Chapter 2
The Geometry of the Drinfeld Curve

Let $\mathbf{Y}$ be the *Drinfeld curve*

$$\mathbf{Y} = \{(x, y) \in \mathbf{A}^2(\mathbb{F}) \mid xy^q - yx^q = 1\}.$$

It is straightforward to verify that:

- G acts linearly on $\mathbf{A}^2(\mathbb{F})$ (via $g \cdot (x, y) = (ax + by, cx + dy)$ if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$) and stabilises $\mathbf{Y}$;
- μ_{q+1} acts on $\mathbf{A}^2(\mathbb{F})$ by homotheties (via $\xi \cdot (x, y) = (\xi x, \xi y)$ if $\xi \in \mu_{q+1}$) and stabilises $\mathbf{Y}$;
- the Frobenius endomorphism $F \colon \mathbf{A}^2(\mathbb{F}) \to \mathbf{A}^2(\mathbb{F})$, $(x, y) \mapsto (x^q, y^q)$ stabilises $\mathbf{Y}$.

Moreover, if $g \in G$ and $\xi \in \mu_{q+1}$, then, as endomorphisms of $\mathbf{A}^2(\mathbb{F})$ (or $\mathbf{Y}$), we have

$$\begin{aligned} g \circ \xi &= \xi \circ g, \\ g \circ F &= F \circ g, \\ F \circ \xi &= \xi^{-1} \circ F. \end{aligned}$$

We can therefore form the monoid

$$G \times (\mu_{q+1} \rtimes \langle F \rangle_{\mathsf{mon}})$$

which acts on $\mathbf{A}^2(\mathbb{F})$ and stabilises $\mathbf{Y}$.

The purpose of this chapter is to assemble the geometric properties of $\mathbf{Y}$ and the action of $G \times (\mu_{q+1} \rtimes \langle F \rangle_{\mathsf{mon}})$ which allows us to calculate its ℓ-adic cohomology (as a module for the monoid $G \times (\mu_{q+1} \rtimes \langle F \rangle_{\mathsf{mon}})$). A large part of this chapter is dedicated to the construction of quotients of $\mathbf{Y}$ by the actions of the finite groups G, U and μ_{q+1}.

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2.1. Elementary Properties

The following proposition is (almost) immediate.

Proposition 2.1.1. *The curve* $\mathbf{Y}$ *is affine, smooth and irreducible.*

Proof. $\mathbf{Y}$ is affine because it is a closed subspace of the affine space $\mathbf{A}^2(\mathbb{F})$. It is irreducible because the polynomial $XY^q - YX^q - 1$ in $\mathbb{F}[X, Y]$ is irreducible (See Exercise 2.1). It is smooth because the differential of this polynomial is given by the 1×2 matrix $\begin{pmatrix} Y^q & -X^q \end{pmatrix}$, which is zero only at $(0,0) \notin \mathbf{Y}$. □

Proposition 2.1.2. *The group* G *acts freely on* $\mathbf{Y}$.

Proof. Let $g \in G$ and $(x, y) \in \mathbf{Y}$ be such that $g \cdot (x, y) = (x, y)$. It follows that 1 is an eigenvalue of g and, after conjugating g by an element of G, we may assume that there exists an $a \in \mathbb{F}_q$ such that

$$g = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Then $x + ay = x$ and, as $y \neq 0$ (since $(x, y) \in \mathbf{Y}$), we conclude that $a = 0$. □

The next proposition is clear.

Proposition 2.1.3. *The group* μ_{q+1} *acts freely on* $\mathbf{A}^2(\mathbb{F}) \setminus \{(0,0)\}$ *and therefore also on* $\mathbf{Y}$.

Note however, that the group $G \times \mu_{q+1}$ does not act freely on $\mathbf{Y}$: the pair $(-I_2, -1)$ acts as the identity. (Even the quotient $(G \times \mu_{q+1})/\langle(-I_2, -1)\rangle$ does not act freely, see Exercise 2.3.)

2.2. Interesting Quotients

We will now describe the quotients of $\mathbf{Y}$ by the finite groups G, U and μ_{q+1}. In order to construct them we will use the following proposition, a proof of which can be found in [Bor, Proposition 6.6]. (Note that the proposition is far from optimal, but will be sufficient for our needs.)

Proposition 2.2.1. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be two smooth and irreducible varieties,* $\varphi : \mathbf{V} \to \mathbf{W}$ *a morphism of varieties, and* Γ *a finite group acting on* $\mathbf{V}$. *Suppose that the following three properties are satisfied:*

(1) φ *is surjective;*
(2) $\varphi(v) = \varphi(v')$ *if and only if* v *and* v' *are in the same* Γ*-orbit;*
(3) *There exists* $v_0 \in \mathbf{V}$ *such that the differential of* φ *at* v_0 *is surjective.*

Then the morphism $\bar{\varphi} : \mathbf{V}/\Gamma \longrightarrow \mathbf{W}$ *induced by* φ *is an isomorphism of varieties.*

2.2.1. Quotient by G

The map

$$\begin{array}{rccc}\gamma\colon & \mathbf{Y} & \longrightarrow & \mathbf{A}^1(\mathbb{F}) \\ & (x,y) & \longmapsto & xy^{q^2}-yx^{q^2}\end{array}$$

is a morphism of varieties. It is $\mu_{q+1}\rtimes\langle F\rangle_{\mathsf{mon}}$-equivariant (for the action of μ_{q+1} on $\mathbf{A}^1(\mathbb{F})$ given by $\xi\cdot z=\xi^2 z$ and the action of F given by $z\mapsto z^q$). An elementary calculation shows that γ is constant on G-orbits. Even better, if we denote by $\bar{\gamma}\colon \mathbf{Y}/G\to\mathbf{A}^1(\mathbb{F})$ the morphism of varieties obtained by passing to the quotient, we have the following.

Theorem 2.2.2. *The morphism of varieties $\bar{\gamma}\colon \mathbf{Y}/G\to\mathbf{A}^1(\mathbb{F})$ is a $\mu_{q+1}\rtimes\langle F\rangle_{\mathsf{mon}}$-equivariant isomorphism.*

Proof. The $\mu_{q+1}\rtimes\langle F\rangle_{\mathsf{mon}}$-equivariance is evident. In order to show that $\bar{\gamma}$ is an isomorphism we must verify points (1), (2) and (3) of Proposition 2.2.1.

Choose $a\in\mathbb{F}$. To show (1) and (2), it is sufficient to show that $|\gamma^{-1}(a)|=|G|$ (as G acts freely on $\mathbf{Y}$ by Proposition 2.1.2). After changing variables $(z,t)=(x,y/x)$, we have a bijection $\gamma^{-1}(a)\xrightarrow{\sim}\mathscr{E}_a$, where

$$\mathscr{E}_a=\{(z,t)\in\mathbb{F}^\times\times\mathbb{F}^\times\mid t^q-t=\frac{1}{z^{q+1}}\text{ and }t^{q^2}-t=\frac{a}{z^{q^2+1}}\}.$$

As $t^{q^2}-t=(t^q-t)^q+(t^q-t)$, we obtain

$$\mathscr{E}_a=\{(z,t)\in\mathbb{F}^\times\times\mathbb{F}^\times\mid t^q-t=\frac{1}{z^{q+1}}\text{ and }\frac{1}{z^{q+1}}+\frac{1}{z^{q^2+q}}=\frac{a}{z^{q^2+1}}\}.$$

Or equivalently

$$\mathscr{E}_a=\{(z,t)\in\mathbb{F}^\times\times\mathbb{F}^\times\mid z^{q^2-1}-az^{q-1}+1=0\text{ and }t^q-t=\frac{1}{z^{q+1}}\}.$$

The polynomial $z^{q^2-1}-az^{q-1}+1$ is coprime to its derivative, and therefore has q^2-1 distinct non-zero roots. For each of these roots, there are q non-zero solutions t to the equation $t^q-t=\dfrac{1}{z^{q+1}}$. Therefore

$$|\gamma^{-1}(a)|=|\mathscr{E}_a|=(q^2-1)q=|G|,$$

as expected.

We now turn to (3). Let $v=(x_0,y_0)\in\mathbf{Y}$. The tangent space $\mathscr{T}_v(\mathbf{Y})$ to $\mathbf{Y}$ at v has equation $y_0^q x-x_0^q y=0$ and the differential $d_v\gamma\colon \mathscr{T}_v(\mathbf{Y})\to\mathbb{F}=\mathscr{T}_{\gamma(v)}(\mathbf{A}^1(\mathbb{F}))$ is given by

$$d_v\gamma(x,y)=y_0^{q^2}x-x_0^{q^2}y.$$

Therefore, if $(x,y) \in \operatorname{Ker} d_v\gamma$, then

$$y_0^q x - x_0^q y = 0 \quad \text{and} \quad y_0^{q^2} x - x_0^{q^2} y = 0.$$

The determinant of this system is $-y_0^q x_0^{q^2} + x_0^q y_0^{q^2} = (x_0 y_0^q - y_0 x_0^q)^q = 1$, therefore $\operatorname{Ker} d_v\gamma = 0$. □

2.2.2. *Quotient by U*

The morphism

$$\begin{array}{rccc} \upsilon\colon & \mathbf{Y} & \longrightarrow & \mathbf{A}^1(\mathbb{F}) \setminus \{0\} \\ & (x,y) & \longmapsto & y \end{array}$$

is well-defined and is a morphism of varieties. It is $\mu_{q+1} \rtimes \langle F \rangle_{\mathsf{mon}}$-equivariant (for the action of μ_{q+1} on $\mathbf{A}^1(\mathbb{F}) \setminus \{0\}$ given by $\xi \cdot z = \xi z$ and the action of F given by $z \mapsto z^q$). An elementary calculation show that υ is constant on U-orbits. Even better, if we denote by $\bar{\upsilon}\colon \mathbf{Y}/U \to \mathbf{A}^1(\mathbb{F}) \setminus \{0\}$ the morphism of varieties induced by passing to the quotient, we have the following.

Theorem 2.2.3. *The morphism of varieties $\bar{\upsilon}\colon \mathbf{Y}/U \to \mathbf{A}^1(\mathbb{F}) \setminus \{0\}$ is a $\mu_{q+1} \rtimes \langle F \rangle_{\mathsf{mon}}$-equivariant isomorphism.*

Proof. The $\mu_{q+1} \rtimes \langle F \rangle_{\mathsf{mon}}$-equivariance is evident. To show that $\bar{\upsilon}$ is an isomorphism, we verify points (1), (2) and (3) of Proposition 2.2.1.

The surjectivity of υ is clear. We also have

$$\upsilon(x,y) = \upsilon(x',y') \iff \exists\, u \in U,\ (x',y') = u \cdot (x,y).$$

Indeed, if $(x,y) \in \mathbf{Y}$ and $(x',y') \in \mathbf{Y}$ are such that $y = y'$, then

$$\left(\frac{x}{y}\right)^q - \frac{x}{y} = \left(\frac{x'}{y}\right)^q - \frac{x'}{y},$$

which shows that $\dfrac{x'-x}{y} \in \mathbb{F}_q$. Now, if we set $a = \dfrac{x'-x}{y}$, then

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

This shows (2). Point (3) is immediate. □

2.2.3. *Quotient by μ_{q+1}*

The morphism

$$\begin{array}{rccc} \pi\colon & \mathbf{Y} & \longrightarrow & \mathbf{P}^1(\mathbb{F}) \setminus \mathbf{P}^1(\mathbb{F}_q) \\ & (x,y) & \longmapsto & [x:y] \end{array}$$

is well-defined and is $G \times \langle F \rangle_{\mathsf{mon}}$-equivariant morphism of varieties (for the action of G induced by the natural action on $\mathbf{P}^1(\mathbb{F})$ and the action of F given by $[x;y] \mapsto [x^q;y^q]$). An elementary calculation show that π is constant on μ_{q+1}-orbits. Even better, if we denote by $\bar{\pi}\colon \mathbf{Y}/\mu_{q+1} \to \mathbf{P}^1(\mathbb{F}) \setminus \mathbf{P}^1(\mathbb{F}_q)$ the morphism of varieties induced by passage to the quotient, we have the following.

Theorem 2.2.4. *The morphism of varieties $\bar{\pi}\colon \mathbf{Y}/\mu_{q+1} \to \mathbf{P}^1(\mathbb{F}) \setminus \mathbf{P}^1(\mathbb{F}_q)$ is a $G \times \langle F \rangle_{\mathsf{mon}}$-equivariant isomorphism.*

Proof. The $G \times \langle F \rangle_{\mathsf{mon}}$-equivariance is evident. To show that $\bar{\pi}$ is an isomorphism, we should verify points (1), (2) and (3) of Proposition 2.2.1, which is straightforward. □

2.3. Fixed Points under certain Frobenius Endomorphisms

In order to get the most out of the Lefschetz fixed-point theorem (see Theorem A.2.7(a) in Appendix A) we will need the following two results. Firstly, note that, if $\xi \in \mu_{q+1}$, we have

$$\mathbf{Y}^{\xi F} = \varnothing. \tag{2.3.1}$$

Indeed, $(\mathbf{Y}/\mu_{q+1})^F = \varnothing$ by Theorem 2.2.4. On the other hand, we have the following.

Theorem 2.3.2. *Let $\xi \in \mu_{q+1}$. Then*

$$|\mathbf{Y}^{\xi F^2}| = \begin{cases} 0 & \text{if } \xi \neq -1, \\ q^3 - q & \text{if } \xi = -1. \end{cases}$$

Proof. Let $(x,y) \in \mathbf{Y}^{\xi F^2}$. We then have

$$x = \xi x^{q^2}, \quad y = \xi y^{q^2} \quad \text{and} \quad xy^q - yx^q = 1.$$

As a consequence,

$$1 = (xy^q - yx^q)^q = x^q y^{q^2} - y^q x^{q^2} = \xi(x^q y - xy^q) = -\xi.$$

This shows that, if $\xi \neq -1$, then $\mathbf{Y}^{\xi F^2} = \varnothing$.

Therefore suppose that $\xi = -1$. We are looking for the number of solutions to the system

$$\begin{cases} x = -x^{q^2} & (1) \\ xy^q - yx^q = 1 & (2) \\ y = -y^{q^2} & (3) \end{cases}$$

However, if the pair (x, y) satisfies (1) and (2), then it also satisfies (3). Indeed, if (x, y) satisfies (1) and (2), then $x \neq 0$, $y^q = \dfrac{1 + yx^q}{x}$ and therefore

$$y^{q^2} = \Big(\frac{1 + yx^q}{x}\Big)^q = \frac{1 + y^q x^{q^2}}{x^q} = \frac{1 - xy^q}{x^q} = \frac{-yx^q}{x^q} = -y.$$

It follows that it is sufficient to find the number of solutions to the system given by equations (1) and (2). Now, x being non-zero, there are $q^2 - 1$ possibilities for x to be a solution of (1). As soon as we have fixed x, there are q solutions to equation (2) (viewed as an equation in y). Indeed, as an equation in y, $xy^q - yx^q - 1$ has derivative $-x^q \neq 0$, and so this polynomial does not admit multiple roots. This gives therefore $(q^2 - 1)q$ solutions to equations (1) and (2), and the theorem follows. □

REMARK – As G acts freely on $\mathbf{Y}$, the set $\mathbf{Y}^{-F^2}$ consists of a single G-orbit. □

2.4. Compactification

We will denote by $[x; y; z]$ homogeneous coordinates on the projective space $\mathbf{P}^2(\mathbb{F})$. We view $\mathbf{A}^2(\mathbb{F})$ as the open subset of $\mathbf{P}^2(\mathbb{F})$ defined by

$$\mathbf{A}^2(\mathbb{F}) \simeq \{[x; y; z] \in \mathbf{P}^2(\mathbb{F}) \mid z \neq 0\}.$$

We identify $\mathbf{P}^2(\mathbb{F}) \setminus \mathbf{A}^2(\mathbb{F})$ with $\mathbf{P}^1(\mathbb{F})$ (using the canonical isomorphism $[x; y] \mapsto [x; y; 0]$). The action of $G \times (\mu_{q+1} \rtimes \langle F \rangle)$ on $\mathbf{A}^2(\mathbb{F})$ extends uniquely to $\mathbf{P}^2(\mathbb{F})$: if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\xi \in \mu_{q+1}$ and $[x; y; z] \in \mathbf{P}^2(\mathbb{F})$, then

$$g \cdot [x; y; z] = [ax + by; cx + dy; z],$$

$$\xi \cdot [x; y; z] = [\xi x; \xi y; z]$$

and

$$F[x; y; z] = [x^q; y^q; z^q].$$

Now let $\overline{\mathbf{Y}}$ be the projective curve defined by

$$\overline{\mathbf{Y}} = \{[x; y; z] \in \mathbf{P}^2(\mathbb{F}) \mid xy^q - yx^q = z^{q+1}\}.$$

The morphism

$$\begin{array}{ccc} \mathbf{Y} & \longrightarrow & \overline{\mathbf{Y}} \\ (x,y) & \longmapsto & [x;y;1] \end{array}$$

is an open immersion and allows us to identify $\mathbf{Y}$ with $\overline{\mathbf{Y}} \cap \mathbf{A}^2(\mathbb{F})$.

Proposition 2.4.1. *The closed subvariety $\overline{\mathbf{Y}}$ of $\mathbf{P}^2(\mathbb{F})$ is the closure of $\mathbf{Y}$ in $\mathbf{P}^2(\mathbb{F})$. It is smooth and stable under the action of of $G \times (\mu_{q+1} \rtimes \langle F \rangle)$. Moreover,*

$$\overline{\mathbf{Y}} \setminus \mathbf{Y} \simeq \mathbf{P}^1(\mathbb{F}_q),$$

with this isomorphism given by $[x;y] \in \mathbf{P}^1(\mathbb{F}_q) \mapsto [x;y;0]$.

Proof. The only point needing a little work is the smoothness. The points of $\mathbf{Y}$ are smooth by Proposition 2.1.1. As G acts transitively on $\overline{\mathbf{Y}} \setminus \mathbf{Y} = \mathbf{P}^1(\mathbb{F}_q) \simeq G/B$ (as a G-set), it is enough to show that $[1;0;0]$ is a smooth point of $\overline{\mathbf{Y}}$. For this, let us consider the open subvariety defined by $x \neq 0$. In this open set (again isomorphic to $\mathbf{A}^2(\mathbb{F})$, this time via the morphism $(y,z) \mapsto [1;y;z]$) $\overline{\mathbf{Y}}$ is defined by the equation $y - y^q - z^{q+1} = 0$ and the differential at $(0,0)$ of this polynomial is the 1×2 matrix

$$(1 \quad 0),$$

which is non-zero. □

We finish with a study of the quotient of $\overline{\mathbf{Y}}$ by μ_{q+1}. Consider the morphism

$$\begin{array}{rccc} \pi_0 \colon & \overline{\mathbf{Y}} & \longrightarrow & \mathbf{P}^1(\mathbb{F}) \\ & [x;y;z] & \longmapsto & [x;y]. \end{array}$$

It is well-defined, G-equivariant, and surjective. Moreover, it is constant on μ_{q+1}-orbits and therefore induces, after passing to the quotient, a morphism of varieties $\bar{\pi}_0 \colon \overline{\mathbf{Y}}/\mu_{q+1} \to \mathbf{P}^1(\mathbb{F})$.

Theorem 2.4.2. *The morphism of varieties $\bar{\pi}_0 \colon \overline{\mathbf{Y}}/\mu_{q+1} \to \mathbf{P}^1(\mathbb{F})$ is a $G \times \langle F \rangle_{\mathsf{mon}}$-equivariant isomorphism.*

Proof. We omit the proof, as it follows the same arguments as those used in the proof of Theorem 2.2.4. □

2.5. Curiosities*

Independent of representation theory, the Drinfeld curve has interesting geometric properties which we discuss briefly here: it has a "large" automorphism group and gives a solution to a particular case of the *Abhyankar's Conjecture* [Abh] about unramified coverings of the affine line in positive characteristic.

2.5.1. *Hurwitz Formula, Automorphisms**

The group μ_{q+1} acts trivially on $\overline{\mathbf{Y}} \setminus \mathbf{Y} = \mathbf{P}^1(\mathbb{F}_q)$. Also, as μ_{q+1} is of order prime to p, the morphism π_0 is tamely ramified: it is only ramified at the points $a \in \mathbf{P}^1(\mathbb{F}_q)$ and ramification index at a is $e_a = q+1$. If we denote by $\mathbf{g}(\overline{\mathbf{Y}})$ the genus of $\overline{\mathbf{Y}}$, then

$$\mathbf{g}(\overline{\mathbf{Y}}) = \frac{q(q-1)}{2} \tag{2.5.1}$$

as $\overline{\mathbf{Y}}$ is a smooth plane curve of degree $q+1$. Note also that π_0 is a morphism of degree $\deg \pi_0 = q+1$. We can therefore verify the Hurwitz formula [Har, Chapter IV, Corollary 2.4]

$$2\mathbf{g}(\overline{\mathbf{Y}}) - 2 = (\deg \pi_0)(2 \cdot \mathbf{g}(\mathbf{P}^1(\mathbb{F})) - 2) + \sum_{a \in \mathbf{P}^1(\mathbb{F}_q)} (e_a - 1),$$

as $\mathbf{g}(\mathbf{P}^1(\mathbb{F})) = 0$.

We will now extend the group $G \times \mu_{q+1}$ to a bigger group $\mathscr{G}$ still acting on $\mathbf{Y}$ (or $\overline{\mathbf{Y}}$). Set

$$\mathscr{G} = \{(g, \xi) \in \mathsf{GL}_2(\mathbb{F}_q) \times \mathbb{F}_{q^2}^\times \mid \det(g) = \xi^{1+q}\}.$$

It is then straightforward to verify that,

$$\text{if } (g, \xi) \in \mathscr{G} \text{ and } (x, y) \in \mathbf{Y}, \text{ then } g \cdot (\xi x, \xi y) \in \mathbf{Y}. \tag{2.5.2}$$

This defines for us an action of $\mathscr{G}$ on $\mathbf{Y}$ which extends naturally to an action on $\overline{\mathbf{Y}}$. Set

$$\mathscr{D} = \begin{cases} \langle(-\mathsf{I}_2, -1)\rangle & \text{if } q \equiv 3 \mod 4, \\ \langle(\sqrt{-1}\,\mathsf{I}_2, -\sqrt{-1})\rangle & \text{if } q \equiv 1 \mod 4. \end{cases}$$

Then $\mathscr{D}$ is a central subgroup of $\mathscr{G}$ contained in the kernel of the action on $\mathbf{Y}$ (and on $\overline{\mathbf{Y}}$). Even better, we have the following.

Lemma 2.5.3. *The group $\mathscr{G}/\mathscr{D}$ acts faithfully on $\mathbf{Y}$ (and $\overline{\mathbf{Y}}$).*

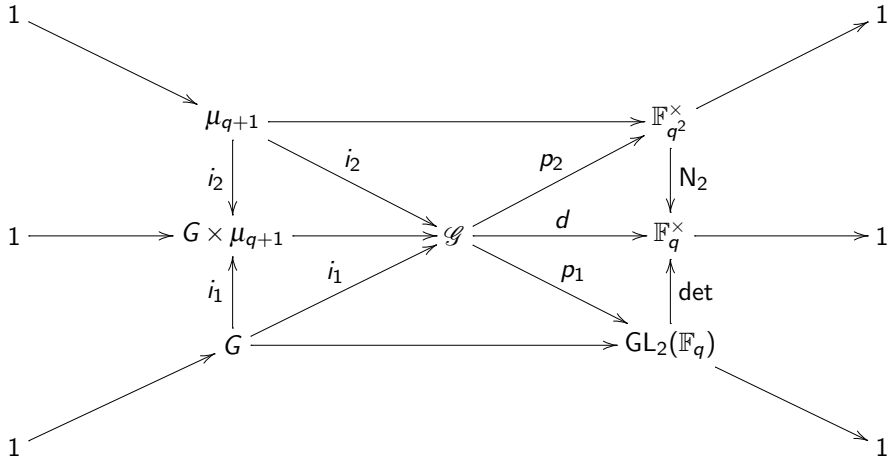
Proof. Let (g, ξ) be an element of $\mathscr{G}$ which acts trivially on $\mathbf{Y}$. Then (g, ξ) acts trivially on $\overline{\mathbf{Y}}$ (as $\mathbf{Y}$ is dense in $\overline{\mathbf{Y}}$) and, after passing to the quotient by $\{1\} \times \mu_{q+1}$ (which is a central subgroup of $\mathscr{G}$), we conclude that g acts trivially on $\mathbf{P}^1(\mathbb{F})$ (by Theorem 2.4.2). Therefore g is a homothety: $g = \lambda \mathsf{I}_2$, with $\lambda \in \mathbb{F}_q^\times$.

Now, if $(x, y) \in \mathbf{Y}$, we have $(g, \xi) \cdot (x, y) = (x, y)$, that is $\lambda\xi = 1$. Therefore $\xi = \lambda^{-1}$. On the other hand, $\det(g) = \xi^{q+1}$, which implies that $\lambda^2 = \xi^{q+1}$ or, in other words, $\lambda^{q+3} = 1$. As $\lambda^{q-1} = 1$, we collude that $\lambda^4 = 1$, which finishes the proof. □

Let $\Delta = \mathscr{D} \cap (G \times \mu_{q+1}) = \langle(-\mathsf{I}_2, -1)\rangle$.

Corollary 2.5.4. *The group* $(G \times \mu_{q+1})/\Delta$ *acts faithfully on* $\mathbf{Y}$.

Denote by $p_1 : \mathscr{G} \to \mathsf{GL}_2(\mathbb{F}_q)$ and $p_2 : \mathscr{G} \to \mathbb{F}_{q^2}^\times$ the canonical projections, and $i_1 : \mu_{q+1} \to \mathscr{G}$, $\xi \mapsto (\mathsf{I}_2, \xi)$ and $i_2 : G \to \mathscr{G}$, $g \mapsto (g, 1)$. The group $G \times \mu_{q+1}$ is contained in $\mathscr{G}$ and we set $d : \mathscr{G} \to \mathbb{F}_q^\times$, $(g, \xi) \mapsto \det(g)$. We have a commutative diagram



in which all straight lines of the form $1 \to \mathscr{X} \to \mathscr{G} \to \mathscr{Y} \to 1$ are exact sequences (which follows essentially from the surjectivity of N_2). In particular,

$$|\mathscr{G}| = q(q^2-1)^2. \tag{2.5.5}$$

It follows from Lemma 2.5.3 that

$$|\operatorname{Aut}\overline{\mathbf{Y}}| \geqslant \begin{cases} \dfrac{q(q^2-1)^2}{2} & \text{if } q \equiv 3 \mod 4, \\[2ex] \dfrac{q(q^2-1)^2}{4} & \text{if } q \equiv 1 \mod 4. \end{cases}$$

In particular, as soon as $q \geqslant 7$, we have, by 2.5.1,

$$|\operatorname{Aut}\overline{\mathbf{Y}}| > 84(\mathbf{g}(\overline{\mathbf{Y}}) - 1) = 42(q-2)(q+1).$$

This illustrates the fact that the "Hurwitz bound" [Har, Chapter IV, Exercise 2.5] is not valid in positive characteristic.

2.5.2. *Abhyankar's Conjecture (Raynaud's Theorem)**

It is not too difficult to show that if a finite group Γ is the Galois group of an unramified covering of the affine line $\mathbf{A}^1(\mathbb{F})$, then Γ is generated by its Sylow p-subgroups. The other implication was conjectured by Abhyankar and shown by Raynaud in a very difficult work [Ray].

Raynaud's theorem (Abhyankar's conjecture). *A finite group Γ is the Galois group of an unramified Galois covering of the affine line $\mathbf{A}^1(\mathbb{F})$ if and only if it is generated by its Sylow p-subgroups.*

EXAMPLE – The morphism $\mathbf{A}^1(\mathbb{F}) \to \mathbf{A}^1(\mathbb{F})$, $x \mapsto x^q - x$ is an unramified Galois covering of $\mathbf{A}^1(\mathbb{F})$ with Galois group $\mathbb{F}_q^+$. □

By Proposition 1.4.1 and Lemma 1.2.2, the group $G = \mathsf{SL}_2(\mathbb{F}_q)$ is generated by its Sylow p-subgroups. By virtue of Raynaud's theorem, G should be the Galois group of an unramified covering of $\mathbf{A}^1(\mathbb{F})$. In fact, in this particular case, the construction of such a covering is easy: the isomorphism $\mathbf{Y}/G \simeq \mathbf{A}^1(\mathbb{F})$ and the fact that G acts freely on $\mathbf{Y}$ (see Proposition 2.1.2) tells us that

(2.5.6) $\mathbf{Y}$ *is an unramified Galois covering of* $\mathbf{A}^1(\mathbb{F})$ *with Galois group* $\mathsf{SL}_2(\mathbb{F}_q)$.

Exercises

2.1. Show that the polynomial $XY^q - YX^q - 1$ in $\mathbb{F}[X, Y]$ is irreducible (***Hint:*** By performing the change of variables $(Z, T) = (X/Y, 1/Y)$ reduce the problem to showing that $T^{q+1} - Z^q - Z$ in $\mathbb{F}[Z, T]$ is irreducible. View this as a polynomial in T with coefficients $\mathbb{F}[Z]$ and use Eisenstein's criterion).

2.2* Let $\mathbb{F}[X, Y]$ a the polynomial ring in two variables, which we identify with the algebra of polynomial functions on $\mathbf{A}^2(\mathbb{F})$. If $g \in G$, $P \in \mathbb{F}[X, Y]$ and $v \in \mathbf{A}^2(\mathbb{F})$, we set $(g \cdot P)(v) = P(g^{-1} \cdot v)$.

(a) Show that this does indeed give an action of G via $\mathbb{F}$-algebra automorphisms.
(b) Show that $XY^{q-1} - X^q$ and Y are algebraically independent and that $\mathbb{F}[X, Y]^U = \mathbb{F}[XY^{q-1} - X^q, Y]$.
(c) Show that $XY^q - YX^q$ divides $XY^{q^2} - YX^{q^2}$.
(d) Show that $D_1 = XY^q - YX^q$ and $D_2 = \dfrac{XY^{q^2} - YX^{q^2}}{XY^q - YX^q}$ are algebraically independent.
(e) Show that $\mathbb{F}[X, Y]^G = \mathbb{F}[D_1, D_2]$ (*Dickson invariants*).
(f) Use this to give another proof of Theorem 2.2.2.

2.3. Denote by Δ the subgroup of $G \times \mu_{q+1}$ generated by $(-I_2, -1)$. The purpose of this exercise is to show that $(G \times \mu_{q+1})/\Delta$ does not act freely on $\mathbf{Y}$. To this end, choose $\xi \in \mu_{q+1} \setminus \{1, -1\}$ and let $v = (x, y) \in \mathbf{A}^2(\mathbb{F})$ be an eigenvector of $\mathbf{d}'(\xi)$ with eigenvalue ξ.

(a) Show that $xy^q - yx^q \neq 0$ (***Hint:*** $xy^q - yx^q = x\prod_{a\in\mathbb{F}_q}(y + ax)$).
(b) Let $\kappa \in \mathbb{F}^\times$ be such that $\kappa^{-1-q} = xy^q - yx^q$. Show that $\kappa v \in \mathbf{Y}$.
(c) Show that $(\mathbf{d}'(\xi), \xi^{-1})$ stabilises $\kappa v \in \mathbf{Y}$.

2.4. † Let $\mathbf{Z} = \{(x, y) \in \mathbf{A}^2(\mathbb{F}) \mid x^{q+1} + y^{q+1} + 1 = 0\}$. We keep the notation F for the restriction to $\mathbf{Z}$ of the Frobenius endomorphism F of $\mathbf{A}^2(\mathbb{F})$. The purpose of this exercise is to construct an isomorphism of $\mathbf{Y}$ and $\mathbf{Z}$ which commutes with F^4.

(a) Show that $\mathbf{Z}^{F^2} \neq \varnothing$. Deduce that there does not exist an isomorphism of varieties $\tau\colon \mathbf{Y} \xrightarrow{\sim} \mathbf{Z}$ such that $\tau \circ F^2 = F^2 \circ \tau$.

Let $z \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and $d \in \mathbb{F}$ be such that $d^{q+1} = -\dfrac{1}{z^q - z}$.

(b) Show that $d \in \mathbb{F}_{q^4}$.
(c) Let $g = \begin{pmatrix} d^q z^q & dz \\ d^q & d \end{pmatrix}$. Show that $g \in \mathrm{GL}_2(\mathbb{F}_{q^4})$ and that $g(\mathbf{Z}) = \mathbf{Y}$.

2.5. Denote by $\tau\colon \mathbf{Y}/U \to \mathbf{Y}/G$ the canonical projection. Set $\tau' = \bar{\gamma} \circ \tau \circ \bar{\upsilon}^{-1}\colon \mathbf{A}^1(\mathbb{F}) \setminus \{0\} \longrightarrow \mathbf{A}^1(\mathbb{F})$, so that the diagram

$$\begin{array}{ccc} \mathbf{Y}/U & \xrightarrow{\ \tau\ } & \mathbf{Y}/G \\ \Big\downarrow{\scriptstyle \bar{\upsilon}} & & \Big\downarrow{\scriptstyle \bar{\gamma}} \\ \mathbf{A}^1(\mathbb{F}) \setminus \{0\} & \xrightarrow{\ \tau'\ } & \mathbf{A}^1(\mathbb{F}) \end{array}$$

commutes. Show that $\tau'(y) = y^{-q}(y^{q^2} + y)$.

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