Preface

If our sole purpose were to calculate the character table of the finite group $G = \mathrm{SL}_2(\mathbb{F}_q)$ (here, q is a power of a prime number p) by ad hoc methods, this book would only amount to a few pages. Indeed, this problem was solved independently by Jordan [Jor] and Schur [Sch] in 1907. The goal of this book is rather to use the group G to give an introduction to the ordinary and modular representation theory of finite reductive groups, and in particular to Harish-Chandra and Deligne-Lusztig theories. It is addressed in particular to students who would like to delve into Deligne-Lusztig theory with a concrete example at hand. The example of $G = \mathrm{SL}_2(\mathbb{F}_q)$ is sufficiently simple to allow a complete description, and yet sufficiently rich to illustrate some of the most delicate aspects of the theory.

There are a number of excellent texts on Deligne-Lusztig theory (see for example Lusztig [Lu1], Carter [Carter], Digne-Michel [DiMi] for the theory of ordinary characters and Cabanes-Enguehard [CaEn] for modular representations). This book does not aim to offer a better approach, but rather to complement the general theory with an illustrated example. We have tried not to rely upon the above books and give full proofs, in the example of the group G, of certain general theorems of Deligne-Lusztig theory (for example the Mackay formula, character formulas, questions of cuspidality etc.). Although it is not always straightforward, we have tried to give proofs which reflect the spirit of the general theory, rather than giving ad hoc arguments. We hope that this shows how general arguments of Deligne-Lusztig theory may be made concrete in a particular case. At the end of the book we have included a chapter offering a very succinct overview (without proof) of Deligne-Lusztig theory in general, as well as making links to what has already been seen (see Chapter 12).

Historically, the example of $\mathrm{SL}_2(\mathbb{F}_q)$ played a seminal role. In 1974, Drinfeld (at age nineteen!) constructed a Langlands correspondence for $\mathrm{GL}_2(\mathbb{K})$, where $\mathbb{K}$ is a global field of equal characteristic [Dri]. In the course of this work, he remarks that the *cuspidal* characters of $G = \mathrm{SL}_2(\mathbb{F}_q)$ may be found in the first ℓ-adic cohomology group (here, ℓ is a prime number different from p) of the curve $\mathbf{Y}$ with equation $xy^q - yx^q = 1$, on which G acts naturally by linear changes of coordinates (we call $\mathbf{Y}$ the *Drinfeld curve*). This example inspired Deligne and Lusztig (see their comments in [DeLu, page 117, lines 22-24]) who then, in their fundamental article [DeLu], established the basis of what has come to be known as *Deligne-Lusztig theory*.

A large part of this book is concerned with unravelling Drinfeld's example. Our principal is to rather shamelessly make use of the fundamental results of ℓ-adic cohomology (for which we provide an overview tailored to our needs in Appendix A) to construct representations of G in characteristic 0 or ℓ. In order to efficiently use this machinery, we conduct a precise study of the geometric properties of the action of G on the Drinfeld curve $\mathbf{Y}$, with particular attention being paid to the construction of quotients by various finite groups.

Having completed this study we do not limit ourselves to character theory. Indeed, a large part of this book is dedicated to the study of modular representation theory, most notably via the study of Broué's abelian defect group conjecture [Bro]. This conjecture predicts the existence of an equivalence of derived categories when the defect group is abelian. For the representations of the group G in characteristic $\ell \notin \{2, p\}$, the defect group is cyclic, and such an equivalence can be obtained by entirely algebraic methods [Ric1], [Lin], [Rou2]. However, in order to stay true to the spirit of this book, we show that it is possible, when ℓ is odd and divides $q+1$, to realise this equivalence of derived categories using the complex of ℓ-adic cohomology of the Drinfeld curve (this result is due to Rouquier [Rou1]).

For completeness we devote a chapter to the study of representations in *equal*, or *natural*, characteristic. Here the Drinfeld curve ceases to be useful to us. We give an algebraic construction of the simple modules by restriction of rational representations of the group $\mathbf{G} = \mathrm{SL}_2(\overline{\mathbb{F}}_q)$, as may be done for an arbitrary finite reductive group. Moreover, in this case the Sylow p-subgroup is abelian, and it was shown by Okuyama [Oku1], [Oku2] (for the principal block) and Yoshii [Yo] (for the nonprincipal block with full defect) that Broué's conjecture holds. Unfortunately, the proof is too involved to be included in this book.

PREREQUISITES – The reader should have a basic knowledge of the representation theory of finite groups (as contained, for example, in [Ser] or [Isa]). In the appendix we recall the basics of *block theory*. He or she should also have a basic knowledge of algebraic geometry over an algebraically closed field (knowledge of the first chapter of [Har], for example, is more than sufficient). An overview of ℓ-adic cohomology is given in Appendix A, while Appendix C contains some basic facts about reflection groups (this appendix will only be used when we discuss some curiosities connected to the groups $\mathrm{SL}_2(\mathbb{F}_q)$ for $q \in \{3, 5, 7\}$).

We have also added a number of sections and subsections (marked with an asterisk) which contain illustrations, provided by G and $\mathbf{Y}$, of related but more geometric subjects (for example the Hurwitz formula, automorphisms of curves, Abyankhar's conjecture, invariants of reflections groups). These sections require a more sophisticated geometric background and are not necessary for an understanding of the main body of this book.

Lastly, for results concerning derived categories, we will not need more than is contained in the economical and efficient summary in Appendix A1 of the book of Cabanes and Enguehard [CaEn]. The sections and subsections requiring some knowledge of derived categories are also marked with an asterisk.

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