is the linear span of vectors of those fields at (m',ϑ').

Proposition 2.9 *The family $\mathsf{H}^E_{(m,\vartheta)}$ introduced in Section 2.1 is a connection on $\pi^{-1}\mathfrak{U}_\alpha$.*

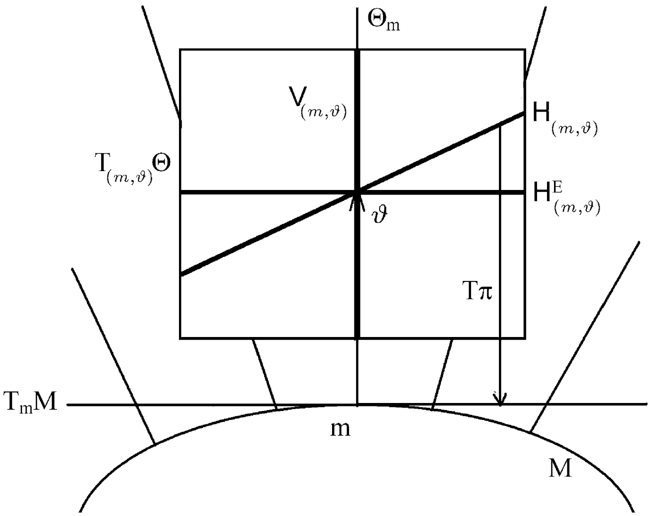
Proof. The presentation $T_{(m,\vartheta)}\Theta = \mathsf{H}^E_{(m,\vartheta)} \oplus \mathsf{V}_{(m,\vartheta)}$ was derived in Section 2.1 from the definition of $\mathsf{H}^E_{(m,\vartheta)}$. Also by definition $\mathsf{H}^E_{(m,\vartheta)}$ is the linear span of smooth linearly independent vectors $\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}$.

At any point $(m,\vartheta) \in \Theta$ the space $\mathsf{H}^E_{(m,\vartheta)}$ is the set of all vectors whose quadruple presentation takes the form $(m,\vartheta,Y_1,0)$ (see Remark 2.6). By formula (2.8) we see that Ta is a one-to-one mapping sending $(m,\vartheta,Y_1,0)$ to the quadruple $(m,a\vartheta,Y_1,0)$. Thus $Ta\mathsf{H}_{(m,\vartheta)} = \mathsf{H}_{(m,a\vartheta)}$. □

Definition 2.10. The family of subspaces $\mathsf{H}^E_{(m,\vartheta)}$ is called the *Euclidean connection* of a given trivialization of $\pi^{-1}\mathfrak{U}_\alpha$.

Indeed, $\mathsf{H}^E_{(m,\vartheta)}$ depends on the trivialization (see Proposition 2.5).

There exist connections $\{\mathsf{H}_{(m,\vartheta)}\}$ that may not be presented as the Euclidean connection of a trivialization. Of course, at a given point (m,ϑ) for a subspace $\mathsf{H}_{(m,\vartheta)}$, complimentary to $\mathsf{V}_{(m,\vartheta)}$, one can find a trivialization such that $\mathsf{H}_{(m,\vartheta)}$ coincides with the Euclidean connection of a trivialization at (m,ϑ), but in general this cannot be achieved for subspaces at all points in a given neighborhood. In order not to exclude general connections, we have not limited ourselves to Euclidean connections.



Proposition 2.11 *$T\pi : \mathsf{H}_{(m,\vartheta)} \to T_mM$ is a linear isomorphism.*

Proof. Recall that $T\pi : T_{(m,\vartheta)}\Theta \to T_m M$ is surjective and (by Proposition 2.7) $\mathsf{V}_{(m,\vartheta)}$ is the kernel of $T\pi$. Thus, the Proposition follows from the general result of linear algebra that a surjective linear operator is one-to-one on a complement to the kernel. □

The above proof is also valid in the analogous case of $\mathsf{H}^E_{(m,\vartheta)}$ in Proposition 2.7. However, we used a coordinate proof there for simplicity.

Combining Propositions 2.7 and 2.11 we see that $T\pi$ is connected with the decomposition $T_{(m,\vartheta)}\Theta = \mathsf{H}_{(m,\vartheta)} \oplus \mathsf{V}_{(m,\vartheta)}$ as follows:

Lemma 2.12

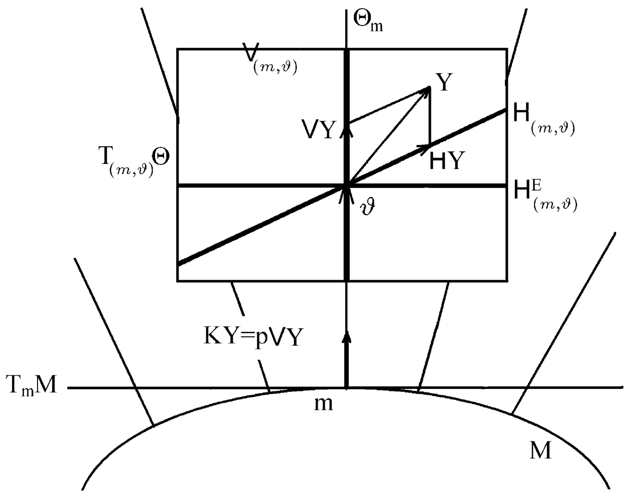
(i) $T\pi : \mathsf{H}_{(m,\vartheta)} \to T_m M$ *is an isomorphism.*
(ii) $\mathsf{V}_{(m,\vartheta)} = \ker T\pi$.

Our next step is to construct a map that is one-to-one on $\mathsf{V}_{(m,\vartheta)}$ and whose kernel is $\mathsf{H}_{(m,\vartheta)}$. Recall that the operator $\mathbf{p}$ that establishes a linear isomorphism between the vector space Θ_m and the tangent space $T_\vartheta \Theta_m$ to it acts by formula (2.11). Hence, for a vector from $\mathsf{V}_{(m,\vartheta)}$ with the quadruple $(m, \vartheta, 0, Y_2)$ (see Remark 2.6) where $Y_2 = \dot{Y}^i \frac{\partial}{\partial \dot{q}^i}$, we have

$$\mathbf{p}(m, \vartheta, 0, Y_2) = (m, \mathbf{p}Y_2) = \dot{Y}^i e_i. \tag{2.12}$$

The decomposition $T_{(m,\vartheta)}\Theta = \mathsf{H}_{(m,\vartheta)} \oplus \mathsf{V}_{(m,\vartheta)}$ yields the decomposition $Y_{(m,\vartheta)} = \mathsf{H}Y \oplus \mathsf{V}Y$ for every $Y_{(m,\vartheta)} \in T_{(m,\vartheta)}\Theta$, where $\mathsf{H}Y \in \mathsf{H}_{(m,\vartheta)}$ and $\mathsf{V}Y \in \mathsf{V}_{(m,\vartheta)}$. The symbols H and V may be considered as projections $\mathsf{H} : T_{(m,\vartheta)}\Theta \to \mathsf{H}_{(m,\vartheta)}$ and $\mathsf{V} : T_{(m,\vartheta)}\Theta \to \mathsf{V}_{(m,\vartheta)}$ in the above decomposition.

Definition 2.13. The map $K = \mathbf{p}\mathsf{V} : T_{(m,\vartheta)}\Theta \to \Theta$ is called the *connector* of the connection H.



Thus $K(Y_{(m,\vartheta)}) = \mathbf{p}(\mathsf{V}Y_{(m,\vartheta)})$. Evidently K on $\mathsf{V}_{(m,\vartheta)}$ coincides with $\mathbf{p}$ and so K maps $\mathsf{V}_{(m,\vartheta)}$ onto Θ_m isomorphically. On the other hand, $\mathsf{V}(\mathsf{H}_{(m,\vartheta)}) = 0 \in \mathsf{V}_{(m,\vartheta)}$ and so $K(\mathsf{H}_{(m,\vartheta)}) = 0 \in \Theta_m$. We summarize these properties in the following lemma:

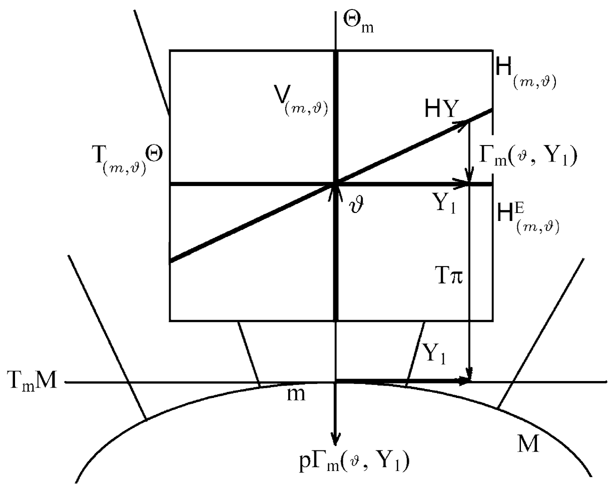
Lemma 2.14

(i) $K : \mathsf{V}_{(m,\vartheta)} \to \Theta_m$ *is a linear isomorphism.*
(ii) $\mathsf{H}_{(m,\vartheta)} = \ker K$.

Compare Lemmas 2.12 and 2.14. Notice the difference: we know that $\mathsf{V}_{(m,\vartheta)}$ and $T\pi$ exist on each vector bundle Θ while $\mathsf{H}_{(m,\vartheta)}$ and K must be given "by hand".

In order to work with $\mathsf{H}_{(m,\vartheta)}$ and K we need to describe them by means of coordinates. The best way to do that is to compare $\mathsf{H}_{(m,\vartheta)}$ with $\mathsf{H}^E_{(m,\vartheta)}$ of a certain trivialization over a chart $\mathfrak{U}_\alpha$ since the coordinate presentation of $\mathsf{H}^E_{(m,\vartheta)}$ is known.

Consider a vector $Y_1 \in T_mM$. Since $T\pi$ sends both $\mathsf{H}_{(m,\vartheta)}$ and $\mathsf{H}^E_{(m,\vartheta)}$ onto T_mM one-to-one, each of the spaces contains a unique vector whose image under $T\pi$ is Y_1. The vector in $\mathsf{H}^E_{(m,\vartheta)}$ is usually denoted by the same symbol Y_1. Denote the vector in $\mathsf{H}_{(m,\vartheta)}$ by $\mathsf{H}Y$. Consider the difference $\Gamma_m(\vartheta, Y_1) = Y_1 - \mathsf{H}Y \in T_{(m,\vartheta)}\Theta$. By construction we have $T\pi\Gamma_m(\vartheta, Y_1) = T\pi(Y_1) - T\pi(\mathsf{H}Y) = Y_1 - Y_1 = 0 \in T_mM$. Hence, $\Gamma_m(\vartheta, Y_1) \in \mathsf{V}_{(m,\vartheta)}$ since $\mathsf{V}_{(m,\vartheta)}$ is the kernel of $T\pi$. Thus we can apply $\mathbf{p}$ to $\Gamma_m(\vartheta, Y_1)$ and obtain the vector $\mathbf{p}\Gamma_m(\vartheta, Y_1) \in \Theta_m$. We have constructed the operator $\mathbf{p}\Gamma_m(\cdot,\cdot) : \Theta_m \times T_mM \to \Theta_m$.



Definition 2.15. The operator $\mathbf{p}\Gamma_m(\cdot,\cdot)$ is called the *local connector* (or *local connection coefficient*) of the connection H.

The word “local” means that the operator is constructed and calculated in a certain chart $\mathcal{U}_\alpha$ on M with respect to a certain trivialization of $\pi^{-1}\mathcal{U}_\alpha$.

Theorem 2.16 *The operator* $\mathbf{p}\Gamma_m(\cdot,\cdot)$ *is linear in the second argument.*

Indeed, $\mathbf{p}\Gamma_m(\vartheta, Y_1) = T\pi^{-1}(Y_1)_{|\mathsf{H}^E_{(m,\vartheta)}} - T\pi^{-1}(Y_1)_{|\mathsf{H}_{(m,\vartheta)}}$. Since the operation $T\pi^{-1}$ and the operation of taking the difference are both linear, $\mathbf{p}\Gamma_m(\vartheta, Y_1)$ is linear in Y_1.

Theorem 2.17 *The operator* $\mathbf{p}\Gamma_m(\cdot,\cdot)$ *is linear in the first argument.*

To prove Theorem 2.17 we need the following:

Lemma 2.18 *Let* $B : E \to E$ *be a map in the vector space* E*, smooth and homogeneous with degree* 1*. Then* B *is a linear operator.*

Proof. (of Lemma 2.18) Recall that B is homogeneous with degree k if for any vector $X \in E$ and any $\lambda \in \mathbb{R}$ we have $B(\lambda X) = \lambda^k B(X)$. From the homogeneity it follows that $B(0) = 0$.

Since B is smooth, we can expand it by the Taylor formula in a neighborhood of $0 \in E$ up to a certain degree greater than 1. Thus, since $B(0) = 0$, $B(X) = B'(X) + \frac{1}{2}B''(X, X) + \dots$ where B' is the first derivative of B at the origin (recall that B' is a linear operator), B'' is the second derivative of B at the origin (recall that B'' is a bilinear operator), etc. On the right-hand side only B' is homogeneous with degree 1; $B''(X, X)$ is homogeneous with degree 2 and the other summands have greater degrees of homogeneity. Thus the left-hand side is homogeneous with degree 1 only if all summands on the right hand side except B' are equal to zero. Hence $B = B'$ and so it is a linear operator. □

Proof. (of Theorem 2.17) Since by Definition 2.8(i) both $\mathsf{H}_{(m,\vartheta)}$ and $\mathsf{H}^E_{(m,\vartheta)}$ are smooth in ϑ, so too is $\mathbf{p}\Gamma_m(\vartheta, Y_1)$. We shall show that $\mathbf{p}\Gamma_m(\vartheta, Y_1)$ is homogeneous with degree 1 in ϑ so that the statement of Theorem 2.17 will follow from Lemma 2.18.

The vector $\mathsf{H}Y = T\pi^{-1}(Y_1) \in \mathsf{H}_{(m,\vartheta)}$ is presented as a quadruple in the form $(m, \vartheta, Y_1, \Gamma_m(\vartheta, Y_1))$. By Definition 2.8(ii) and by formula (2.8) (describing Ta) the vector $Ta(\mathsf{H}Y) = (m, a\vartheta, Y_1, a\Gamma_m(\vartheta, Y_1))$ belongs to $\mathsf{H}_{(m,a\vartheta)}$. Using formula (2.3) we get $T\pi((m, a\vartheta, Y_1, a\Gamma_m(\vartheta, Y_1)) = (m, Y_1)$. Since $T\pi$ is one-to-one on $\mathsf{H}_{(m,a\vartheta)}$, it is the unique vector in $\mathsf{H}_{(m,a\vartheta)}$ whose image under $T\pi$ is (m, Y_1). But the vector $\mathsf{H}Y = T\pi^{-1}(Y_1) \in \mathsf{H}_{(m,a\vartheta)}$, whose quadruple takes the form $(m, a\vartheta, Y_1, \Gamma_m(a\vartheta, Y_1))$, also has this property: $T\pi(m, a\vartheta, Y_1, \Gamma_m(a\vartheta, Y_1)) = (m, Y_1)$. Hence,

$$(m, a\vartheta, Y_1, a\Gamma_m(\vartheta, Y_1)) = (m, a\vartheta, Y_1, \Gamma_m(a\vartheta, Y_1))$$

and so, since $\mathbf{p}$ is linear, $\mathbf{p}\Gamma_m(a\vartheta, Y_1) = a\mathbf{p}\Gamma_m(\vartheta, Y_1)$. □

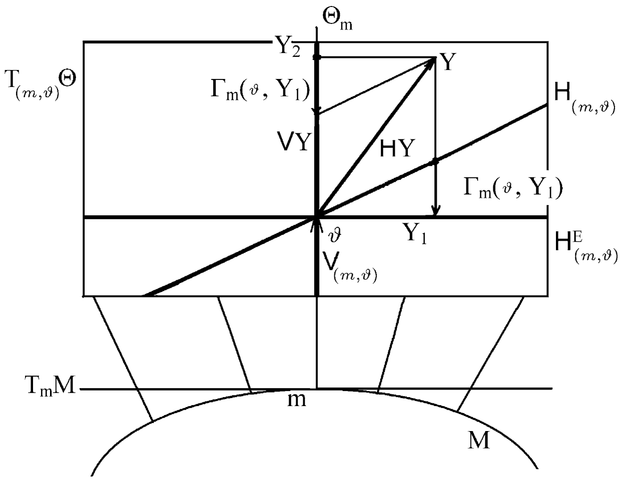
So, $\mathbf{p}\Gamma_m(\cdot,\cdot)$ is linear in both arguments. It is useful to find its values on basis vectors. Consider $\mathbf{p}\Gamma_m(e_i, \frac{\partial}{\partial q^j})$. It is a vector from Θ_m and so it can be expanded in coordinates Γ_{ij}^k with respect to the basis $e_1, \ldots, e_d$: $\mathbf{p}\Gamma_m(e_i, \frac{\partial}{\partial q^j}) = \Gamma_{ij}^k e_k$. The coordinates Γ_{ij}^k depend on $m \in \mathfrak{U}_\alpha$ (as well as on a trivialization), this means that they are real-valued functions of $m \in \mathfrak{U}_\alpha$.

Definition 2.19. The functions Γ_{ij}^k are called *Christoffel symbols of the second kind* for the connection H.

Knowing Γ_{ij}^k, we can calculate the values $\mathbf{p}\Gamma_m(X, Y)$ for any $X \in \Theta_m$, $Y \in T_m M$. Indeed, let $X = X^i e_i$ and $Y = Y^j \frac{\partial}{\partial q^j}$, then by linearity we get

$$\mathbf{p}\Gamma(X, Y) = X^i Y^j \Gamma_{ij}^k e_k. \tag{2.13}$$

Now let us turn back to the connector K. Recall that $K(Y) = \mathbf{p}\mathsf{V}Y$ for $Y \in T_{(m,\vartheta)}\Theta$. Thus we need to describe $\mathbf{p}\mathsf{V}Y$. For Y we have two decompositions:



$Y = Y_1 + Y_2$ and $Y = \mathsf{H}Y + \mathsf{V}Y$. Hence $Y_1 + Y_2 = \mathsf{H}Y + \mathsf{V}Y$ and so $\mathsf{V}Y - Y_2 = Y_1 - \mathsf{H}Y = \Gamma_m(\vartheta, Y_1)$. Thus $\mathsf{V}Y = Y_2 + \Gamma_m(\vartheta, Y_1)$ and consequently $\mathsf{V}Y = \mathsf{V}(m, \vartheta, Y_1, Y_2) = (m, \vartheta, 0, Y_2 + \Gamma_m(\vartheta, Y_1))$. Finally we obtain the formula for K in the form:

$$K(m, \vartheta, Y_1, Y_2) = \mathbf{p}\mathsf{V}(m, \vartheta, Y_1, Y_2) = (m, \mathbf{p}Y_2 + \mathbf{p}\Gamma(\vartheta, Y_1)). \tag{2.14}$$

Compare (2.14) with (2.3) and (2.5).

Let $Y_2 = \dot{Y}^k \frac{\partial}{\partial \vartheta^k}$ and $\vartheta = \dot{q}^i e_i$. Using (2.12) we describe (2.14) in coordinates as follows

$$K(m, \vartheta, Y_1, Y_2) = (\dot{Y}^k + \dot{q}^i Y^j \Gamma_{ij}^k) e_k. \tag{2.15}$$

Remark 2.20. If we choose arbitrary functions $\Gamma^k_{ij}(m)$ on $\mathfrak{U}_\alpha$ for all possible values of i, j and k, we shall be able to define a local connector $\mathbf{p}\Gamma_m(\cdot,\cdot)$ by formula (2.13) and consequently a connector K by formula (2.14) or (2.15) and then define the corresponding connection H on $\pi^{-1}\mathfrak{U}_\alpha$ as kernels of K in all tangent spaces.

Remark 2.21. We say that $\mathbf{p}\Gamma_m(\vartheta, Y_1)$ is a vector in Θ_m. If we change the trivialization, this vector will no longer correspond to the local connector. Indeed, the Euclidean connection will be changed (see Proposition 2.5) and the old $\mathbf{p}\Gamma_m(\vartheta, Y_1)$ will not be the difference between $\mathsf{V}Y$ and the new Y_1. So, the change of $\mathbf{p}\Gamma_m(\vartheta, Y_1)$ under a change of coordinates on M and of a trivialization is described by a complicated "non-tensorial" formula that follows from (2.2). We shall derive it in explicit form for some special cases below (see formula (2.19)).

The covariant derivative and parallel translation

Here we present the general construction by which every connection defines its own method of differentiating a cross-section of Θ along a vector field on M. Notice that the use of a Euclidean connection of a natural trivialization of $\mathbb{R}^n \times \mathbb{R}^d$ gives the standard method of differentiating typically introduced in a classical course in mathematical analysis.

Let X be a smooth vector field on M and Y be a cross-section of a vector bundle Θ equipped with a connection H.

Definition 2.22. The *covariant derivative* $\nabla_X Y$ of a cross-section Y along a vector field X is the cross-section of Θ determined by the formula $\nabla_X Y = K \circ TY(X)$.

Let us discuss this definition. The cross-section Y can be considered as a smooth map $Y : M \to \Theta$. Its tangent map TY sends the vector $X \in T_m M$ to the tangent space $T_{(m,Y)}\Theta$. On applying K we again map into Θ_m.

Example 2.23. Consider the Euclidean connection of a natural trivialization of $\mathbb{R}^n \times \mathbb{R}^d$. The section Y can be presented as the map $m \mapsto (m, Y_m)$. We express the tangent map TY in coordinates and find the vector $TY(X)$. Here V coincides with the projection along H^E. One can easily see that the obtained covariant derivative coincides with the ordinary derivative of Y along the field X.

Theorem 2.24 *The covariant derivative has the following properties for any vector fields X, X_1 and X_2, smooth cross-sections Y, Y_1 and Y_2, smooth function $f : M \to \mathbb{R}$ and $\varkappa, \lambda \in \mathbb{R}$:*

(i) $\nabla_{(\varkappa X_1 + \lambda X_2)} Y = \varkappa \nabla_{X_1} Y + \lambda \nabla_{X_2} Y$;
(ii) $\nabla_{fX} Y = f \nabla_X Y$;
(iii) $\nabla_X (\varkappa Y_1 + \lambda Y_2) = \varkappa \nabla_X Y_1 + \lambda \nabla_X Y_2$;

(iv) $\nabla_X fY = (Xf)Y + f\nabla_X Y,$

where Xf *is the derivative of* f *along* X.

Proof. Properties (i) and (ii) follow immediately from the linearity of $TY : T_m M \to T_{(m,Y)}\Theta$, of the projection V and of $\mathbf{p}$ (see Definition 2.13 of K). In order to prove (iii) one should recall the action of TY derived in (2.6) and the representation of K via $\mathbf{p}\Gamma_m(\cdot,\cdot)$ in (2.14). Now (iii) follows from the fact that $\mathrm{d}_m Y$ is linear in Y and from the linearity of $\mathbf{p}\Gamma_m(\cdot,\cdot)$ in the first argument (Theorem 2.17). For the proof of (iv), we find a formula for $\mathrm{d}_m(fY)$ according to the usual rules of differentiation as follows:

$$\mathrm{d}_m(fY) = \begin{pmatrix} I \\ \left(\frac{\partial fY^i}{\partial q^j}\right) \end{pmatrix} = \begin{pmatrix} I \\ Y\mathrm{d}f + f\left(\frac{\partial Y^i}{\partial q^j}\right) \end{pmatrix}$$

where $\mathrm{d}f = \frac{\partial f}{\partial q^i}\mathrm{d}q^i$ is the differential of f (see (1.14)). Thus, taking into account that $Xf = \mathrm{d}f(X)$ (see formula (1.16)), we get

$$\begin{aligned} T(fY)X_m = T(fY)(m,X) &= \left(m, fY, X, (Xf)Y + \left(\frac{\partial Y^i}{\partial q^j}\right)X\right) \\ &= (m, fY, 0, (Xf)Y) + \left(m, fY, X, f\left(\frac{\partial Y^i}{\partial q^j}\right)X\right). \end{aligned}$$

By definition $K\left(\left(m, fY, X, f\left(\frac{\partial Y^i}{\partial q^j}\right)X\right)\right) = f\nabla_X Y$. Since $(m, fY, 0, (Xf)Y)$ is vertical (i.e., belongs to $\mathsf{V}_{(m,fY)}$), $K((m, fY, 0, (Xf)Y)) = (m, (Xf)Y)$. □

Using the expression of K via Christoffel symbols (2.15), we find the expression for $\nabla_X Y$ in local coordinates in the form:

$$\nabla_X Y = \left(\frac{\partial Y^k}{\partial q^j}X^j + Y^i X^j \Gamma^k_{ij}\right)e_k. \tag{2.16}$$

Notice that $\frac{\partial Y^k}{\partial q^j}X^j e_k$ is the ordinary derivative of Y along X as in a trivial bundle. Under a change of trivialization this term transforms incorrectly. Only after adding $Y^i X^j \Gamma^k_{ij} e_k$ does (2.16) retain its form under a change of coordinates and trivialization. In the language used by physicists, this means that formula (2.16) is *covariant*. This is why we call the operation $\nabla_X Y$ the covariant derivative.

For further applications we also need a covariant construction for differentiating a cross-section in the "time" variable t along a certain curve $m(t)$ in M.

Let $m(t)$ be a smooth curve on M and $Y(t)$ be a cross-section of Θ over $m(\cdot)$. This means that at any point $m(t)$ there is associated a vector $Y(t) \in \Theta_{m(t)}$, and $Y(t)$ is smooth in t. The vector $\frac{\mathrm{d}}{\mathrm{d}t}Y(t)$ at any t belongs to the tangent space $T_{(m(t),Y(t))}\varTheta$. Consider the vector $\frac{\mathrm{D}}{\mathrm{d}t}Y(t) = K \circ \frac{\mathrm{d}}{\mathrm{d}t}Y(t)$ in $\varTheta_{m(t)}$.

Definition 2.25. The vector $\frac{\mathrm{D}}{\mathrm{d}t}Y(t) = K \circ \frac{\mathrm{d}}{\mathrm{d}t}Y(t)$ is called the *covariant derivative* of $Y(t)$ along $m(t)$ in t.

Let us discuss the relation between the operations ∇ and $\frac{\mathrm{D}}{\mathrm{d}t}$. We might hope that $\frac{\mathrm{D}}{\mathrm{d}t}Y(t)$ would be equal to $\nabla_{\dot{m}(t)}Y = K \circ TY(\dot{m}(t))$ if the latter expression were well-defined. Unfortunately this is not the case since the cross-section $Y(t)$ is given only at the points of the curve $m(t)$ while, when determining TY, it is necessary that Y is defined in a neighborhood of $m(t)$.

This is why we have to apply the following trick. On a subinterval of the domain, where the curve has neither self intersections nor periods where it is constant, define an auxiliary smooth vector field $\tilde{Y}$ in a neighborhood of $m(t)$ such that at the points of $m(t)$ it coincides with $Y(t)$: $\tilde{Y}_{m(t)} = Y(t)$. Various constructions of such fields are typically described in textbooks on differential geometry and topology. The expression $\nabla_{\dot{m}(t)}\tilde{Y} = K \circ T\tilde{Y}(\dot{m}(t))$ therefore makes sense.

Theorem 2.26 $\nabla_{\dot{m}(t)}\tilde{Y} = \frac{\mathrm{D}}{\mathrm{d}t}Y(t)$ *and so it does not depend on the choice of smooth vector field* $\tilde{Y}$.

Proof. Since the curve $m(t)$ and the map $\tilde{Y} : M \to TM$ are smooth, the curve $\tilde{Y}_{m(t)}$ in $\varTheta$ is smooth and by the construction of the tangent map $T\tilde{Y}(\dot{m}(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}_{m(t)}$. But $\tilde{Y}_{m(t)} = Y(t)$, hence $T\tilde{Y}(\dot{m}(t)) = \frac{\mathrm{d}}{\mathrm{d}t}Y(t)$ and so $\nabla_{\dot{m}(t)}\tilde{Y} = K \circ T\tilde{Y}(\dot{m}(t)) = K \circ \frac{\mathrm{d}}{\mathrm{d}t}Y(t) = \frac{\mathrm{D}}{\mathrm{d}t}Y(t)$. In particular $\nabla_{\dot{m}(t)}\tilde{Y}$ does not depend on the choice of $\tilde{Y}$. □

Remark 2.27. Taking into account Theorem 2.26 we shall sometimes use the expression $\frac{\mathrm{D}}{\mathrm{d}t}Y(t) = \nabla_{\dot{m}}Y(t)$ where it is understood that in the right hand side $Y(t)$ represents some $\tilde{Y}$ such that $\tilde{Y}_{m(t)} = Y(t)$. This will simplify the formulae and arguments below.

Thus, in order to obtain a representation of $\frac{\mathrm{D}}{\mathrm{d}t}$ in terms of a local connector, analogous to (2.16), we should replace the vector field X by the velocity vector $\dot{m}(t) = \frac{\mathrm{d}m^j}{\mathrm{d}t}\frac{\partial}{\partial q^j}$ and $T\tilde{Y}(\dot{m}(t))$ by $\frac{\mathrm{d}}{\mathrm{d}t}Y(t)$. So, the analog of (2.16) takes the form

$$\frac{\mathrm{D}}{\mathrm{d}t}Y(t) = \left(\frac{\mathrm{d}Y^k}{\mathrm{d}t} + \varGamma^k_{ij}Y^i\frac{\mathrm{d}m^j}{\mathrm{d}t}\right)e_k. \tag{2.17}$$

If our vector bundle $\varTheta$ is trivial and a trivialization is specified, the notion of a constant cross-section of $\varTheta$ is well-defined. Indeed, since $\varTheta$ is represented as a direct product $M \times \mathbb{R}^d$, the cross-section $M \times Y_0$ corresponding to the layer of a fixed $Y_0 \in \mathbb{R}^d$ can be considered where at any point $m \in M$ the same vector in $\varTheta_m$ is applied. The visual image here is that all vectors of the cross-section are parallel to each other. The derivative of such a cross-section along any smooth curve in M is equal to zero.

In a general non-trivial bundle the idea of "applying the same vector" at each point of M cannot be realized. Nevertheless we still have a covariant derivative along a curve (rather than an ordinary derivative, which is not convenient, see above) and so we can consider cross-sections along curves with zero covariant derivatives and say that they consist of vectors parallel to each other. Let us give the exact definition.

Definition 2.28. A cross-section $Y(t)$ along a curve $m(t)$, $t \in [0, l]$, is called *parallel* if $\frac{\mathrm{D}}{\mathrm{d}t} Y(t) = 0$ for all $t \in [0, l]$.

It follows from (2.17) that a parallel cross-section is described by the system of first order linear differential equations

$$\frac{\mathrm{d}Y^k}{\mathrm{d}t} + \Gamma^k_{ij} Y^i \frac{\mathrm{d}m^j}{\mathrm{d}t} = 0. \tag{2.18}$$

Theorem 2.29 *For any initial vector $Y_0 \in \Theta_{m(0)}$ there exists a unique solution $Y(t)$ of the system (2.18), well-defined for all $t \in [0, l]$.*

Indeed, this is a well-known existence and uniqueness theorem for linear first order differential equations. The only modification needed here is that one should prove the existence and uniqueness in a finite number of charts since (2.18) is given in terms of local coordinates.

Definition 2.30. The solution $Y(t)$ whose existence is asserted in Theorem 2.29 is called the *parallel translation of vector Y_0 along $m(\cdot)$*.

The idea of parallel translation can also be expressed in another language. Let a vector field X be given on M. At any point $m \in M$ consider the fiber Θ_m and the horizontal subspaces $\mathsf{H}_{(m,\vartheta)}$ at all points $(m, \vartheta) \in \Theta_m$. Recall that (see Proposition 2.7) $T\pi : \mathsf{H}_{(m,\vartheta)} \to T_m M$ is one-to-one and so at any (m, ϑ) we can define the vector $\tilde{X}_{(m,\vartheta)} = T\pi^{-1}(X_m)_{|\mathsf{H}_{(m,\vartheta)}}$.

Definition 2.31. The vector field $\tilde{X}$ on Θ is called the *horizontal lift* of the field X.

Now restrict the bundle Θ to the curve $m(\cdot)$ and consider on $\Theta_{m(\cdot)}$ the horizontal lift of the field $\dot{m}(t)$. This gives a smooth vector field on $\Theta_{m(\cdot)}$ and, taking the initial value $Y_0 \in \Theta_{m(0)}$, we can find the unique integral curve $Y(t)$ of this vector field. One can easily see that $Y(t)$ is the parallel translation of Y_0 according to Definition 2.30.

Let $m(t)$, $t \in [0, T]$, be a smooth curve on M and $\vartheta(t)$ be a cross-section of Θ along $m(\cdot)$ (i.e., $\vartheta(t)$ belongs to the fiber $\Theta_{m(t)}$ for all $t \in [0, T]$). Denote by $\Gamma_{s,t}$ the linear operator of parallel translation along $m(\cdot)$ from $\Theta_{m(t)}$ to $\Theta_{m(s)}$. Consider $\bar{\vartheta}(t) = \Gamma_{s,t}\vartheta(t)$, a curve in the fiber $\Theta_{m(s)}$. Its derivative $\frac{\mathrm{d}}{\mathrm{d}t}\bar{\vartheta}(t)_{|t=s}$ belongs to $T_{\bar{\vartheta}(s)}\Theta_{m(s)}$. Applying to it the operator $\mathbf{p}$, we obtain a vector in the fiber $\Theta_{m(s)}$. Everywhere below we regard $\frac{\mathrm{d}}{\mathrm{d}t}\bar{\vartheta}(t)_{|t=s}$ as a free vector lying in $\Theta_{m(s)}$ and so we do not distinguish in notation between $\mathbf{p}\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t)_{|t=s}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t)_{|t=s}$.

Theorem 2.32 $\frac{\mathrm{D}}{\mathrm{d}t}\vartheta(t)_{|t=s} = \frac{\mathrm{d}}{\mathrm{d}t}(\Gamma_{s,t}\vartheta(t))_{|t=s}$.

Proof. Since the curve $\Gamma_{s,t}\vartheta(t)$ lies in the fiber $\Theta_{m(s)}$, its derivative is vertical. Clearly $\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{s,t}\vartheta(t) = T\Gamma_{s,t}\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t)$. Note that for any given t the vector $\Gamma_{s,t}\vartheta(t)$ is the value at s of the horizontal lift of $m(t)$ that at t goes through $q(t) \in \Theta_{m(t)}$. Then the vector tangent to the horizontal lift belongs to the kernel of the tangent mapping $T\Gamma_{s,t}$. But this vector is the horizontal component of $\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t)$. In particular, this means that $\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{s,t}\vartheta(t)_{|t=s}$ is the vertical component of $\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t)_{|t=s}$. Hence $\mathbf{p}\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t)_{|t=s} = \frac{\mathrm{D}}{\mathrm{d}t}\vartheta(t)_{|t=s}$. Since (see above) we do not distinguish between $\mathbf{p}\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t)_{|t=s}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t)_{|t=s}$, the Theorem follows. □

2.3 Connections on Manifolds

Since the tangent bundle TM of a manifold M is a particular case of a vector bundle, all the constructions of Section 2.2 are also valid for tangent bundles.

Definition 2.33. A connection as in Section 2.2, given on the vector bundle TM, is called a *connection* on the manifold M.

Connections on manifolds have special features since here the fiber of the bundle is also a tangent space to the manifold (the base of the bundle). For this reason some constructions are simplified and some operators acquire new properties. In this Section we describe these special features. We use the notation and constructions from Section 2.1.

The vertical subspace $\mathsf{V}_{(m,X)} \subset T_{(m,X)}TM$ turns out to be the tangent space to the fiber of the tangent bundle, i.e. $\mathsf{V}_{(m,X)} = T_X T_m M$. This is why the operator $\mathbf{p}$, introduced by formula (1.2), is an isomorphism of $\mathsf{V}_{(m,X)}$ to $T_m M$.

When we specify a connection H on the tangent bundle, we introduce a subspace $\mathsf{H}_{(m,X)}$ in each $T_{(m,X)}TM$ that is complementary to $\mathsf{V}_{(m,X)}$ in such a way that the collection H satisfies Definition 2.8.

Recall that the tangent bundle of TM is called the second tangent bundle to M and is denoted by TTM or T^2M (see Definition 2.3). So, the connector K sends TTM onto TM and in particular it transforms each $T_{(m,X)}TM$ into $T_m M$. The subspaces $\mathsf{H}_{(m,X)}$ are kernels of K and the mapping K on $\mathsf{V}_{(m,X)}$ coincides with $\mathbf{p}$. As in the general case, $T\pi$ sends $\mathsf{H}_{(m,X)}$ isomorphically onto $T_m M$ and $\mathsf{V}_{(m,X)}$ is the kernel of $T\pi$. Thus for any vector $Y \in T_m M$ at any point $(m, X) \in TM$ there exists a unique vector $Y^l \in \mathsf{V}_{(m,X)}$ such that $\mathbf{p}Y^l = Y$, and a unique vector $Y^T \in \mathsf{H}_{(m,X)}$ such that $T\pi Y^T = Y$.

Definition 2.34. The vector Y^l is called the *vertical lift* of Y at the point (m, X), and the vector Y^T is called the *horizontal lift* of Y at the point (m, X).

Recall that a Euclidean connection and the local connector corresponding to it depend on a trivialization in $\pi^{-1}\mathfrak{U}_\alpha$. We retain the notation $\mathsf{H}^E_{(m,X)}$ for a trivialization by coordinate frames $\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}$ (see Sections 1.1 and 2.1). For corresponding objects with respect to other trivializations we shall introduce the special notation below. For the sake of simplicity we denote by $\boldsymbol{\Gamma}_m(\cdot,\cdot)$ the local connector with respect to this trivialization, i.e., $\boldsymbol{\Gamma}_m(\cdot,\cdot) = \mathbf{p}\Gamma_m(\cdot,\cdot)$.

The local connector $\boldsymbol{\Gamma}_m(\cdot,\cdot)$ is a bilinear operator $\boldsymbol{\Gamma}_m : T_mM \times T_mM \to T_mM$. In particular, in this case the condition that $\boldsymbol{\Gamma}_m$ is symmetric is reasonable. The Christoffel symbols of the second kind Γ^k_{ij} are well-defined for indices $i, j, k = 1, \dots, n$. We emphasize that in the natural coordinate systems $\Gamma_m\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right) = \Gamma^k_{ij}\frac{\partial}{\partial q^k}$ while $\boldsymbol{\Gamma}_m\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right) = \Gamma^k_{ij}\frac{\partial}{\partial q^k}$ where Γ^k_{ij} are Christoffel symbols of the second kind.

Since the operator $g_{\beta\alpha}$ in TM equals $\varphi'_{\beta\alpha}$ and $\Gamma_m(X, Y_1)$ as a quadruple is presented in the form $(m, X, Y_1, \Gamma_m(X, Y_1))$, from formula (2.10) it follows that under a change of coordinates $\varphi_{\beta\alpha}$ the local connector of a connection on a manifold transforms in the following manner

$$\boldsymbol{\Gamma}_m(X, Y_1)^\beta = -\varphi''_{\beta\alpha}(m^\alpha)(X^\alpha, Y_1^\alpha) + \varphi'_{\beta\alpha}(\boldsymbol{\Gamma}_m(X, Y_1)^\alpha). \tag{2.19}$$

The geometric interpretation of formula (2.19) is the same as that given in Remark 2.21.

Proposition 2.35 *The difference $\boldsymbol{\Gamma}(\cdot,\cdot) - \bar{\boldsymbol{\Gamma}}(\cdot,\cdot)$ of local connectors $\boldsymbol{\Gamma}(\cdot,\cdot)$ and $\bar{\boldsymbol{\Gamma}}(\cdot,\cdot)$ of different connections is a $(1,2)$-tensor.*

Indeed, by formula (2.19) the difference transforms under coordinate changes by the rule

$$\boldsymbol{\Gamma}_m(\cdot,\cdot)^\beta - \bar{\boldsymbol{\Gamma}}_m(\cdot,\cdot)^\beta = \varphi'_{\beta\alpha}[\boldsymbol{\Gamma}_m(\cdot,\cdot)^\alpha - \bar{\boldsymbol{\Gamma}}_m(\cdot,\cdot)^\alpha].$$

Since the cross-sections of a tangent bundle are vector fields on M, the covariant derivative $\nabla_X Y$ differentiates the vector field Y in the direction of the vector field X and $\frac{\mathrm{D}}{\mathrm{d}t}X(t)$ differentiates the vector field $X(t)$ in the time parameter along the curve $m(t)$ (see Section 2.2).

Equations (2.16) and (2.17) take the forms

$$\nabla_X Y = \left(\frac{\partial Y^k}{\partial q^j}X^j + \Gamma^k_{ij}Y^iX^j\right)\frac{\partial}{\partial q^k}, \tag{2.20}$$

$$\frac{\mathrm{D}}{\mathrm{d}t}Y(t) = \left(\frac{\mathrm{d}Y^k}{\mathrm{d}t} + \Gamma^k_{ij}Y^i\frac{\mathrm{d}m^j}{\mathrm{d}t}\right)\frac{\partial}{\partial q^k}. \tag{2.21}$$

Since in each chart the basis vectors $\frac{\partial}{\partial q^i}$ have constant coordinates in the decomposition with respect to the same basis (the i-th coordinate is 1 and all others equal zero), from formula (2.20) it follows that

$$\nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = \Gamma^k_{ij} \frac{\partial}{\partial q^k}. \tag{2.22}$$

Theorem 2.36 *Let ∇ and $\bar{\nabla}$ be covariant derivatives of two different connections. Then there exists a unique $(1, 2)$-tensor $S(\cdot, \cdot)$, determined by the connections, such that for any pair of smooth vector fields X and Y the equality $\nabla_X Y - \bar{\nabla}_X Y = S(X, Y)$ holds.*

Theorem 2.36 follows from formula (2.20) and Proposition 2.35.

If $\frac{\mathrm{D}}{\mathrm{d}t} X(t) = 0$, by analogy with Definition 2.28 we say that $X(t)$ is a *parallel vector field* along the curve $m(t)$. From (2.21) it follows that a parallel vector field satisfies the system of equations

$$\frac{\mathrm{d}Y^k}{\mathrm{d}t} + \Gamma^k_{ij} Y^i \frac{\mathrm{d}m^j}{\mathrm{d}t} = 0. \tag{2.23}$$

A parallel vector field along a curve is an analog of a constant vector field in a linear space. We refer the reader to Section 2.2 where the analogy between a "constant" cross-section of a trivial vector bundle and a parallel cross-section along a curve is described. Notice that the Euclidean connection H^E on a linear space has zero local connector and so the covariant derivative generated by it coincides with the ordinary derivative of a vector field along a vector field or in time along a curve. Thus, on a linear space, parallel vector fields become constant.

Applying Theorem 2.29 to equation (2.23) we obtain that for every smooth curve $m(t)$ and for a specified initial vector $X \in T_{m_0}M$, there exists a unique parallel vector field $X(t)$ with initial condition $X(0) = X$ that is well-defined for all t in the domain of the curve. This vector field is called the *parallel translation* of X along $m(t)$.

Remark 2.37. In the case of a Riemannian manifold M, there is another commonly used trivialization of $\pi^{-1}\mathfrak{U}_\alpha$, namely by a field of orthonormal frames. Let in each tangent space T_mM, $m \in \mathfrak{U}_\alpha$, an orthonormal frame be specified that consists of vectors $e_1, \ldots, e_n$ and let each vector field e_i, $i = 1, \ldots, n$, be smooth. This frame field generates the trivialization in which a point $(m, X^i e_i) \in \pi^{-1}\mathfrak{U}_\alpha$ transforms into the point $(m, (X^1, \ldots, X^n)) \in \mathfrak{U}_\alpha \times \mathbb{R}^n$. The corresponding local connector is called a *tetrad connector* and is denoted by $\mathbf{p} \overset{\circ}{\Gamma}_m (\cdot, \cdot)$ (the term "tetrad" derives from general relativity where $n = 4$). The tetrad Christoffel symbols are denoted by $\overset{\circ}{\Gamma}{}^k_{ij}$ and are defined by the equality $\nabla_{e_i} e_j = \overset{\circ}{\Gamma}{}^k_{ij} e_k$. Since $\mathbf{p} \overset{\circ}{\Gamma}_m (\cdot, \cdot)$ is bilinear, it is uniquely determined by the tetrad symbols. For more detail, see e.g. [57].

2.4 Geodesics

The notion of a parallel vector field along a curve leads to another important notion.

Definition 2.38. A curve $m(t)$ along which its velocity vector field $\dot{m}(t)$ is parallel is called a *geodesic*.

On a manifold with connection the geodesics are analogs of straight lines in a vector space. Indeed, since a parallel vector field along a curve is an analog of a constant vector field in linear space, the property of a curve possessing a parallel velocity vector field is analogous to the property of a curve in a vector space possessing constant velocity. In a vector space the straight lines with natural parametrization, and only these lines, have the latter property.

From Definition 2.38 and the definition of parallel translation it follows that a curve $m(t)$ is a geodesic if and only if at each of its points the equality

$$\frac{\mathrm{D}}{\mathrm{d}t}\dot{m}(t) = 0 \tag{2.24}$$

holds. Equation (2.24) describes an analog of the property that straight lines in linear spaces have zero second derivative.

We now derive the equation of geodesics in local coordinates. For this purpose, in equation (2.23) we replace the coordinates of the vector Y by the coordinates of the vector $\dot{m}(t)$, since in our case the latter is parallel along $m(t)$. Then we obtain

$$\frac{\mathrm{d}^2 m^k}{\mathrm{d}t^2} + \Gamma^k_{ij}\frac{\mathrm{d}m^i}{\mathrm{d}t}\frac{\mathrm{d}m^j}{\mathrm{d}t} = 0. \tag{2.25}$$

Unlike (2.18) and (2.23), (2.25) is a non-linear second order differential equation (recall that (2.18) and (2.23) are linear first order differential equations). This is why we can apply only the most general existence of solution theorem for second order differential equations with smooth right-hand sides, from which we obtain the following statement of local existence and uniqueness of geodesics with given initial data.

Theorem 2.39 *For every point $m \in M$ and every vector $X \in T_mM$ there exists a unique geodesic $m(t)$, with initial conditions $m(0) = m$ and $\dot{m}(0) = X$, that is defined for $t \in [0, \varepsilon)$ where $\varepsilon > 0$ is a sufficiently small positive number.*

Theorem 2.39 is much weaker than existence Theorem 2.29 but it mirrors the physical situation if no additional hypotheses are assumed. For example, on an open manifold (e.g., consisting of only one open chart) the geodesic exists for $t \in [0, \varepsilon)$ where ε is the instant of time when the geodesic reaches the boundary, but it does not exist at any later time.

Definition 2.40. If each geodesic of a connection H exists for $t \in (-\infty, \infty)$, the connection H on M is said to be *complete*.

Let $X \in T_m M$ be a tangent vector at a point m. Denote by $m_X(t)$ the geodesic with initial data $m(0) = m$ and $\dot{m}(0) = X$ (which we know exists for $t \in [0, \varepsilon)$ by Theorem 2.39). Specify a positive number $\lambda < 1$. One can easily see that $m(\lambda t)$ is a geodesic with initial vector λX that exists for $t \in [0, \frac{1}{\lambda}\varepsilon)$. Thus, if X is close enough to the origin, the geodesic $m_X(t)$ exists at $t = 1$.

Definition 2.41. The mapping $\exp : \mathcal{O} \to M$, where $\mathcal{O}$ is a neighborhood of the origin in $T_m M$, is given by the formula $\exp(X) = m_X(1)$, and is called the *exponential mapping* of the connection H.

It is clear that if H is a complete connection, the exponential mapping is well-defined on $T_m M$. Sometimes, when dealing with exponential mappings from tangent spaces at various points of M, we shall use the notation $\exp_m : \mathcal{O}_m \to M$.

Theorem 2.42 *There exists a neighborhood $\mathcal{O}_m$ of the origin in $T_m M$ such that $\exp_m$ is a diffeomorphism of $\mathcal{O}_m$ onto $\exp_m \mathcal{O}_m$ and the exponential mapping is smooth on the neighborhood $\bigcup_{m \in M} \mathcal{O}_m$ of the zero-section in TM.*

A proof of Theorem 2.42 can be found, for example, in [26] and [161].

Notice that the pair $(\mathcal{O}_m, \exp_m)$ satisfies the definition of chart. This pair is called the *normal chart* (or *normal neighborhood*) of the connection H at the point m. In this chart at m the connection space $\mathsf{H}_{(m,X)}$ at each $X \in T_m M$ coincides with the Euclidean connection space $\mathsf{H}^E_{(m,X)}$ and so $\boldsymbol{\Gamma}_m(\cdot, \cdot) = 0$. Hence in a normal chart at m all Christoffel symbols of the second kind $\Gamma^k_{ij}(m)$ for H at this point are equal to zero.

Suppose that the connection is complete and $\mathcal{O}_m$ is the maximal domain on which $\exp_m$ is one-to-one, i.e., such that the exponential map is one-to-one on $\mathcal{O}_m$ but not on the boundary $\partial \mathcal{O}_m$ in $T_m M$.

Definition 2.43. The set $\partial \mathcal{O}_m \subset T_m M$ is called the *cut locus* corresponding to the point m. The same term is also used to designate the image of $\partial \mathcal{O}_m$ under the mapping $\exp_m$.

All points of M besides the cut locus belong to the image of $\mathcal{O}_m$ under the diffeomorphism $\exp_m$. From this it follows that each manifold can be constructed from an open ball in a vector space by "gluing" the points of the boundary (according to a rule, determined by the manifold) so that the corresponding cut locus is obtained (see [140]).

Let the points m_0 and m_1 be connected by a geodesic $a(\cdot)$ of a connection H. This means that $m_1 = \exp_{m_0} X$ for some vector $X \in T_{m_0} M$.

Definition 2.44. If the differential $\mathrm{d}_X \exp : T_X T_{m_0} M \to T_{m_1} M$ at X is degenerate, we say that $m_1 = \exp_{m_0} X$ is *conjugate* with m_0 along the geodesic $a(\cdot)$ joining them.

2.5 Curvature and Torsion Tensors

Let X and Y be smooth vector fields on a manifold M with connection. These vector fields determine a transformation of an arbitrary smooth vector field Z by the formula

$$\mathsf{R}_{XY}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{2.26}$$

Observe that the value $\mathsf{R}_{XY}Z$ at $m \in M$ depends only on the values of the vector fields X, Y, Z at m (it does not depend on their values in a neighborhood of m), i.e., $\mathsf{R}_{XY}Z$ is a tensor. (In particular this means that, in spite of the definition, (2.26) is well-defined for non-smooth vector fields X, Y and Z.)

Definition 2.45. $\mathsf{R}_{XY}Z$ is called the *curvature tensor*.

If $\mathsf{R}_{XY}Z = 0$ for all $X, Y, Z,$, the connection is called *flat*. An example of a flat connection is a Euclidean connection of any coordinate system.

The curvature is a $(1,3)$-tensor and its description as a polylinear form takes the form $\mathsf{R}(\alpha, X, Y, Z) = \alpha(\mathsf{R}_{XY}Z)$, where α is a covector field (1-form). We denote the components of the curvature tensor by R^i_{jkl}.

For two vector fields X and Y on M one can consider a third vector field

$$\mathsf{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{2.27}$$

Observe that the value $\mathsf{T}(X,Y)$ at $m \in M$ depends only on the values of X and Y at m (it does not depend on their values in a neighborhood of m), i.e., $\mathsf{T}(X,Y)$ is a tensor.

Definition 2.46. $\mathsf{T}(X,Y)$ is called the *torsion tensor*.

The curvature and torsion tensors together "measure" how the vector can be transformed under parallel translation along a closed infinitesimal loop (for details see, e.g., [26]).

Torsion is a $(1,2)$-tensor, i.e., its description as a polylinear form takes the form $\mathsf{T}(\alpha, X, Y) = \alpha(\mathsf{T}(X,Y))$ where α is a covector field (1-form). Denote the components of T by the symbols T^k_{ij}. To calculate these components we substitute into (2.27) the coordinate expressions of $\nabla_X Y$ and $\nabla_Y X$ from formula (2.20) as well as the coordinate expression for $[X,Y]$ from Proposition 1.7. We then obtain

$$\begin{aligned}\mathsf{T}(X,Y) &= \left\{\left(\frac{\partial Y^k}{\partial q^j}X^j + Y^i X^j \Gamma^k_{ij}\right) - \left(\frac{\partial X^k}{\partial q^j}Y^j + X^i Y^j \Gamma^k_{ji}\right)\right. \\ &\qquad \left. - \left(\frac{\partial Y^k}{\partial q^j}X^j - \frac{\partial X^k}{\partial q^j}Y^j\right)\right\}\frac{\partial}{\partial q^k} \\ &= Y^i X^j \Gamma^k_{ij} - X^i Y^j \Gamma^k_{ji}.\end{aligned}$$

Hence,

$$T^k_{i,j} = \Gamma^k_{ij} - \Gamma^k_{ji}. \tag{2.28}$$

Formula (2.28) immediately yields:

Proposition 2.47 *The equality* $\mathsf{T} = 0$ *holds at all points* $m \in M$ *if and only if in all charts* $\Gamma^k_{ij} = \Gamma^k_{ji}$*, i.e., the local connector* $\boldsymbol{\Gamma}_m(\cdot,\cdot)$ *is a symmetric bilinear operator.*

2.6 Riemannian Connections. The Levi-Civitá Connection

From all the connections on a Riemannian manifold M we select one whose covariant derivative properties are the closest to those of the ordinary derivative in Euclidean space.

If on a manifold M a Riemannian metric and a connection are given independently, one should not expect, for the covariant derivative, to find an analog of the Leibnitz formula for differentiating the inner product. Nevertheless for every Riemannian manifold there exists a class of connections having this property.

Let a Riemannian or semi-Riemannian metric $\langle\cdot,\cdot\rangle$ be given on M. For two smooth vector fields Y and Z on M we consider the smooth function $\langle Y, Z\rangle$ that assigns the value of the Riemannian inner product $\langle Y_m, Z_m\rangle$ of the vectors of Y and Z at m to the point m. We find the derivative $X\langle Y, Z\rangle$ of the function $\langle Y, Z\rangle$ in the direction of a smooth vector field X.

Definition 2.48. A connection on M is said to be *Riemannian* if for all smooth vector fields X, Y and Z on M the following equality holds:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle. \tag{2.29}$$

Taking into account the interrelation between ∇ and $\frac{\mathrm{D}}{\mathrm{d}t}$ (see Remark 2.27) one can easily derive the following version of formula (2.29) for $\frac{\mathrm{D}}{\mathrm{d}t}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Y(t), Z(t)\rangle = \left\langle \frac{\mathrm{D}}{\mathrm{d}t}Y(t), Z(t)\right\rangle + \left\langle Y(t), \frac{\mathrm{D}}{\mathrm{d}t}Z(t)\right\rangle, \tag{2.30}$$

where $Y(t)$ and $Z(t)$ are smooth vector fields along a smooth curve $m(t)$.

An existence theorem for Riemannian connections will be proved below (see Remark 2.55).

Specify a Riemannian connection on a Riemannian manifold M.

Theorem 2.49 *Let* $Y(t)$ *and* $Z(t)$ *be parallel vector fields along a smooth curve* $m(t)$*. Then* $\langle Y(t), Z(t)\rangle = \mathrm{const}$*.*

Proof. By the definition of a parallel vector field, $\frac{\mathrm{D}}{\mathrm{d}t}Y(t) = 0$ and $\frac{\mathrm{D}}{\mathrm{d}t}Z(t) = 0$. Having substituted these expressions into (2.30) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Y(t), Z(t)\rangle = \left\langle \frac{\mathrm{D}}{\mathrm{d}t}Y(t), Z(t)\right\rangle + \left\langle Y(t), \frac{\mathrm{D}}{\mathrm{d}t}Z(t)\right\rangle = 0.$$

This means that the function $\langle Y(t), Z(t)\rangle$ is constant. □

Corollary 2.50 *If $Y(t)$ is a parallel vector field along a smooth curve $m(t)$, $\|Y(t)\| = \mathrm{const}$.*

Indeed, $\|Y(t)\| = \sqrt{\langle Y(t), Y(t)\rangle}$ and the assertion of Corollary 2.50 follows from Theorem 2.49.

Corollary 2.51 *If $Y(t)$ and $Z(t)$ are parallel vector fields along a smooth curve $m(t)$, the cosine of the angle between those vectors is constant.*

Since the cosine of the angle between $Y(t)$ and $Z(t)$ equals $\frac{\langle Y(t), Z(t)\rangle}{\|Y(t)\|\|Z(t)\|}$, the assertion of Corollary 2.51 follows from Theorem 2.49 and Corollary 2.50.

Definition 2.52. The functions $\Gamma_{ij,k} = \langle \nabla_{\partial q^i}\partial q^j, \partial q^k\rangle$ in a chart of a Riemannian manifold M are called *Christoffel symbols of the first kind.*

We now describe the interrelation between Christoffel symbols of the first and second kinds. By formula (2.22) $\nabla_{\frac{\partial}{\partial q^i}}\frac{\partial}{\partial q^j} = \Gamma_{ij}^l \frac{\partial}{\partial q^l}$. Thus

$$\Gamma_{ij,k} = \left\langle \Gamma_{ij}^l \frac{\partial}{\partial q^l}, \frac{\partial}{\partial q^k}\right\rangle = g_{lk}\Gamma_{ij}^l. \tag{2.31}$$

Applying the same arguments as in the derivation of formula (1.21), from (1.20) we obtain

$$\Gamma_{ij}^k = g^{lk}\Gamma_{ij,l}. \tag{2.32}$$

In particular, if the torsion tensor equals zero, i.e., $\Gamma_{ij}^k = \Gamma_{ji}^k$, then also $\Gamma_{ij,k} = \Gamma_{ji,k}$.

Note that here we use only the fact that the matrix (g_{ij}) is invertible, not that it is positive-definite. Thus formula (2.32) is well-defined both for Riemannian and semi-Riemannian metrics.

Lemma 2.53 (The principal lemma of Riemannian geometry) *On every manifold M with Riemannian or semi-Riemannian metric $\langle\cdot,\cdot\rangle$ there exists a unique Riemannian connection whose torsion tensor equals zero at all $m \in M$.*

Proof. Recall that by definition $g_{ij} = \left\langle \frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right\rangle$. Since the connection that we are looking for is Riemannian, from formula (2.29) it follows that

$$\frac{\partial}{\partial q^l}g_{ij} = \frac{\partial}{\partial q^l}\left\langle \frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right\rangle = \left\langle \nabla_{\frac{\partial}{\partial q^l}}\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right\rangle + \left\langle \frac{\partial}{\partial q^i}, \nabla_{\frac{\partial}{\partial q^l}}\frac{\partial}{\partial q^j}\right\rangle.$$

So, taking into account Definition 2.52, we obtain $\frac{\partial}{\partial q^l} g_{ij} = \Gamma_{li,j} + \Gamma_{lj,i}$. Considering all rearrangements of the given indices i, j, l we obtain a system of three equations of the same kind as above:

$$\begin{cases} \frac{\partial}{\partial q^l} g_{ij} = \Gamma_{li,j} + \Gamma_{lj,i} \\ \frac{\partial}{\partial q^i} g_{lj} = \Gamma_{il,j} + \Gamma_{ij,l} \\ \frac{\partial}{\partial q^j} g_{il} = \Gamma_{ji,l} + \Gamma_{jl,i}. \end{cases} \tag{2.33}$$

Recall that the torsion tensor equals zero, i.e., the Christoffel symbols of the first kind are symmetric in the first two indices. The system (2.33) of three linear algebraic equations has three unknowns. Adding the second equation to the third one and subtracting the first one from the sum, we obtain

$$\Gamma_{ij,l} = \frac{1}{2}\left(\frac{\partial}{\partial q^i} g_{lj} + \frac{\partial}{\partial q^j} g_{li} - \frac{\partial}{\partial q^l} g_{ij}\right). \tag{2.34}$$

Then by formula (2.32)

$$\Gamma_{ij}^k = \frac{1}{2}\left(\frac{\partial}{\partial q^i} g_{lj} + \frac{\partial}{\partial q^j} g_{li} - \frac{\partial}{\partial q^l} g_{ij}\right) g^{kl}. \tag{2.35}$$

Formula (2.35) uniquely determines the Christoffel symbols of the second kind. From this it follows that the connection we are looking for is unique. The existence is proved by an elementary verification that the connection with Christoffel symbols (2.35) has the properties described in the hypothesis. □

Definition 2.54. The connection whose existence is asserted in Lemma 2.53 is called the *Levi-Civitá connection* of the metric $\langle \cdot, \cdot \rangle$.

It is easy to see that in the Euclidean space $\mathbb{R}^n$ the Levi-Civitá connection of the standard inner product coincides with the Euclidean connection of the standard coordinate system.

Remark 2.55. If a connection is Riemannian but the torsion is not zero, system (2.33) consists of three equations but has six unknowns. This system has an infinite set of solutions, each of them determining a Riemannian connection.

Remark 2.56. The tetrad Christoffel symbols (see Remark 2.37) of the Levi-Civitá connection are determined by the formula

$$\overset{\circ}{\Gamma}{}_{ij}^k = \frac{1}{2}\left(c_{kj}^i + c_{ki}^j + c_{ij}^k\right), \tag{2.36}$$

where c_{pq}^l can be found from the equalities $[e_p, e_q] = c_{pq}^l e_l$, see [57].

The next property of the Levi-Civitá connection follows from the fact that its torsion tensor equals zero and so the property does not hold for other Riemannian connections.

Let $\gamma(t,s)$ be a smooth mapping from the rectangle $[a,b]\times(c,d)$ into M. Then one can consider the vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ on $\gamma([a,b]\times(c,d))$.

Lemma 2.57 (Lemma on the second covariant derivative)

$$\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s}=\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t}.$$

Proof. By construction, s and t are coordinates on $[a,b]\times(c,d)$. Hence on $\gamma([a,b]\times(c,d))$ the fields $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ commute, i.e., $\left[\frac{\partial}{\partial t},\frac{\partial}{\partial s}\right]=0$ (see Section 1.7). Then $\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s}-\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t}=(\Gamma^k_{ij}-\Gamma^k_{ji})\frac{\partial}{\partial q^k}=\mathsf{T}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial s}\right)$ and the assertion of the Lemma follows from the fact that the torsion tensor T equals zero. □

Lemma 2.57 is an analog of the classical equality $\frac{\partial^2}{\partial x\partial y}=\frac{\partial^2}{\partial y\partial x}$.

For the curvature tensor of the Levi-Civitá connection of a Riemannian or semi-Riemannian metric we consider the following constructions. As above denote by R^i_{jkl} the components of the curvature tensor R. Contract R by the only contravariant and the second covariant indices (see Section 1.5). The result is a tensor Ric called the *Ricci curvature*. Its components take the form $R_{jl}=R^k_{jkl}$. If $\mathsf{R}=0$, it is evident that $\mathrm{Ric}=0$, but not vice versa. Ric is a symmetric $(0,2)$-tensor.

Let $\widehat{\mathrm{Ric}}$ be the $(1,1)$-tensor with components R^l_j that is physically equivalent to Ric. The contraction of $\widehat{\mathrm{Ric}}$, i.e., the scalar $S=R^j_j$, is called the *Gaussian* or *scalar curvature*. If $\mathrm{Ric}=0$, then $S=0$ but not vice versa.

Nevertheless, if $\dim M=2$ the Gaussian curvature determines both the Ricci curvature and the curvature tensor and if $\dim M=3$ the Ricci curvature determines the curvature tensor. If $\dim M\geq 4$ no such determinations are valid.

Definition 2.58. The operator $\nabla^2=\nabla\nabla^*$, where ∇^* is the operator conjugate to the operation of covariant derivation of the Levi-Civitá connection ∇, is called the *Laplace-Beltrami operator.*

In local coordinates of a chart the operator ∇^2 is described by the formula $\nabla^2=g^{ij}\nabla_i\nabla_j=-g^{ij}\Gamma^k_{ij}\frac{\partial}{\partial q^k}+g^{ij}\frac{\partial^2}{\partial q^i\partial q^j}$ where ∇_k is the covariant derivative in the direction of $\frac{\partial}{\partial q^k}$ and g^{ij} are the components of the metric tensor (g^{ij}). From this one can easily see that in a Euclidean space $\mathbb{R}^n$ with standard basis, ∇^2 coincides with the ordinary Laplacian. Note also that the above coordinate representation of ∇^2 defines its action on functions.

In general the Laplace-Beltrami operator does not coincide with the Laplace-de Rham operator $\Delta=\mathrm{d}\delta+\delta\mathrm{d}$ (see Definition 1.73) in spite of the fact that in $\mathbb{R}^n$, modulo the sign, they both give the Laplacian. On functions both operators on all Riemannian manifolds take the same value. In the general case of differential forms (polyvectors) on manifolds the relation between the operators is described by the so-called *Weitzenbök formulae* (a special formula exists for each degree of the form), see [135]. For the material below we need the following Weitzenbök formula for 1-forms (and so for vector fields):

$$\Delta X = -\nabla^2 X + \widehat{\mathrm{Ric}} \circ X. \qquad (2.37)$$

2.7 Connections on Principal Bundles

Let $\underline{G}$ be a principal bundle with fiber (structure group) G over base M. The fiber at $m \in M$ will be denoted G_m. Recall that G_m is homeomorphic to the group G. In the tangent space $T_g\underline{G}$ to $\underline{G}$ at $g \in \underline{G}$, as is the case for any bundle, we can consider the subspace that consists of the vectors tangent to the fiber $G_{\pi g}$. As usual we call this space the *vertical subspace* and denote it by $\mathsf{V}_g \subset T_g\underline{G}$. The vectors in V_g are said to be*vertical.*

The collection of vertical subspaces at all points forms the bundle $\mathsf{V} \to \underline{G}$ with fibers V_g.

Theorem 2.59 *The bundle* $\mathsf{V} \to \underline{G}$ *is trivial.*

Proof. Every point $g \in \underline{G}$ determines a diffeomophism of the group G onto the fiber $G_{\pi g}$ (which will be denoted by the same symbol g) by the formula $g \circ G$, where $\circ$ is the right action of G on $\underline{G}$ (see Section 1.3). It is clear that the diffeomorphism g sends the unit $e \in G$ to the point $g \in \underline{G}$. Since this is a diffeomorphism, the tangent map $Tg : TG \to TG_g$ is a linear isomorphism of $T_eG = \mathfrak{g}$ onto the tangent space to the fiber $G_{\pi g}$ at g, i.e., onto V_g. Thus every vertical subspace V_g is linearly isomorphic to the Lie algebra $\mathfrak{g}$ and the isomorphism smoothly depends on the point $g \in \underline{G}$. Hence, having specified a vector $X \neq 0 \in \mathfrak{g}$, we obtain the smooth vector field $\bar{X}_g = TgX \neq 0$ on $\underline{G}$. In particular, taking vectors of a basis in $\mathfrak{g}$, we obtain a basis at every V_g. So, we have represented $\mathsf{V} \to \underline{G}$ in the form of a direct product $\underline{G} \times \mathfrak{g}$. □

Definition 2.60. The vector field $\bar{X}$ on $\underline{G}$, constructed in the proof of Theorem 2.59 from the vector $X \in \mathfrak{g}$, is called a *fundamental vector field.*

Thus the fundamental vector fields trivialize the bundle $\mathsf{V} \to \underline{G}$ since they determine the frames in the fibers of V_g.

Recall that for a vector bundle Θ we also constructed vertical subspaces $\mathsf{V}_{(m,\vartheta)} \subset T_\vartheta\Theta_m$ that were sent onto the fibers $\Theta_{\pi q}$ by the linear isomorphism $\mathbf{p}$. In the case of a principal bundle an isomorphism like $\mathbf{p}$ does not exist since the fibers of the bundle are not vector spaces. However, Theorem 2.59 provides us with something that was not available in the case of vector bundles: all V_g are in a standard way isomorphic to a unique vector space $\mathfrak{g}$ while in the case of a vector bundle vertical subspaces at the points of Θ_m were sent onto "their own" fiber Θ_m.

Notation 2.61 *The isomorphism* $\mathsf{V}_g \to \mathfrak{g}$*, described above, is denoted by* p.

Definition 2.62. We say that a connection H is given on a principal bundle $\underline{G}$ if to every $T_g\underline{G}$ is associated a subspace H_g that is complementary to V_g, smoothly depends on the point $g \in \underline{G}$ and is such that the connection H is invariant with respect to the right action of the group G on $\underline{G}$ (i.e., for every element h of G and every $g \in \underline{G}$ the equality $TR_h\mathsf{H}_g = \mathsf{H}_{g\circ h}$ holds where $R_h(g) = g \circ h$ is the right action of h on $\underline{G}$). The subspaces H_g comprising a connection are called *horizontal*, as are the vectors belonging to them.

As in the case of a vector bundle, $Y \in T_g\underline{G}$ is uniquely represented as the sum $Y = \mathsf{H}Y + \mathsf{V}Y$ where $\mathsf{H}Y \in \mathsf{H}_g$ and $\mathsf{V}Y \in \mathsf{V}_g$.

Definition 2.63. The mapping $\varphi : T\underline{G} \to \mathfrak{g}$ whose value at $Y \in T\underline{G}$ is given by the formula $\varphi(Y) = \mathsf{p}\mathsf{V}Y$ is called the *connection form* of H .

It is clear that the connection form is a direct analog of the connector (connection map) on vector bundles. It turns out that connection forms have a much richer collection of properties than connectors and their use allows one to obtain much deeper results. We refer the reader, e.g., to [26] and [161] for a more detailed exposition of the theory of general principal bundles and their connection forms. Here we only describe some objects and constructions that are used later.

For every k-form α on $\underline{G}$ with values in $\mathfrak{g}$ the so-called *covariant differential*

$$\mathrm{D}\alpha(\cdot, \dots, \cdot) = \mathrm{d}\alpha(\mathsf{H}\cdot, \dots, \mathsf{H}\cdot) \tag{2.38}$$

is introduced where, as above, the symbol H denotes the projection onto the connection subspace (i.e., H of a vector is the horizontal component of the vector).

Definition 2.64. The 2-form $\Phi = \mathrm{D}\varphi = \mathrm{d}\varphi(\mathsf{H}\cdot, \mathsf{H}\cdot)$ is called the *curvature form* of the connection H.

Since φ takes values in the Lie algebra $\mathfrak{g}$, the composition $[\varphi, \varphi]$ of the operators φ and bracket $[\cdot, \cdot]$ is well-defined.

The so-called *Bianchi identity*

$$\mathrm{D}\Phi = 0 \tag{2.39}$$

and the *structure equation*

$$\mathrm{d}\varphi = -\frac{1}{2}[\varphi, \varphi] + \Phi \tag{2.40}$$

hold (for the proofs see, e.g., [26]). Note that for a matrix group G

$$-\frac{1}{2}[\varphi, \varphi] = -\varphi^2 \tag{2.41}$$

(for details see, e.g., [26, 146]).

As in the case of vector bundles, by construction $T\pi : \mathsf{H}_g \to T_{\pi g}M$ is a linear isomorphism. Via this we also obtain the well-defined notion of a horizontal lift $\tilde{X}$ of a vector field X from the base M onto $\underline{G}$: $\tilde{X}_g = T\pi^{-1}_{|\mathsf{H}_g} X_{\pi g}$. Consider a smooth curve $m(t)$ on the base. The pull-back of the bundle $\underline{G}$ over this curve is a manifold on which the vector field $\tilde{\dot{m}}(t)$, the horizontal lift of the velocity vector field $\dot{m}(t)$ of $m(t)$, is given. Take a point $g_0 \in G_{m(0)}$ and consider the integral curve $g(t)$ of $\tilde{\dot{m}}(t)$ with initial data $g(0) = g_0$.

Definition 2.65. The curve $g(t)$ is called the *parallel translation* of g_0 along the curve $m(t)$.

Let Θ be a bundle with fiber F associated with a principal bundle $\underline{G}$. As on any other bundle, we can consider vertical subspaces in the tangent spaces to Θ, i.e., the subspaces tangent to fibers. The mapping $\lambda : \underline{G} \times F \to \Theta$ (see Notation 1.36) sends horizontal subspaces on $\underline{G}$ into subspaces of tangent spaces to Θ complementary to vertical subspaces. The collection of subspaces that we obtain in this way is called the *connection on* Θ. The parallel translation in the associated bundle Θ is defined by analogy with Definition 2.65.

If G is $GL(k, \mathbb{R})$, or one of its subgroups, with the standard action on $\mathbb{R}^k$, the associated bundle is a vector bundle.

Proposition 2.66 *Every connection on a vector bundle by means of Section 2.2 is an image of some connection on the corresponding principal bundle under the mapping* λ*.*

Now let us consider what is for us the most important case of a principal bundle, the frame bundle BM (see Definition 1.37). Recall that the tangent bundle TM is associated with BM, i.e., by the last statement every connection on the manifold M is obtained from some connection on BM as explained above.

We introduce a connection H on BM by means of Definition 2.62. Consider the bundle $\mathsf{H} \to BM$ whose fiber at every point $b \in BM$ is H_b.

Theorem 2.67 *The bundle* $\mathsf{H} \to BM$ *is trivial.*

Proof. Specify a vector $X \in \mathbb{R}^n$, i.e., a column with coordinates $X^1, \dots, X^n$. Every $b \in BM$, i.e., a frame $b = e_1, \dots, e_n$ in $T_{\pi b}M$, can be considered as a linear mapping $b : \mathbb{R}^n \to T_{\pi b}M$ defined by the formula $bX = X^i e_i$ (see Section 1.3). Denote by $\mathsf{E}_b(X)$ the vector in H_b of the form $\mathsf{E}_b(X) = T\pi^{-1}_{|\mathsf{H}_b} bX$. One can easily see that the mapping $\mathsf{E}_b : \mathbb{R}^n \to \mathsf{H}_b$ is a linear isomorphism and smoothly depends on $b \in BM$. In particular a basis in $\mathbb{R}^n$ determines a corresponding basis in every H_b so that, using coordinate decomposition of vectors of H_b with respect to this basis, we can represent $\mathsf{H} \to BM$ in the form $BM \times \mathbb{R}^n$. □

Definition 2.68. The vector field $\mathsf{E}(X)$ on BM that is equal to $\mathsf{E}_b(X)$ at $b \in BM$ is called the *basic vector field*.

It is clear that basic vector fields are smooth. The basic vector fields trivialize the bundle $\mathsf{H} \to BM$ just as the fundamental vector fields trivialize $\mathsf{V} \to BM$.

Theorem 2.69 *The tangent bundle TBM is trivial.*

This statement is a corollary to Theorem 2.59 and Theorem 2.67. Indeed, by construction, for every $b \in BM$ we have $T_b BM = \mathsf{H}_b \oplus \mathsf{V}_b$, but by the theorems mentioned above the bundles $\mathsf{H} \to BM$ and $\mathsf{V} \to BM$ are trivial.

We introduce a mapping from TBM to $\mathbb{R}^n$ as follows. For $b \in BM$, where $b = (e_1, \ldots, e_n)$ is a basis in $T_{\pi b}M$, consider a vector $X \in T_b BM$. Then $T\pi X \in T_{\pi b}M$ has the coordinate decomposition $T\pi X = \omega^i e_i$. The mapping $X \mapsto (\omega^1, \ldots, \omega^n) \in \mathbb{R}^n$ is considered as a 1-form ω with values in $\mathbb{R}^n$ and is called the *displacement form*. Note that the displacement form ω exists without having to introduce a connection on BM. But if a connection H on BM is specified, we can consider the covariant differential $\mathrm{D}\omega = \mathrm{d}\omega(\mathsf{H}\cdot, \mathsf{H}\cdot)$ (see formula (2.38)) of ω with respect to this connection (cf. Definition 2.64).

Definition 2.70. The 2-form $\Omega = \mathrm{D}\omega$ is called the *torsion form* of H.

The curvature form Φ and the torsion form Ω determine the curvature and torsion tensors, respectively (see details, e.g., in [26]). In addition to (2.40) there is another structure equation for H on BM in terms of ω and Ω:

$$\mathrm{d}\omega = -\varphi\omega + \Omega \tag{2.42}$$

where φ is the connection form. The composition $\varphi\omega$ makes sense since φ is a transformation of $\mathbb{R}^n$ (a matrix from $\mathfrak{gl}(n, \mathbb{R})$) and ω takes values in $\mathbb{R}^n$ (see [26, 146] for details).

At the moment we have two constructions of a parallel vector field along a curve on a manifold: by general Definition 2.28 applied to connections on manifolds (see Section 2.3) and by analogy with Definition 2.65 for the case of associated bundles. Here we describe a third construction.

Let $m(t)$ be a smooth curve on M and $b(t)$ be the parallel translation of a basis $b_0 = b(0)$ in the tangent space $T_{m(0)}M$ along $m(t)$ by means of Definition 2.65. As said above, every basis $b(t)$ is a linear isomorphism $b(t) : \mathbb{R}^n \to T_{m(t)}M$. Let $X_0 \in T_{m(0)}M$ and consider the vector field $X(t) = b(t)(b_0^{-1}X_0) \in T_{m(t)}M$ along $m(t)$. Notice that $X(0) = b_0(b_0^{-1}X_0) = X_0$.

Proposition 2.71 *The vector field $X(t)$ along $m(t)$, introduced above, does not depend on the initial basis b_0 of the parallel translation $b(t)$.*

Proof. Specify another basis $\bar{b}_0$ in $T_{m(0)}M$ and let $\bar{b}(t)$ be the parallel translation of this basis along $m(t)$. It is clear that there exists an $h \in GL(n, \mathbb{R})$ such that $\bar{b}_0 = b_0 \circ h$ where $\circ$ denotes the right action of h on BM. Since by definition a connection H on BM is invariant with respect to the right action of $GL(n, \mathbb{R})$ (see Definition 2.62), one can easily see that $\bar{b}(t) = b(t) \circ h$. Then $\bar{b}(t)(\bar{b}_0^{-1}X_0) = b(t) \circ h((b_0 \circ h)^{-1}X_0) = b(t) \circ h((h^{-1} \circ b_0^{-1})X_0) = X(t)$. □

Thus the formula $X(t) = b(t)(b_0^{-1}X_0)$ uniquely determines the translation of X_0 along $m(t)$. The following statement holds:

Theorem 2.72 *Let a connection on M be obtained from a connection on BM as described above. Then the parallel translation by means of Definition 2.28 applied to connections on manifolds, the parallel translation introduced analogously to Definition 2.65 for the case of associated bundles, and the translation by formula $b(t)(b_0^{-1}X_0)$ coincide.*

Remark 2.73. Let a connection on M be obtained from a connection on BM as described above. It is clear that the geodesics of this connection, and only these geodesics, are projections onto M of integral curves of basic vector fields on BM (see Definition 2.68).

Consider the bundle of orthonormal frames OM on a Riemannian manifold M. This is a principal bundle with a structure group $O(n)$ of orthogonal matrices. If a connection is given on BM, one can consider the spaces H_b of this connection at the points $b \in OM$. However, this collection of subspaces becomes a connection on OM only if it is invariant with respect to the right action of $O(n)$ on OM. In addition, a connection on a Riemannian manifold M is Riemannian if and only if it is obtained from some connection on OM as the image of the mapping λ.

Among the connections on OM there is unique connection with zero torsion form. This connection corresponds to the Levi-Civitá connection M. A detailed description of this material can be found in [26] and [161].

2.8 A Connection on the Total Space of a Vector Bundle

In this section we describe a construction that allows one to create a connection on the total space of a vector bundle (as on a manifold) from a connection of the bundle and a connection on the base (again as on a manifold). A more detailed presentation of this material (at least for the case of a tangent bundle) can be found in [23].

Denote by $\pi : \Theta \to M$ the vector bundle and by Θ_m its fiber at $m \in M$. Let a connection H^π be given on Θ by means of Section 2.2. Denote the connector of this connection by $K^\pi : T\Theta \to \Theta$.

In order to avoid confusion, in this section we denote the projection of a tangent bundle TM on M by $\tau : TM \to M$. Let a connection be given on the manifold M; for this connection we introduce the notation H^τ and denote its connector by $K^\tau : T^2M \to TM$ (recall that according to Section 2.3 a connection on a manifold M is a connection on its tangent bundle TM).

Using connections H^τ and H^π, we construct a connection H^Θ on the total space of Θ (i.e., on the manifold Θ) as follows. We define the connector $K : T^2\Theta \to T\Theta$ of this connection by the formula $K = K^H \oplus K^V$ with $K^H : T^2\Theta \to \mathsf{H}^\pi$ and $K^V : T^2\Theta \to \mathsf{V}$ where V is the vertical subspace at the corresponding point (recall that the fibers of the bundle V over Θ are the subspaces in $T\Theta$ that are tangent to the fibers of Θ, see above). These connectors we define in the form $K^H = \Gamma^\pi \circ K^\tau \circ T^2\pi$ where $\Gamma^\pi = T\pi^{-1}$ is a linear isomorphism of tangent spaces to M onto H^π (see Lemma 2.12) while $K^V = \mathbf{p}^{-1} \circ K^\pi \circ TK^\pi$ where $\mathbf{p} : \mathsf{V}_q \to \Theta_{\pi q}$ is the natural isomorphism of the tangent space V_q to the vector space $\Theta_{\pi q}$ onto the space $\Theta_{\pi q}$ introduced in (1.2) (see also (2.11)).

The covariant derivative on a manifold Θ corresponding to K will be denoted by $\frac{\mathrm{D}}{\mathrm{d}t} = K \circ \frac{\mathrm{d}}{\mathrm{d}t}$. By construction, $\frac{\mathrm{D}}{\mathrm{d}t} = \frac{\mathrm{D}}{\mathrm{d}t}^H + \frac{\mathrm{D}}{\mathrm{d}t}^V$ where $\frac{\mathrm{D}}{\mathrm{d}t}^H = K^H \circ \frac{\mathrm{d}}{\mathrm{d}t}$ and $\frac{\mathrm{D}}{\mathrm{d}t}^V = K^V \circ \frac{\mathrm{d}}{\mathrm{d}t}$. Notice the following important feature: $T\pi\frac{\mathrm{D}}{\mathrm{d}t} = T\pi\frac{\mathrm{D}}{\mathrm{d}t}^H$ and it is equal to the covariant derivative of the connection H^τ on M. From this it follows that for a parallel translation $X(t)$ along a curve $q(t)$ in Θ with respect to the connection H^Θ, the vector field $T\pi X(t)$ along the curve $m(t) = \pi q(t)$ in M is the parallel translation with respect to the connection H^τ. In particular the geodesics of the connection H^Θ on Θ are projected by π onto the geodesics of the connection H^τ on M.

2.9 Second Order Tangent Vectors and Connections

Definition 2.74. A *second order tangent vector* to a manifold M at a point $m \in M$ is a second order differential operator on M at m with zero constant term and a symmetric matrix of coefficients at second order derivatives in local coordinates. The linear space of second order tangent vectors at a point $m \in M$ is called the *second order tangent space* and is denoted by $\tau_m M$.

Usually the fact that the constant term of a second order differential operator $\mathcal{A}$ equals zero is expressed by the condition $\mathcal{A}1 = 0$ where 1 is the function identically equal to unity.

Recall that a vector (i.e., a first order vector) may be considered as a first order differential operator without constant term (the derivative in the direction of a vector, see Section 1.1). By analogy, second order differential operators without constant terms are called *second order tangent vectors*.

The set of all second order tangent vectors has the structure of a fiber bundle with fiber $\tau_m M$ and is called the *second order tangent bundle* τM.

In local coordinates every second order tangent vector $\mathcal{A} \in \tau_m M$ is uniquely represented in the form: $\mathcal{A}x = b^i \frac{\partial}{\partial q^i} + \beta^{ij} \frac{\partial^2}{\partial q^j \partial q^j}$ where the matrix (β^{ij}) is symmetric since $\frac{\partial^2 f}{\partial q^j \partial q^j} = \frac{\partial^2 f}{\partial q^j \partial q^i}$ for a smooth real-valued f. Thus $\frac{\partial}{\partial x^i}$ and $\frac{\partial^2}{\partial x^i \partial x^j}$, $i, j = 1, 2, \ldots, n$ form a basis in $\tau_m M$. The transformation of the components of a second order vector under coordinate changes is described by the formulae (see, e.g., [148])

$$\beta^{i'j'} = \frac{\partial q^{i'}}{\partial q^i}\frac{\partial q^{j'}}{\partial q^j}\beta^{ij},$$
$$b^{k'} = \frac{\partial q^{k'}}{\partial q^k} b^k + \frac{\partial^2 q^{k'}}{\partial q^i \partial q^j}\beta^{ij}. \tag{2.43}$$

From (2.43) it follows that at every $m \in M$ the first order tangent space $T_m M$ is a subspace in $\tau_m M$ consisting of vectors with zero matrix (β^{ij}). However, if this matrix is not zero, the column (b^i) is not a first order tangent vector since it has another transformation rule. On the other hand, by (2.43) the field of matrices (β^{ij}) is a symmetric $(2,0)$-tensor field and it is symmetric in every coordinate system.

There is an analogous construction of second order differential forms.

The theory of second order vectors and differential forms is presented in detail, for example, in [69, 179, 180, 204, 205]. In these works one can also find an interesting approach to stochastic differential equations on manifolds.

At every $m \in M$ there is a canonical isomorphisms between the space $T_m M \odot T_m M$ (where $\odot$ denotes the symmetric tensor product, see Section 1.5) and the quotient space $\tau_m M / T_m M$, and hence between $TM \odot TM$ and $\tau M / TM$ (see [205]). Taking into account this factorization, we construct the morphism $\mathcal{Q} : \tau M \to TM \odot TM$, i.e., the field of linear projectors $\mathcal{Q}_m : \tau_m M \to T_m M \odot T_m M$ such that

$$\mathcal{Q}B(t,m) = \mathcal{Q}\left(b^i \frac{\partial}{\partial q^i} + \beta^{ij}\frac{\partial^2}{\partial q^i \partial q^j}\right) = \beta^{ij}\frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial q^j}. \tag{2.44}$$

Every connection H on M determines a linear operator from $\tau_m M$ to $T_m M$ at any point $m \in M$ as follows:

$$\mathcal{H}\left(b^k \frac{\partial}{\partial q^k} + \beta^{ij}\frac{\partial^2}{\partial q^j \partial q^j}\right) = \left(b^k + \Gamma^k_{ij}\beta^{ij}\right)\frac{\partial}{\partial q^k}, \tag{2.45}$$

where Γ^k_{ij} are the Christoffel symbols of the connection H. Thus connections, and only connections, are smooth cross-sections of the bundle $\mathrm{Hom}(\tau M, TM)$ of fiber-wise linear operators from τM to TM.

Let $m(t) = (q^1(t), \ldots, q^n(t))$ be a smooth curve in a chart $\mathcal{U}$. The second order vector $\mathrm{D}^2 m(t) = \ddot{q}^k \frac{\partial}{\partial q^k} + \dot{q}^i \dot{q}^j \frac{\partial^2}{\partial q^i \partial q^j}$ is called the *acceleration* of $m(t)$.

Proposition 2.75 *For any smooth curve the equality* $\frac{\mathrm{D}}{\mathrm{d}t}\dot{m}(t) = \mathcal{H}\mathrm{D}^2 m(t)$ *holds where* $\frac{\mathrm{D}}{\mathrm{d}t}$ *is the covariant derivative of a connection* H.

Indeed, by formula (2.45) we obtain that $\mathcal{H}\mathrm{D}^2 m(t) = (\ddot{q}^k + \Gamma^k_{ij}\dot{q}^i\dot{q}^j)\frac{\partial}{\partial q^k}$.

Corollary 2.76 *A curve* $m(t)$ *is a geodesic of a connection* H *if and only if* $\mathcal{H}D^2 m(t) = 0$.

Proof. By Proposition 2.75 the equality $\mathcal{H}\mathrm{D}^2 m(t) = 0$ means that for $m(t)$ the geodesic equation (2.25) holds. $\square$

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