Preface

Magic Labelings

Magic squares are among the more popular mathematical recreations. Their origins are lost in antiquity; over the years, a number of generalizations have been proposed. In the early 1960s, Sedláček asked whether "magic" ideas could be applied to graphs.

Shortly afterward, Kotzig and Rosa formulated the study of graph labelings or *valuations* as they were first called. A labeling is a mapping whose domain is some set of graph elements—the set of vertices, for example, or the set of all vertices and edges—and whose range was a set of positive integers. Various restrictions can be placed on the mapping. The case that we shall find most interesting is where the domain is the set of all vertices and edges of the graph, and the range consists of the positive integers from 1 up to the number of vertices and edges. No repetitions are allowed.

In particular, one can ask that the set of labels associated with any edge—the label on the edge itself and those on its endpoints—always add to the same sum. Kotzig and Rosa called such a labeling, and the graph possessing it, *magic*. To avoid confusion with the ideas of Sedláček and the many possible variations, we would call it an *edge-magic total labeling*. A related concept, a *vertex-magic total labeling*, is one in which the sum of the label on any vertex and the labels on the edges containing that vertex is always constant. Any labeling that has both these properties (usually with two different constants) is called *totally magic*.

Magic labelings were studied briefly, but they were overshadowed by other graph valuations, in particular by *graceful labelings*, which have a number of applications and are related to the problems of decomposing graphs into trees. However, since the 1990s there has been a resurgence of interest in magic labelings.

One reason for this resurgence is the deceptively simple question, “does every tree have an edge-magic total labeling?” Although it is so easy to ask, no progress has been made toward answering this question for four decades. A number of interesting results about other families of graphs have been discovered, but trees remain elusive.

Several mathematicians have become intrigued by the problem of discovering which graphs are magic. A number of small theorems of combinatorics and graph theory get used in the study of these graphs. For example, in the text, we need to discuss coloring problems and Vizing’s Theorem. The construction of magic arrays other than squares is needed. Small structural graph-theoretic theorems need to be invented. And some of the problems seem to be deeper and more difficult than one would at first expect.

Some applications have been studied, mainly in network-related areas. Suppose it is required to assign addresses to the possible links in a communications network. It is required that the addresses are all different, and that the address of a link can be deduced from the identities of the two nodes linked, without the need of using a lookup table. This has been modeled using edge-magic labelings. Another application is in the construction of *ruler models*, which have been applied to the study of *radar pulse codes*.

However, the main reasons for a monograph studying magic labelings are threefold:

1. Magic labelings provide an introduction to the more general topic of graph labelings. (The topic is large and growing; the interested reader should consult such sources as Gallian’s online survey [31] and the book [8] by Bača and Miller on anti-magic labelings.)

2. A focused book, on one particular problem such as this, is a good guide for graduate students beginning research, so they can see how new mathematics comes into existence. In fact, a draft version of the first edition was used as notes for a graduate “special topics” course. Students see some small graph-theoretic proofs and get some idea of how different areas of graph theory interact (as, for example, when Vizing’s Theorem on edge-chromatic number is used).

3. In recent years a number of researchers have found the topic fascinating; unfortunately, they have not all communicated very well with each other, and we hope this volume will obviate unnecessary repetition of intellectual effort and help unify the notation, which is currently diverse and self-contradictory.

About this Book

The book begins with a survey of the main ingredients. Magic properties are introduced by a discussion of magic squares, also touching on the related Latin squares and on Latin rectangles, and the basics of graph theory are covered briefly. We then define graph labelings in general and magic labelings in particular. The first chapter also includes a brief sketch of applications. Subsequent chapters explore the three main types of magic labelings—edge-magic, vertex-magic, and totally magic—in turn. Finally, magic labelings of directed graphs are discussed.

Throughout the text there are exercises and research problems. The exercises are designed to aid understanding. Some are quite easy; some ask the reader to do a complete search for labelings of a particular graph or labelings of a particular type; a few are quite difficult. Some of the research problems require very little work, but a few are substantial. A brief commentary on the research problems is included in the volume.

There is an extensive bibliography and solutions to the majority of the exercises. The book closes with an index, in which the convention of italicizing the entries where a definition occurs has been followed.

Some knowledge of groups and fields is assumed in the preliminary chapter, in the discussion of magic squares and Latin squares, but these details can be skipped if desired. Most readers will have a background in graph theory, but a summary has been provided. So there are not many mathematical prerequisites. However, the reader is assumed to have some mathematical maturity, to understand proofs, and to use matrices and modular arithmetic with reasonable facility.

The Second Edition

There have been a number of developments in graph labeling over the last decade—for example, the work on directed graphs—and this is why a new edition seemed appropriate. While every new edition endeavors to contain the most comprehensive and current results, we have chosen to focus on the problems that best introduce readers to magic graphs. Some of the research problems from the first edition have been solved, and we have commented on these results and used them in concocting some new exercises. We are not trying to produce an encyclopedia, so some proofs have been omitted; the reader can always consult the original papers.

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