2

Edge-Magic Total Labelings

2.1 Basic Ideas

2.1.1 Definitions

An *edge-magic total labeling* on a graph G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \ldots, v + e$, where $v = |V(G)|$ and $e = |E(G)|$, with the property that, given any edge xy,

$$\lambda(x) + \lambda(xy) + \lambda(y) = k$$

for some constant k. In other words, $\mathrm{wt}(xy) = k$ for any choice of edge xy. Then k is called the *magic sum* of G. Any graph with an edge-magic total labeling will be called *edge-magic*.

As an example of edge-magic total labelings, Fig. 2.1 shows an edge-magic total labeling of $K_4 - e$.

An edge-magic total labeling will be called *(V-)super edge-magic* if it has the property that the vertex-labels are the integers $1, 2, \ldots, v$, the *smallest* possible labels. A graph with a super edge-magic total labeling will be called *super edge-magic*. Recall that some authors may call such a edge-magic labeling *strong* and the resulting graph a *strongly edge-magic graph*.

A.M. Marr and W.D. Wallis, *Magic Graphs*, DOI 10.1007/978-0-8176-8391-7_2,

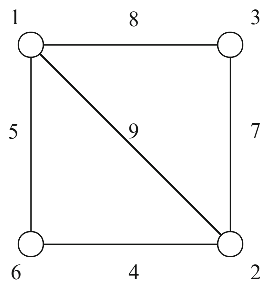


Fig. 2.1. An edge-magic total labeling of $K_4 - e$ with $k = 12$

2.1.2 *Some Elementary Counting*

As a standard notation, assume the graph G has v vertices $\{x_1, x_2, \ldots, x_v\}$ and e edges. For convenience, we always say vertex x_i has degree d_i and receives label a_i. As we shall frequently refer to the sum of consecutive integers, we define

$$\sigma_i^j = (i+1) + (i+2) + \cdots j = i(j-i) + \binom{j-i+1}{2}. \tag{2.1}$$

The necessary conditions in order that $\{a_1, a_2, \ldots, a_v\} = \lambda(V(G))$, where λ is an edge-magic total labeling of a graph G with magic sum k, are

1. $a_h + a_i + a_j = k$ cannot occur if any two of x_h, x_i, x_j are adjacent.
2. The sums $a_i + a_j$, where $x_i x_j$ is an edge, are all distinct.
3. $0 < k - (a_i + a_j) \leq v + e$ when x_i is adjacent to x_j.

Suppose λ is an edge-magic labeling of a given graph with sum k. If x and y are adjacent vertices, then edge xy has label $k - \lambda(x) - \lambda(y)$. Since the sum of all these labels plus the sum of all the vertex labels must equal the sum of the first $v + e$ positive integers, k is determined. So the vertex labels specify the complete labeling.

Of course, not every possible assignment will result in an edge-magic labeling: the above process may give a non-integral value for k, or give repeated labels.

Among the labels, write S for the set $\{a_i : 1 \leq i \leq v\}$ of vertex labels, and s for the sum of elements of S. Then S can consist of the v smallest labels, the v largest labels, or somewhere in between, so

$$\sigma_0^v \leq s \leq \sigma_e^{v+e},$$

$$\binom{v+1}{2} \leq s \leq ve + \binom{v+1}{2}. \tag{2.2}$$

Clearly, $\sum_{xy \in E}(\lambda(xy) + \lambda(x) + \lambda(y)) = ek$. This sum contains each label once, and each vertex label a_i an additional $d_i - 1$ times. So

$$ke = \sigma_0^{v+e} + \sum (d_i - 1)a_i. \tag{2.3}$$

If e is even, every d_i is odd and $v + e \equiv 2(\bmod 4)$ then (2.3) is impossible. We have

Theorem 2.1 [82] *If G has e even and $v + e \equiv 2(\bmod 4)$, and every vertex of G has odd degree, then G has no edge-magic total labeling.*

(A generalization of this Theorem will be proven in Sect. 2.2).

Corollary 2.1.1 *The complete graph K_v is not magic when $v \equiv 4(\bmod 8)$. The n-spoke wheel W_n, formed from C_n by adding a new vertex and joining it to every existing vertex, is not magic when $n \equiv 3(\bmod 4)$.*

(We shall see in Sect. 2.3.2 that K_n is never magic for $n > 6$, so the first part of the Corollary really only eliminates K_4.)

If G is super edge-magic, then S must consist of the first v integers. In many cases equation (2.3) can be used to show that this is impossible, because there is no assignment of these integers to the vertices such that $\sigma_0^{v+e} + \sum (d_i - 1)a_i$ is divisible by e. For example, for the cycle C_v where v is even, (2.3) is

$$kv = \sigma_0^{2v} + \sigma_0^{v} = v(2v + 1) + \tfrac{1}{2}v(v + 1),$$

so $k = 2v + 1 + \frac{1}{2}(v + 1)$, which is not integral. Thus no even cycle is super edge-magic.

Exercise 2.1 *Prove that the graph tK_4, consisting of t disjoint copies of K_4, has no edge-magic total labeling when t is odd. Is the same true of tK_v when $v \equiv 4(\bmod 8)$?*

Exercise 2.2 *Prove that the union of t n-wheels, tW_n, has no edge-magic total labeling when t is odd and $n \equiv 3(\bmod 4)$.*

Research Problem 2.1 *Investigate graphs G for which equation (2.3) implies the nonexistence of an edge-magic total labeling of $2G$.*

Equation (2.3) may be used to provide bounds on k. Suppose G has v_j vertices of degree j, for each i up to Δ, the largest degree represented in G. Then ke cannot be smaller than the sum obtained by applying the v_Δ smallest labels to the vertices of degree Δ, the next-smallest values to the vertices of degree $\Delta - 1$, and so on; in other words,

$$ke \geq (d_\Delta - 1)\sigma_0^{v_\Delta} + (d_{\Delta-1} - 1)\sigma_{v_\Delta}^{v_\Delta+(v_{\Delta-1})} + \cdots \\ +\sigma_{v_\Delta+(v_{\Delta-1})+\cdots+v_3}^{v_\Delta+(v_{\Delta-1})+\cdots+v_2} + \binom{v+e+1}{2}.$$

An upper bound is achieved by giving the *largest* labels to the vertices of highest degree, and so on.

Exercise 2.3 *Suppose a regular graph G of degree d is edge-magic. Prove*

$$ke = (d-1)s + \sigma_0^{v+e} = (d-1)s + \tfrac{1}{2}(v+e)(v+e+1), \tag{2.4}$$

$$kdv = 2(d-1)s + (v+e)(v+e+1). \tag{2.5}$$

Avadayappan, Vasuki and Jeyanthi [5] define the *magic strength* of an edge-magic graph G to be the smallest value k such that G has a magic labeling with magic sum k. From equations (2.4) and (2.5) we deduce that for an edge-magic regular graph

$$ke \leq \left(\frac{(d-1)v(v+1) + (v+e)(v+e+1)}{2}\right),$$

so

Theorem 2.2 *An edge-magic regular graph of degree d, with v vertices and e edges, has magic sum at least*

$$\left\lceil \frac{(d-1)v(v+1) + (v+e)(v+e+1)}{2e} \right\rceil. \tag{2.6}$$

Exercise 2.4 *Prove that the magic strength of a cycle C_n is at least* $\lceil \frac{1}{2}(5v+3) \rceil$.

2.1.3 Duality

Given a labeling λ, its *dual* labeling λ' is defined by

$$\lambda'(x_i) = (v+e+1) - \lambda(x_i),$$

and for any edge xy,

$$\lambda'(xy) = (v+e+1) - \lambda(xy).$$

It is easy to see that if λ is a magic labeling with magic sum k, then λ' is a magic labeling with magic sum $k' = 3(v+e+1) - k$. The sum of vertex labels is $s' = v(v+e+1) - s$.

Either s or s' will be less than or equal to $\frac{1}{2}v(v+e+1)$. This means that, in order to see whether a given graph has an edge-magic total labeling, it suffices to check either all cases with $s \leq \frac{1}{2}v(v+e+1)$ or all cases with $s \geq \frac{1}{2}v(v+e+1)$ (equivalently, check either all cases with $k \leq \frac{3}{2}(v+e+1)$ or all with $k \geq \frac{3}{2}(v+e+1)$).

2.2 Graphs with no Edge-Magic Total Labeling

The results of this section are from the unpublished technical report [51].

2.2.1 Main Theorem

Theorem 2.3 *Suppose G is a graph with v vertices and e edges, where e is even, and suppose every vertex of G has odd degree. Select a positive integer δ such that for each vertex x_i*

$$d(x_i) = 2^\delta d_i + 1$$

for some nonnegative integer d_i. If $T = \sum d_i$, define τ and Q by

$$T = 2^\tau \cdot Q,\ \ \tau \text{ integral, } Q \text{ odd.}$$

If G has an edge-magic total labeling λ, then

$$\begin{aligned}
\tau = 0 \Rightarrow v &= 2^\delta V \text{ for some } V \equiv 1 \bmod 2\\
e &= 2^\delta E \text{ for some } E \equiv 1 \bmod 2\\
\tau = 1 \Rightarrow v &= 2^{\delta+1} V' \text{ for some } V' \equiv 1 \bmod 2\\
e &= 2^{\delta+1} E' \text{ for some } E' \equiv 1 \bmod 2\\
\tau \geq 2 \Rightarrow &\ 2^{\delta+2} \text{ divides } v \text{ and } 2^{\delta+2} \text{ divides } e
\end{aligned}$$

Proof. Since e is even, let us write $e = 2^\nu E$ for some odd integer E. The familiar equation $2e = \sum d(x_i)$ yields

$$2e = v + 2^\delta T,$$

so

$$v + e = 3e - 2^{\delta+\tau} \cdot Q = 2^\nu \cdot 3E - 2^{\delta+\tau} \cdot Q.$$

So (2.3) is

$$(2^\nu \cdot 3E - 2^{\delta+\tau} \cdot Q)(2^\nu \cdot 3E - 2^{\delta+\tau} \cdot Q + 1)/2 = 2^\nu \cdot Ek - 2^\delta \cdot \sum d_i \lambda(x_i),$$

which is of the form

$$2^\delta \cdot (2R - 2^\tau \cdot X) = 2^\nu \cdot Y$$

where $R = \sum d_i \lambda(x_i)$, $X = (2^{\delta+\tau} Q - 3 \cdot 2^{\nu+1} - 1)Q$ and $Y = 2Ek - 2^\nu \cdot 9E^2 - 3E$. The actual values of X and Y are unimportant; what matters is that both are odd, and the result follows. □

Corollary 2.3.1 *Suppose $e(G)$ is even and $e(G) + v(G) \equiv 2(\bmod 4)$, and suppose each vertex of G has odd degree. Then G is not edge-magic.*

2.2.2 Forests

Theorem 2.4 *Suppose G is a forest with c component trees and suppose G satisfies the conditions of Theorem 2.3. Then*

$$\begin{aligned}
\tau = 0 &\Rightarrow c \equiv 0 \bmod 2^{\delta+1} \text{ and } e \equiv 2^{\delta} \bmod 2^{\delta+1}\\
\tau = 1 &\Rightarrow c \equiv 0 \bmod 2^{\delta+2} \text{ and } e \equiv 2^{\delta+1} \bmod 2^{\delta+2}\\
&\quad\ \text{or } c \equiv 2^{\delta+1} \bmod 2^{\delta+2} \text{ and } e \equiv 0 \bmod 2^{\delta+2}\\
\tau \geq 2 &\Rightarrow c \equiv e \equiv x \bmod 2^{\delta+2} \text{ where } x = 0 \text{ or } 2^{\delta+1}.
\end{aligned}$$

Proof. For a forest, $v = c + e$; the results follow from Theorem 2.3. □

A family $\mathcal{F}$ of forests is defined as follows. If δ is any positive integer, a δ-tree is a tree in which each vertex x_i has degree $d(x_i) = 1+2^{\delta}d_i$ for some nonnegative integer d_i, and a δ-forest is a forest *with an even number of edges* in which every component tree is a δ-tree. Figure 2.2 shows a 2-forest—the vertices have degrees 1, 5 ($= 2^2 + 1$) and 9 ($= 2^3 + 1$).

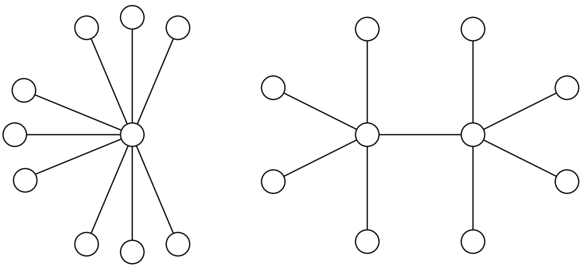


Fig. 2.2. A 2-forest

If F is a δ-forest define $\tau(F) = \tau$ to be the nonnegative integer such that $\sum_i d_i = 2^{\tau} \cdot T$ where T is odd, then $\mathcal{F}_{\delta}$ is the set of all δ-forests F that satisfy one of the following conditions, where e is the total number of edges and c is the number of component trees of F:

(i) $\tau(F) = 0$ and $c \not\equiv 0 \bmod 2^{\delta+1}$ or $e \not\equiv 0 \bmod 2^{\delta}$;

(ii) $\tau(F) = 1$ and $c \not\equiv 0 \bmod 2^{\delta+1}$ or $e \not\equiv 0 \bmod 2^{\delta+1}$ or $c + e \not\equiv 2^{\delta+1} \bmod 2^{\delta+2}$;

(iii) $\tau(F) \geq 2$ and $c \not\equiv e \bmod 2^{\delta+2}$ or $c \not\equiv 0 \bmod 2^{\delta+1}$.

Finally, $\mathcal{F}$ is the union of all the sets $\mathcal{F}_{\delta}$.

Theorem 2.5 *For every nonnegative integer τ there are infinitely many forests F belonging to $\mathcal{F}$ that have $\tau(F) = \tau$.*

Proof. Select natural numbers p, q and t such that $p \geq q$. F_{pqt} is the disjoint union of $(2t+1)\cdot 2^q$ copies of the star with 2^p+1 edges. This forest has $c=(2t+1)\cdot 2^q$ components. Vertex x_i of F_{pqt} has degree $2^p d_i + 1$, where d_i = 0 or 1 and $\sum d_i = (2t+1)\cdot 2^q$. In terms of the definition of $\mathcal{F}$, $\delta = p$ and $\tau = q$. Clearly c is not divisible by 2^{p+1}, since the largest power of 2 dividing c is 2^q, so $F_{pqt} \in \mathcal{F}$. So for any natural number τ we have constructed infinitely many members of $\mathcal{F}$ with that τ-value.

For $\tau = 0$, consider the graph F_n consisting of the disjoint union of a 5-star with $(4n-3)$ 3-stars. One vertex has degree $2^1{\cdot}2+1$ ($d_i = 2$), $4n-3$ vertices have degree $2^1{\cdot}1+1$ ($d_i = 1$), and the other vertices have degree $2^1{\cdot}0+1$ ($d_i = 0$). So $\delta = 1$ and $\tau = 0$. Moreover $c = 4x-2$ is not divisible by $4 = 2^{\delta+1}$, so $F_x \in \mathcal{F}$, and we have constructed infinitely many members of $\mathcal{F}$ that have $\tau = 0$. $\square$

It is clear from Theorem 2.4 that a member of $\mathcal{F}$ cannot have an edge-magic total labeling. So we have:

Corollary 2.5.1 *There are infinitely many forests with no edge-magic total labeling.*

2.2.3 Regular Graphs

Theorem 2.6 *Suppose G is a regular graph of odd degree d with v vertices and e edges, where e is even. Write*

$$d = 2^r s + 1$$

where s is an odd positive integer and r is a positive integer. If G is edge-magic, then 2^{r+2} divides v.

Proof. As usual, $2e = dv$. Since e is even and d is odd, 4 divides v; write $v = 4V$, so that $e = 2dV$. Then (2.3) becomes

$$(4V + 2dV)(4V + 2dV + 1)/2 = 2dVk - (d-1)\sum_x \lambda(x),$$

and

$$V[(d+2)(4V + 2dV + 1) - 2dk] = 2^r s \sum_x \lambda(x).$$

The coefficient of V on the left-hand side is odd, so 2^r divides V, giving the result. $\square$

Remark. This theorem could also be expressed as follows. If a regular graph with v vertices is edge-magic, then

$$\begin{array}{ll} d \equiv 3 \pmod 4 & \Rightarrow v \equiv 2 \pmod 4 \text{ or } v \equiv 0 \pmod 8 \\ d \equiv 5 \pmod 8 & \Rightarrow v \equiv 2 \pmod 4 \text{ or } v \equiv 0 \pmod{16} \\ & \dots \\ d \equiv 2^{p-1} + 1 \pmod{2^p} & \Rightarrow v \equiv 2 \pmod 4 \text{ or } v \equiv 0 \pmod{2^{p+1}}. \end{array}$$

A family $\mathcal{R}$ of regular graphs is now defined as follows. If G is regular of degree d and has v vertices, then $G \in \mathcal{R}$ if and only if there exist natural numbers α, β, γ and δ satisfying

$$\begin{aligned} &\delta < 2^\alpha, \\ &d = 2^{\alpha+1}\beta - 2^\alpha + 1, \\ &v = 4(2^\alpha \cdot \gamma - \delta). \end{aligned}$$

Theorem 2.7 *No member of $\mathcal{R}$ is edge-magic.*

Proof. Suppose α, β, γ, and δ are natural numbers and $\delta < 2^\alpha$. Consider a regular graph G of degree $d = 2^{\alpha+1}\beta - 2^\alpha + 1$ with $v = 4(2^\alpha \cdot \gamma - \delta)$ vertices (a typical element of $\mathcal{R}$). Writing $r = \alpha$ and $s = 2\beta - 1$ we have $d = 2^r s + 1$ so by Theorem 2.6 G can have an edge-magic total labeling only if v is divisible by $2^{r+2} = 2^{\alpha+2}$. However $v \equiv -4\delta \not\equiv 0 \pmod{2^{\alpha+2}}$ as $0 < \delta < 2^\alpha$. So G has no edge-magic total labeling. □

Lemma 2.8 *If d is any odd natural number greater than 1, there are infinitely many graphs of degree d that belong to $\mathcal{R}$.*

Remark. Lemma 3 of [51] allows the case $d = 1$. The first line of the proof there is "For the case $d = 1$, lemma 4 (*sic*) follows immediately from Theorem 3." (His Theorem 3 is our Theorem 2.6.) However, if $d = 1$, the definition of $\mathcal{R}$ would require $2^{\alpha+1}\beta = 2^\alpha$, which is impossible for natural β.

Proof of Lemma 2.8. We construct solutions in the special case $\delta = 1$ (so that $\delta < 2^\alpha$ is obviously true).

Assume d is an odd positive integer greater than 1. There exist unique integers α and b such that b is odd and $d = 2^\alpha b + 1$. Define $\beta = (b + 1)/2$. Then $d = 2^{\alpha+1}\beta - 2^\alpha + 1$.

If d is any natural number and v is any even integer greater than d, there exists a regular graph of degree d on v vertices (select any one-factorization of K_v and delete any $v - d - 1$ factors). If d is odd, then among the even integers greater than d, there will be infinitely many of the form $4(2^\alpha \cdot \gamma - 1)$ where α was defined above, and each gives rise to a member of $\mathcal{R}$. □

Corollary 2.8.1 *If d is any odd natural number, there are infinitely many regular graphs of degree d that have no edge-magic total labeling.*

Proof. For $d = 1$, the graphs eK_2, e even, suffice. For larger values of d, use Lemma 2.8. □

2.3 Cliques and Complete Graphs

2.3.1 Sidon Sequences and Labelings

Suppose the graph G has an edge-magic total labeling λ, and suppose G contains a complete subgraph (or *clique*) H on n vertices. Let us write $x_1, x_2, \ldots, x_n$ for the vertices of H, and denote $\lambda(x_i)$ by a_i. Without loss of generality we can assume the names x_i to have been chosen so that $a_1 < a_2 < \cdots < a_n$.

If k is the magic sum, then $\lambda(x_i x_j) = k - a_i - a_j$, so the sums $a_i + a_j$ must all be distinct. This property is called being *well-spread*; in particular, a *Sidon sequence* or *well-spread sequence* $A = (a_1, a_2, \ldots, a_n)$ of length n is a sequence with the properties:

1. $0 < a_1 < a_2 < \cdots < a_n$.
2. If a_i, a_j, a_k, a_ℓ are all different, then $a_i + a_j \neq a_k + a_\ell$.

Such sequences are related to the work of Sidon [84]. Their study was initiated by Erdös and Turan [27]. A survey of work on them appears in [41].

In discussing Sidon sequences (or, equivalently, cliques in edge-magic graphs), the difference $d_{ij} = |a_j - a_i|$ (the absolute difference between the labels on the endpoints of the edge $x_i x_j$) will be important.

Lemma 2.9 *Suppose A is a Sidon sequence of length n. If $d_{ij} = d_{pq}$, then $\{a_i, a_j\}$ and $\{a_p, a_q\}$ have a common member. No three of the differences d_{ij} are equal.*

Proof. Suppose $d_{ij} = d_{pq}$. We can assume that $i > j$ and $p > q$. If $p = i$, we get $a_i = a_p$ and $a_j = a_q$ and the result is trivial. Thus, without loss of generality we can also assume $p > i$. Then $a_i - a_j = a_p - a_q$, so $a_i + a_q = a_j + a_p$. Therefore $a_i = a_q$ and $i = q$ ($a_j = a_p$ is impossible), and $p > i > j$—the common element is the middle one in order of magnitude.

Now suppose three pairs have the same difference. By the above reasoning there are two possibilities: the pairs must have a common element, or form a triangle. In the former case, suppose the differences are d_{ij}, d_{ik} and $d_{i\ell}$.

From $d_{ij} = d_{ik}$ we must have either $k > i > j$ or $j > i > k$; let's assume the former. Then $d_{ij} = d_{i\ell}$ implies $i > j > \ell$. So j is greater than both k and ℓ. But $d_{ik} = d_{i\ell}$ must mean that either $k > j > \ell$ or $\ell > j > k$, both of which are impossible. On the other hand, suppose they form a triangle, say $d_{ij} = d_{ik} = d_{jk}$. We can assume $i > j$. Then $d_{ij} = d_{jk}$ implies $i > j > k$, and $j > k$ and $d_{ik} = d_{jk}$ imply $k > i$, again a contradiction. $\square$

Lemma 2.10 *Suppose A is a Sidon sequence of length n. If $d_{ij} = d_{ik}$, then $d_{ij} \leq \frac{1}{2}d_{1n}$.*

Proof. Suppose $d_{ij} = d_{ik}$, and assume $j < k$. Then $d_{jk} = a_k - a_j = a_i - a_j + a_k - a_i = d_{ij} + d_{ik} = 2d_{ij}$. But $a_1 \leq a_j$ and $a_k \leq a_n$, so $d_{jk} \leq d_{1n}$, giving the result. $\square$

Theorem 2.11 *In any Sidon sequence of length n, $\binom{n}{2} \leq \lfloor \frac{3}{2}d_{1n} \rfloor$, or equivalently $d_{1n} \geq \lceil \frac{1}{3}n(n-1) \rceil$.*

Proof. There are $\binom{n}{2}$ unordered pairs of elements in the sequence, so there are $\binom{n}{2}$ differences. From Lemmas 2.9 and 2.10, the collection of values of these differences can contain the integers $1, 2, \ldots, \lfloor \frac{1}{2}r \rfloor$ at most twice each, and $\lfloor \frac{1}{2}r \rfloor + 1, \ldots, d_{1n}$ at most once each. The result follows. $\square$

2.3.2 Complete Graphs

Theorem 2.12 [90] *Suppose K_v has an edge-magic total labeling with magic sum k. The number p of vertices that receive even labels satisfies the following conditions:*

(i) *If $v \equiv 0$ or $3 \pmod 4$ and k is even, then $p = \frac{1}{2}(v - 1 \pm \sqrt{v+1})$.*

(ii) *If $v \equiv 1$ or $2 \pmod 4$ and k is even, then $p = \frac{1}{2}(v - 1 \pm \sqrt{v-1})$.*

(iii) *If $v \equiv 0$ or $3 \pmod 4$ and k is odd, then $p = \frac{1}{2}(v + 1 \pm \sqrt{v+1})$.*

(iv) *If $v \equiv 1$ or $2 \pmod 4$ and k is odd, then $p = \frac{1}{2}(v + 1 \pm \sqrt{v+3})$.*

Proof. Suppose λ is an edge-magic total labeling of K_v with magic sum k. Let V_{e} denote the set of all vertices x such that $\lambda(x)$ is even, and V_{o} the set of vertices x with $\lambda(x)$ odd; define p to be the number of elements of V_{e}. Write E_1 for the set of edges with both endpoints in the same set, either V_{o} or V_{e}, and E_2 for the set of edges joining the two vertex-sets, so that $|E_1| = \binom{p}{2} + \binom{v-p}{2}$ and $|E_2| = p(v-p)$.

If k is even, then $\lambda(yz)$ is even whenever yz is an edge in E_1 and odd when yz is in E_2, so there are precisely $p + \binom{p}{2} + \binom{v-p}{2}$ even labels. But these labels must be the even integers from 1 to $\binom{v+1}{2}$, taken once each; so

$$p + \binom{p}{2} + \binom{v-p}{2} = \left\lfloor \frac{1}{2}\binom{v+1}{2} \right\rfloor. \tag{2.7}$$

If $\binom{v+1}{2}$ is even, this equation has solutions $p = \frac{1}{2}(v - 1 \pm \sqrt{v+1})$, while $\binom{v+1}{2}$ odd gives solutions $p = \frac{1}{2}(v - 1 \pm \sqrt{v-1})$.

If k is odd, the edges in E_1 are those that receive the odd labels, and instead of (2.7) we have

$$p + p(v-p) = \left\lfloor \frac{1}{2}\binom{v+1}{2} \right\rfloor \tag{2.8}$$

with solutions $p = \frac{1}{2}(v + 1 \pm \sqrt{v+1})$ when $\binom{v+1}{2}$ is even and $p = \frac{1}{2}(v + 1 \pm \sqrt{v+3})$ when $\binom{v+1}{2}$ is odd.

Using the fact that $\binom{v+1}{2}$ is even when $v \equiv 0$ or $3 (\bmod 4)$ and odd otherwise, we have the result. □

Now p must be an integer, so the functions whose roots are taken must always be perfect squares. Therefore:

Corollary 2.12.1 *Suppose K_v has an edge-magic total labeling. If $v \equiv 0$ or $3(\bmod 4)$, then $v+1$ is a perfect square. If $v \equiv 1$ or $2(\bmod 4)$, then either $v-1$ is a perfect square and the magic sum of the labeling is even, or $v+3$ is a perfect square and the magic sum of the labeling is odd.*

This corollary rules out edge-magic total labelings of K_4 (again!—see Corollary 2.1.1) and K_7, as well as infinitely many larger values. The larger values, however, will all be excluded by the next theorem.

Suppose there is an edge-magic total labeling of K_v, where $v \geq 8$. The vertex labels will form a Sidon sequence of length v, A say. Let us denote the edge labels $b_1, b_2, \ldots, b_e$, where $b_1 < b_2 < \cdots < b_e$; of course, $e = \binom{v}{2}$. If the magic sum is k, then

$$\begin{aligned} k &= a_1 + a_2 + b_e &(2.9)\\ &= a_1 + a_3 + b_{e-1} &(2.10)\\ &= a_v + a_{v-1} + b_1 &(2.11)\\ &= a_v + a_{v-2} + b_2. &(2.12) \end{aligned}$$

Subtracting (2.9) from (2.10),

$$a_3 - a_2 = b_e - b_{e-1}, \tag{2.13}$$

while (2.11) and (2.12) yield

$$a_{v-1} - a_{v-2} = b_2 - b_1. \tag{2.14}$$

Suppose labels $1, 2, v+e-1$ and $v+e$ were all edge labels. Then $b_1 = 1, b_2 = 2, b_{e-1} = v+e-1$ and $b_e = v+e$. So, from (2.13) and (2.14), $a_3 - a_2 = a_{v-1} - a_{v-2} = 1$. But 2, 3, $v-2$ and $v-1$ are all distinct, so this contradicts Lemma 2.9. So one of $1, 2, v+e-1, v+e$ is a vertex label. Without loss of generality we can assume either 1 or 2 is a vertex label (otherwise, the dual labeling will have this property). So $a_1 = 1$ or 2.

Equations (2.9) and (2.11) give

$$a_v = b_e - (a_{v-1} - a_2) - (b_1 - a_1). \tag{2.15}$$

Since $(a_2, a_3, \ldots, a_{v-1})$ is a Sidon sequence of length $v-2$ (any subsequence of a Sidon sequence is also well-spread), Lemma 2.11 applies to it, and $(a_{v-1} - a_2) \geq \lceil \frac{1}{3}(v-2)(v-3) \rceil$, which is at least 10 because $v \geq 8$. Also $(b_1 - a_1) \geq -1$ ($b-1$ is at least 1 and a_1 is at most 2), and $b_e \leq v+e$. So, from (2.15),

$$a_v \leq v+e-9.$$

So the six largest labels are all edge labels:

$$b_{e-5} = v+e-5, b_{e-4} = v+e-4, \ldots, b_e = v+e.$$

From (2.9) and (2.10), we get

$$k = a_1 + a_2 + v + e = a_1 + a_3 + v + e - 1,$$

so $a_3 = a_2 + 1$. The next smallest sum of two vertex-labels, after $a_1 + a_2$ and $a_1 + a_3$, may be either $a_2 + a_3$ or $a_1 + a_4$.

If it is $a_2 + a_3$, then

$$k = a_2 + a_3 + v + e - 2$$

and by comparison with (2.10), $a_2 = a_1 + 1$. The next-smallest sum is $a_1 + a_4$, so

$$k = a_1 + a_4 + v + e - 3$$

and $a_4 = a_3 + 2$. Two cases arise. If $a_1 = 1$, then $a_2 = 2, a_3 = 3, a_4 = 5$. Also, a_5 cannot equal 6, because that would imply $a_1 + a_5 = 7 = a_2 + a_4$, contradicting the well-spread property. Every integer up to $v+e$ must occur as a label, so $b_1 = 4$ and $b_2 = 6$. So (2.14) is $a_{v-1} - a_{v-2} = b_2 - b_1 = 2$. But $a_4 - a_3 = 2$, so $d_{v-1,v-2} = d_{34}$, in contradiction of Lemma 2.9. In the other case, $a_1 = 2$, we obtain $a_2 = 3, a_3 = 4, a_4 = 6$, so $b_1 = 1, b_2 = 5$, and $a_{v-1} - a_{v-2} = 4 = a_4 - a_1$, again a contradiction.

If $a_1 + a_4$ is the next-smallest difference, we have

$$k = a_1 + a_4 + v + e - 2,$$

so $a_4 = a_3 + 1$. If $a_1 = 1$ and $a_2 = 3$, it is easy to see that $b_1 = 2, b_2 = 6$, and we get the contradiction $a_{v-1} - a_{v-2} = a_4 - a_1 = 4$. Otherwise $a_2 \geq 4$, so 3 is an edge-label. If $a_1 = 1$, then $b_1 = 2, b_2 = 3$, and $a_{v-1} - a_{v-2} = 1 = a_3 - a_2$. If $a_1 = 2$, then $b_1 = 1, b_2 = 3$, and $a_{v-1} - a_{v-2} = 2 = a_4 - a_2$. In every case, a contradiction is obtained. So we have

Theorem 2.13 *The complete graph K_v does not have an edge-magic total labeling if $v > 6$.*

This theorem was first proven in [55] (see also [54]) but the above proof follows that in [22].

2.3.3 All Edge-Magic Total Labelings of Complete Graphs

The proof in the preceding section used the fact that $v > 7$. There are edge-magic total labelings for all smaller complete graphs except K_4 and K_7 (which were excluded by Corollarys 2.1.1 and 2.12.1). A complete search has been made for smaller orders, and we now list all edge-magic total labelings for complete graphs.

In each case we list the possible values for the magic sum k, the corresponding sum of vertex labels s, and the set S of vertex labels that realize that value s and give an edge-magic total labeling.

The values to be considered are determined by equations (2.2) and (2.4). For K_v, $e = \binom{v}{2}$. So (2.2) becomes $\frac{1}{2}v(v+1) \leq s \leq \frac{1}{2}v(v^2+1)$, while (2.4) is

$$k = \frac{v(v+1)(v^2+v+2) + 8(v-2)s}{4v(v-1)}.$$

For example, when $v = 6$, we have $21 \leq s \leq 111$ and $k = \frac{1}{15}(231 + 4s)$. As k is an integer, $s \equiv 6 (\mathrm{mod}\, 15)$, and the possibilities are $s = 21, 36, 51, 66, 81, 96, 111$, $k = 21, 25, 29, 33, 37, 41, 45$.

- K_2 is trivially possible. Label 1, 2, or 3 can be given to the edge; in each case $k = 6$.
- K_3 The magic sums to be considered are $k = 9, 10, 11, 12$.

  $k = 9$, $s = 6$, $S = \{1, 2, 3\}$.
  $k = 10$, $s = 9$, $S = \{1, 3, 5\}$.
  $k = 11$, $s = 12$, $S = \{2, 4, 6\}$.
  $k = 12$, $s = 15$, $S = \{4, 5, 6\}$.
- K_4 No solutions, by Corollary 2.1.1.

- K_5 The magic sums to be considered are $k = 18, 21, 24, 27, 30$. Theorem 2.12 tells us that no solutions exist when k is odd, so only 18, 24 and 30 are listed.

  $k = 18,\ s = 20,\ S = \{1, 2, 3, 5, 9\}$.
  $k = 24,\ s = 40,\ S = \{1, 8, 9, 10, 12\}$.
  $k = 24,\ s = 40,\ S = \{4, 6, 7, 8, 15\}$.
  $k = 30,\ s = 60,\ S = \{7, 11, 13, 14, 15\}$.

- K_6 The magic sums to be considered are $k = 21, 25, 29, 33, 37, 41, 45$.

  $k = 21,\ s = 21$, no solutions.
  $k = 25,\ s = 36,\ S = \{1, 3, 4, 5, 9, 14\}$.
  $k = 29,\ s = 51,\ S = \{2, 6, 7, 8, 10, 18\}$.
  $k = 33,\ s = 66$, no solutions.
  $k = 37,\ s = 81,\ S = \{4, 12, 14, 15, 16, 20\}$.
  $k = 41,\ s = 96,\ S = \{8, 11, 17, 18, 19, 21\}$.
  $k = 45,\ s = 111$, no solutions.

From this it follows that the magic strengths of K_2, K_3, K_5, and K_6 are 6, 9, 18, and 25, respectively.

2.3.4 Complete Subgraphs

If $A = (a_1, a_2, \ldots, a_n)$ is any Sidon sequence of length n, we define

$$\begin{aligned}
\sigma(A) &= a_n - a_1 + 1\\
\rho(A) &= a_n + a_{n-1} - a_2 - a_1 + 1\\
&= \sigma(A) + a_{n-1} - a_2\\
\sigma^*(n) &= \min \sigma(A)\\
\rho^*(n) &= \min \rho(A)
\end{aligned}$$

where the minima are taken over all Sidon sequences A of length n. σ is called the *size* of the sequence. Without loss of generality one can assume $a_1 = 1$ when constructing a sequence, and then the size equals the largest element.

We now revert to the more general case, where the graph G has an edge-magic total labeling λ and G contains a complete subgraph H on n vertices. $x_1, x_2, \ldots, x_n$ are the vertices of H, and $a_i = \lambda(x_i)$. We assume $a_1 < a_2 < \cdots < a_n$, so $A = (a_1, a_2, \ldots, a_n)$ is a Sidon sequence of length n. Then

$$\lambda(x_n x_{n-1}) = k - a_n - a_{n-1},$$

and since $\lambda(x_n x_{n-1})$ is a label,

$$k - a_n - a_{n-1} \geq 1. \tag{2.16}$$

Similarly

$$\lambda(x_2x_1) = k - a_2 - a_1,$$

and since $\lambda(x_2x_1)$ is a label,

$$k - a_2 - a_1 \leq v + e. \tag{2.17}$$

Combining (2.16) and (2.17) we have

$$v + e \geq a_n + a_{n-1} - a_2 - a_1 + 1 = \rho(A) \geq \rho^*(n).$$

Theorem 2.14 [55] *If the edge-magic graph G contains a complete subgraph with n vertices, then the number of vertices and edges in G is at least $\rho^*(n)$.*

Exercise 2.5 *Suppose $G = K_n \cup tK_1$. In other words, G consists of a K_n together with t isolated vertices. Prove that if G is edge-magic, then*

$$t \geq \rho^*(n) - n - \binom{n}{2}.$$

The *magic number* $M(n)$ of K_n is defined to be the smallest t such that $K_n \cup tK_1$ is edge-magic. So the preceding exercise shows that

$$M(n) \geq \rho^*(n) - n - \binom{n}{2}.$$

Evaluation, bounds

In view of the above theorem, it is interesting to know more about Sidon sequences. Some bounds on $\sigma^*(n)$ and $\rho^*(n)$ are known:

Theorem 2.15 [53] $\sigma^*(n) \geq 4 + \binom{n-1}{2}$ *when* $n \geq 7$.

The proof appears in [53].

Theorem 2.16 [53] $\rho^*(n) \geq 2\sigma^*(n-1)$ *when* $n \geq 4$.

Proof. Consider the sequences

$$\begin{aligned} A &= (a_1, a_2, \ldots, a_n) \\ B &= (a_1, a_2, \ldots, a_{n-1}) \\ C &= (a_2, a_3, \ldots, a_n) \end{aligned}$$

where $n \geq 4$. Clearly

$$\begin{aligned}\rho^*(n) &\geq \rho(A)\\ &= a_n + a_{n-1} - a_2 - a_1 + 1\\ &= (a_n - a_2 + 1) + (a_{n-1} - a_1 + 1) - 1\\ &= \sigma(B) + \sigma(C) - 1\\ &\geq 2\sigma^*(n-1) - 1.\end{aligned}$$

Moreover, equality can apply only if $\sigma(B) = \sigma(C) = \sigma^*(n-1)$. But

$$\begin{aligned}\sigma(B) = \sigma(C) &\Rightarrow a_n - a_2 = a_{n-1} - a_1\\ &\Rightarrow a_{n-1} + a_2 = a_n + a_1,\end{aligned}$$

which is impossible for a Sidon sequence A. Since σ^* and ρ^* are integral,

$$\rho^*(n) \geq 2\sigma^*(n-1).$$ □

Exercise 2.6 *Prove that, when $n \geq 7$,*

$$\rho^*(n) \geq n^2 - 5n + 14. \tag{2.18}$$

From Theorems 2.16 and 2.15, $\rho^*(n) \geq 2\sigma^*(n-1) \geq 2(4 + \binom{n-2}{2}) = 8 + (n-2)(n-3) = n^2 - 5n + 14$.

In practice, values of $\sigma^*(n)$ and $\rho^*(n)$ have been calculated using an exhaustive, backtracking approach. The following result proves helpful in restricting the search for $\rho^*(n)$, once some σ^* values are known.

Theorem 2.17 *Suppose the sequence $A = (1, x, \ldots, y, z)$ satisfies $\rho(A) = \rho^*(n)$, and suppose B is any sequence for which $\rho(B)$ is known. Then*

$$\sigma^*(n) \leq z \leq \rho(B) - \sigma^*(n-2) + 1$$

and

$$x \leq (\rho(B) - \sigma^*(n-2) + 1) - \sigma^*(n-1) + 1.$$

Proof. Since $(x, \ldots, y)$ is a Sidon sequence,

$$y - x + 1 \geq \sigma^*(n-2).$$

But

$$\rho^*(n) = z + y - x$$

so

$$\begin{aligned}z &= \rho^*(n) - (y - x)\\ &\leq \rho^*(n) - \sigma^*(n-2) + 1\\ &\leq \rho(B) - \sigma^*(n-2) + 1.\end{aligned}$$

Also $(x, \ldots, y, z)$ is Sidon, so

$$z - x \geq \sigma^*(n-1) - 1$$

and the second part of the theorem follows from the upper bound for z. □

We know the following small values of the two functions. The values for $n \leq 8$ are calculated in [53] and listed in [55].

$$\begin{array}{rlcrl}
\sigma^*(3) = & 3 & \quad & \rho^*(3) = & 3 \\
\sigma^*(4) = & 5 & & \rho^*(4) = & 6 \\
\sigma^*(5) = & 8 & & \rho^*(5) = & 11 \\
\sigma^*(6) = & 13 & & \rho^*(6) = & 19 \\
\sigma^*(7) = & 19 & & \rho^*(7) = & 30 \\
\sigma^*(8) = & 25 & & \rho^*(8) = & 43 \\
\sigma^*(9) = & 35 & & \rho^*(9) = & 62 \\
\sigma^*(10) = & 46 & & \rho^*(10) = & 80 \\
\sigma^*(11) = & 58 & & \rho^*(11) = & 110 \\
\sigma^*(12) = & 72 & & \rho^*(12) = & 137
\end{array}$$

Sample sequences attaining the σ^* values are:

$\sigma^*(1)$ through $\sigma^*(6)$: 1 2 3 5 8 13 (or part thereof);

$\sigma^*(7)$: 1 2 3 5 9 14 19;

$\sigma^*(8)$: 1 2 3 5 9 15 20 25;

$\sigma^*(9)$: 1 2 3 5 9 16 25 30 35;

$\sigma^*(10)$: 1 2 8 11 14 22 27 42 44 46;

$\sigma^*(11)$: 1 2 6 10 18 32 34 45 52 55 58;

$\sigma^*(12)$: 1 2 3 8 13 23 38 41 55 64 68 72.

The same sequences attain $\rho^*(n)$ for $n, = 1, 2, 3, 4, 5, 6, 8$. For the other values, examples are

$\rho^*(7)$: 1 6 8 10 11 14 22;

$\rho^*(9)$: 1 5 7 9 12 17 26 27 40;

$\rho^*(10)$: 1 2 3 5 9 16 25 30 35 47;

$\rho^*(11)$: 1 2 3 5 9 16 25 30 35 47 65.

$\rho^*(12)$: 1 3 5 8 11 21 30 39 51 62 63 77.

Note: The only other sequence of length 7 with $\rho = 30$ is 1, 9, 12, 13, 15, 17, 22.

From Theorem 2.17, we have

For $n = 7$	$x \leq 12$	$19 \leq z \leq 24$
For $n = 8$	$x \leq 13$	$25 \leq z \leq 31$
For $n = 9$	$x \leq 21$	$35 \leq z \leq 45$
For $n = 10$	$x \leq 30$	$46 \leq z \leq 64$
For $n = 11$	$x \leq 32$	$58 \leq z \leq 77$
For $n = 12$	$x \leq 36$	$72 \leq z \leq 93$

and these bounds were used in calculating the example sequences for $\rho^*(n)$ when $n \geq 7$.

A greedy approach

Here is a simple observation. If $(a_1, a_2, \ldots, a_{n-1})$ is well spread, then none of its sums can exceed $a_{n-2} + a_{n-1}$. Put $a_n = a_{n-1} + a_{n-2}$. Then all the sums $a_i + a_n$ are new, and (since the sequence is strictly monotonic) they are all different. So we have a new well-spread sequence. This will be useful in constructing the smallest well-spread sequences for small orders: for example, after observing that $(1, 2, 3, 5, 8, 13)$ is a minimal example for $n = 6$, one need not test any sequence in the case $n = 7$ which has size greater than 21. (Unfortunately, 21 is not small enough.)

Suppose $(a_1, a_2, \ldots, a_{n-1})$ had minimal size, and put $a_1 = 1$. Then $\sigma^*(n-1) = a_{n-1}$. Clearly $a_{n-2} < a_{n-1}$, so we have the (bad) bound

$$\sigma^*(n) \leq 2\sigma^*(n-1) - 1.$$

Another application of this idea comes from noticing that the recursive construction $a_1 = 1, a_2 = 2, a_n = a_{n-1} + a_{n-2}$ gives a well-spread sequence. This is the Fibonacci sequence, except that the standard notation for the Fibonacci numbers has $f_1 = f_2 = 1, f_3 = 2$, etc. So we have a well-spread sequence with its size equal to the $(n+1)$-th term of the Fibonacci sequence: $a_n = f_{n+1}$. Therefore

$$\sigma^*(n) \leq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

The same reasoning shows that

$$\rho^*(n) \leq f_{n+1} + f_n - 2 = f_{n+2} - 2.$$

Note. For further information on the Fibonacci numbers, see, for example, Sect. 7.1 of [19].

Exercise 2.7 (Continuation of Exercise 2.5.) *The **magic number** $M(n)$ of K_n was defined to be the smallest t such that $K_n \cup tK_1$ is edge-magic. We know that*

$M(n) \geq \rho^*(n) - n - \binom{n}{2}$. *Find an upper bound for* $M(n)$. *(It need not be a good upper bound. The point is to show that* some *upper bound exists.)*

Research Problem 2.2 (Continuation of Exercise 2.7.) *Find* $M(7)$. *Find* $M(8)$.

Research Problem 2.3 *Find good lower and upper bounds on* $M(n)$*, as a function of* n.

Exercise 2.8 *Use the results on the evaluation of Sidon sequences to prove Theorem* 2.13. *(Note: this was the original proof of Theorem* 2.13*, as given in* [54, 55].*)*

2.3.5 Split Graphs

A *split graph* $K_{m+n\backslash m}$ consists of a complete graph K_n, a null graph $\overline{K}_m$, and the $K_{m,n}$ joining the two vertex-sets.

Suppose $K_{m+n\backslash m}$ is edge-magic. This graph has $v = m + n$, $e = mn + \binom{n}{2}$, and $v + e = m + n + mn + \binom{n}{2} = \frac{1}{2}(2m + n)(n + 1)$, so by Theorem 2.14

$$\frac{1}{2}(2m + n)(n + 1) \geq \rho^*(n).$$

This gives no useful information for small n. However, provided $n > 6$, (2.18) applies and from

$$\frac{1}{2}(2m + n)(n + 1) \geq n^2 - 5n + 14$$

we find that there is no edge-magic total labeling of $K_{m+n\backslash m}$ unless

$$m \geq \frac{n - 12}{2} + \frac{20}{n + 1}. \tag{2.19}$$

Research Problem 2.4 *Observe that* $K_{2n\backslash n}$ *always satisfies* (2.19). *Does it always have an edge-magic total labeling?*

2.4 Cycles

The cycle C_v is regular of degree 2 and has v edges. So (2.2) becomes

$$v(v + 1) \leq 2s \leq 2v^2 + v(v + 1) = v(3v + 1),$$

and (2.4) is

$$kv = s + v(2v+1),$$

whence v divides s; in fact $s = (k - 2v - 1)v$. When v is odd, s has $v+1$ possible values $\frac{1}{2}v(v+1), \frac{1}{2}v(v+3), \ldots, \frac{1}{2}v(v+2i-1), \ldots, \frac{1}{2}v(3v+1)$, with corresponding magic sums $\frac{1}{2}(5v+3), \frac{1}{2}(5v+5), \ldots, \frac{1}{2}(5v+2i+1), \ldots, \frac{1}{2}(7v+3)$. For even v, there are v values $s = \frac{1}{2}v^2+v, \frac{1}{2}v^2+2v, \ldots, \frac{1}{2}v^2+iv, \ldots, \frac{3}{2}v^2$, with corresponding magic sums $\frac{5}{2}v+2, \frac{5}{2}v+3, \ldots, \frac{5}{2}v+i+1, \ldots, \frac{7}{2}v+1$.

Kotzig and Rosa [54] proved that all cycles are edge-magic, producing examples with $k = 3v+1$ for v odd, $k = \frac{5}{2}v+2$ for $v \equiv 2(\bmod 4)$ and $k = 3v$ for $v \equiv 0(\bmod 4)$. In [33], labelings are exhibited for the minimum values of k in all cases. For convenience we give proofs for all cases, not exactly the same as the proofs in the papers cited. In each case the proof consists of exhibiting a labeling. If vertex-names need to be cited, we assume the cycle to be $(u_1, u_2, \ldots, u_v)$.

Theorem 2.18 *If v is odd, then C_v has an edge-magic total labeling with $k = \frac{1}{2}(5v+3)$.*

Proof. Say $v = 2m+1$. Consider the cyclic vertex labeling $(1, m+1, 2m+1, m, \ldots, m+2)$, where each label is derived from the preceding one by adding $m(\mod 2m+1)$. The successive pairs of vertices have sums $m+2, 3m+2, 3m+1, \ldots, m+3$, which are all different. If $k = 5m+4$, the edge labels are $4m+2, 2m+2, 2m+3, \ldots, 4m+1$, as required. We have an edge-magic total labeling with $k = 5m+4 = \frac{1}{2}(5v+3)$ and $s = \frac{1}{2}v(v+1)$ (the smallest possible values). □

By duality, we have:

Corollary 2.18.1 *Every odd cycle has an edge-magic total labeling with $k = \frac{1}{2}(7v+3)$.*

Theorem 2.19 *Every odd cycle has an edge-magic total labeling with $k = 3v+1$.*

Proof. Again write $v = 2m+1$. Consider the cyclic vertex labeling $(1, 2m+1, 4m+1, 2m-1, \ldots, 2m+3)$; in this case each label is derived from the preceding one by adding $2m(\mod 4m+2)$. The construction is such that the second, fourth, $\ldots$, $2m$-th vertices receive labels between 2 and $2m+1$ inclusive, while the third, fifth, $\ldots$, $(2m+1)$-th receive labels between $2m+2$ and $4m+1$. The successive pairs of vertices have sums $2m+2, 6m+2, 6m, 6m-2, \ldots, 2m+4$; if $k = 3v+1 = 6m+4$, the edge labels are $4m+2, 2, 4, \ldots, 4m$. We have an edge-magic total labeling with $k = 3v+1$ and $s = v^2$ (the case $i = \frac{1}{2}(v+1)$ in the list). □

Corollary 2.19.1 *Every odd cycle has an edge-magic total labeling with $k = 3v+2$.*

Figure 2.3 shows examples with $v = 7$ of the constructions in Theorems 2.18 and 2.19; they have $k = 19$ and 22, respectively. (Only the vertex labels are shown in the figure; the edge labels can be found by subtraction.)

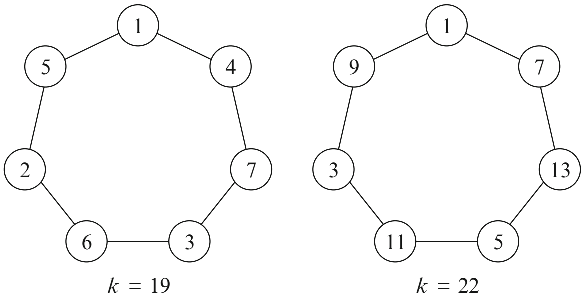


Fig. 2.3. Two edge-magic total labelings of C_7

Theorem 2.20 *If v is even, then C_v has an edge-magic total labeling with $k = \frac{1}{2}(5v + 4)$.*

Proof. Write $v = 2m$. If m is even,

$$\lambda(u_i) = \begin{cases} (i+1)/2 & \text{for } i = 1, 3, \ldots, m+1 \\ 3m & \text{for } i = 2 \\ (2m+i)/2 & \text{for } i = 4, 6, \ldots, m \\ (i+2)/2 & \text{for } i = m+2, m+4, \ldots, 2m \\ (2m+i-1)/2 & \text{for } i = m+3, m+5, \ldots, 2m-1, \end{cases}$$

while if m is odd,

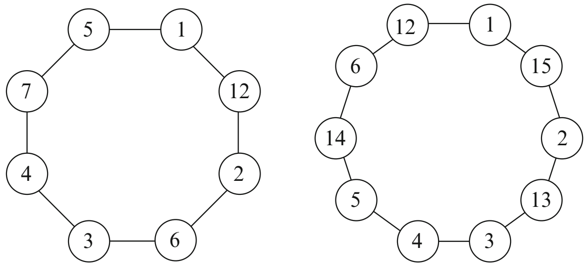
$$\lambda(u_i) = \begin{cases} (i+1)/2 & \text{for } i = 1, 3, \ldots, m \\ 3m & \text{for } i = 2 \\ (2m+i+2)/2 & \text{for } i = 4, 6, \ldots, m-1 \\ (m+3)/2 & \text{for } i = m+1 \\ (i+3)/2 & \text{for } i = m+2, m+4, \ldots, 2m-1 \\ (2m+i)/2 & \text{for } i = m+3, m+5, \ldots, 2m-2 \\ m+2 & \text{for } i = 2m. \end{cases}$$

□

From Theorems 2.2, 2.18 and 2.20, it follows that the magic strength of a cycle of length v is $\frac{1}{2}\lceil(5v + 3)\rceil$.

Corollary 2.20.1 *Every cycle of even length has an edge-magic total labeling with $k = \frac{1}{2}(7v + 2)$.*

In Fig. 2.4 we present two examples, for $v = 8$ and 10. The constructions of Theorem 2.20 yield $k = 22$ and 27.



$k = 22$ $k = 27$

Fig. 2.4. Edge-magic total labelings of C_8 and C_{10}

Theorem 2.21 *Every cycle of length divisible by 4 has an edge-magic total labeling with $k = 3v$.*

Proof. For $v = 4$ the result is given by Theorem 2.20. So assume $v \geq 8$, write $v = 4m, m > 1$. The required labeling is

$$\lambda(u_i) = \begin{cases} i & \text{for } i = 1, 3, \ldots, 2m-1 \\ 4m+i+1 & \text{for } i = 2, 4, \ldots, 2m-2 \\ i+1 & \text{for } i = 2m, 2m+2, \ldots, 4m-2 \\ 4m+i & \text{for } i = 2m+1, 2m+3, \ldots, 4m-3 \\ 2 & \text{for } i = 4m-1 \\ 2v-2 & \text{for } i = 4m. \end{cases}$$

□

Corollary 2.21.1 *Every cycle of length divisible by 4 has an edge-magic total labeling with $k = 3v + 3$.*

Research Problem 2.5 [33] *If v is odd, does C_v have an edge-magic total labeling for every magic sum k satisfying $\frac{1}{2}(5v+3) \leq k \leq \frac{1}{2}(7v+3)$? If v is even, does C_v have an edge-magic total labeling for every magic sum k satisfying $\frac{5}{2}v + 2 \leq k \leq \frac{7}{2}v + 1$?*

2.4.1 Small Cycles

We list all edge-magic total labelings of cycles up to C_6.

There are four labelings of C_3: see under K_3, in Sect. 2.3.3.

For C_4, the possibilities are $k = 12, 13, 14, 15$, with $s = 12, 16, 20, 24$, respectively. The unique solution for $k = 12$ is the cyclic vertex-labeling $(1, 3, 2, 6)$. For $k = 13$ there are two solutions: $(1, 5, 2, 8)$ and $(1, 4, 6, 5)$. The other cases are duals of these two.

For C_5, one must consider $k = 14, 15, 16$ $(s = 15, 20, 25)$ and their duals. The unique solution for $k = 14$ is $(1, 4, 2, 5, 3)$ (the solution from Theorem 2.18). There are no solutions for $k = 15$. For $k = 16$, one obtains $(1, 5, 9, 3, 7)$ (the solution from Theorem 2.19) and also $(1, 7, 3, 4, 10)$. Many other possible sets S must be considered when $k = 15$ or 16, but all can be eliminated using the following observation. The set S cannot contain three labels that add to k: for, in C_5, some pair of the corresponding vertices must be adjacent (given any three vertices of C_5, at least two must be adjacent), and the edge joining them would require the third label.

C_6 has possible sums $k = 17, 18, 19$ $(s = 24, 30, 36)$ and duals. For $k = 17$ there are three solutions: $(1, 5, 2, 3, 6, 7)$, $(1, 6, 7, 2, 3, 5)$, and $(1, 5, 4, 3, 2, 9)$. Notice that two non-isomorphic solutions have the same set of vertex labels. There is one solution for $k = 18$, $(1, 8, 4, 2, 5, 10)$, and six for $k = 19$, namely $(1, 6, 11, 3, 7, 8)$, $(1, 7, 3, 12, 5, 8)$, $(1, 8, 7, 3, 5, 12)$, $(1, 8, 9, 4, 3, 11)$, $(2, 7, 11, 3, 4, 9)$, and $(3, 4, 5, 6, 11, 7)$.

In the case of C_7, the possible magic sums run from 19 to 26, and Godbold and Slater [33] found that all can be realized; there are 118 labelings up to isomorphism. The corresponding numbers for C_8, C_9, and C_{10} are 282, 1540 and 7092, [33].

2.4.2 Generalizations of Cycles

Paths

The path P_n can be viewed as a cycle C_n with an edge deleted.

Say λ is an edge-magic total labeling of C_n with the property that label $2n$ appears on an edge. If that edge is deleted, the result is a P_n with an edge-magic total labeling.

For every n, there is an edge-magic total labeling of C_n in which $2n$ appears on an edge—the labelings in Theorems 2.18 and 2.20 have this property. Deleting this edge yields a path, on which the labeling is edge-magic. So:

Theorem 2.22 *All paths have edge-magic total labelings.*

However, this simple method does not yield the smallest possible magic sum. Avadayappan et al. [5] give the following simple construction for a super edge-magic labeling of a path. Say $n = 2m$ or $2m+1$. Label the vertices $x_1, x_2, \dots, x_{2m}$, preceded by x_0 if n is odd. Vertices $x_1, x_3, x_5, \dots, x_{2n-1}$ receive labels 1, 2, 3, $\dots$, n, while even vertices ($x_0, x_2, x_4, \dots, x_{2n}$ when n is odd, $x_2, x_4, x_6, \dots, x_{2n}$ in the even case) receive $n+1, n+2, \dots$ in order. When this labeling is completed in the obvious way, the magic sum is $\lceil \frac{1}{2}(5n+1) \rceil$, the theoretical minimum. So the magic strength of P_n is $\lceil \frac{1}{2}(5n+1) \rceil$.

Exercise 2.9 *Prove that the minimum possible magic sum for the path P_n is $\left\lceil \frac{1}{2}(5n+1) \right\rceil$.*

Suns

An n-sun is a cycle C_n with an edge terminating in a vertex of degree 1 attached to each vertex. A 4-sun is shown in Figure 2.5.

Theorem 2.23 *All suns are edge-magic.*

Proof. First we treat the odd case. Denote by λ the edge-magic total labeling of C_n given in Theorem 2.18. We construct a labeling μ which has $\mu(u) = \lambda(u) + n$ whenever u is a vertex or edge of the cycle. If a vertex has label x, then the new vertex attached to it has label a_x, where $a_x \equiv x - \frac{1}{2}(n-1) (\bmod n)$ and $1 \le a_x \le n$, and the edge joining them has label b_x, where $b_x \equiv n + 1 - 2x (\bmod n)$ and $3n+1 \le b_x \le 4n$. Then μ is an edge-magic total labeling with $k = \frac{1}{2}(11n+3)$.

In the even case, λ is the edge-magic total labeling of C_n given in Theorem 2.20. The labeling μ again has $\mu(u) = \lambda(u) + n$ whenever u is an element of the cycle. The vertex with label x is adjacent to a new vertex with label a_x, and the edge joining them has label b_x, where:

- If $1 \le x \le \frac{1}{2}n$, then
  - $a_x \equiv x + \frac{1}{2}n (\bmod n)$ and $1 \le a_x \le n$,
  - $b_x \equiv 2 - 2x (\bmod n)$ and $3n+1 \le b_x \le 4n$;
- If $1 + \frac{1}{2}n \le x < n$ then
  - $a_x \equiv x + \frac{1}{2}n + 1 (\bmod n)$ and $1 \le a_x \le n$,
  - $b_x \equiv 1 - 2x (\bmod n)$ and $3n+1 \le b_x \le 4n$;
- $a_{\frac{3n}{2}} = b_{\frac{3n}{2}} = 1$.

Then μ is an edge-magic total labeling with $k = \frac{1}{2}(11n+4)$. □

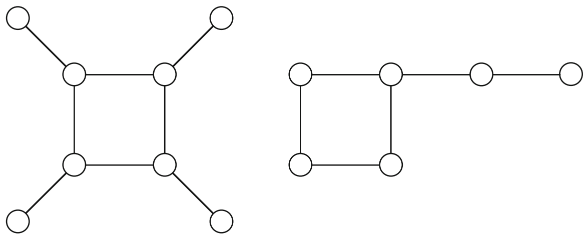


Fig. 2.5. A 4-sun and a $(4, 2)$-kite

Kites

An (n, t)-kite consists of a cycle of length n with a t-edge path (the *tail*) attached to one vertex. A (4,2) kite is shown in Figure 2.5. We write its labeling as the list of labels for the cycle (ending on the attachment point), separated by a semicolon from the list of labels for the path (starting at the vertex nearest the cycle).

Theorem 2.24 *An $(n, 1)$-kite (a kite with tail length 1) is edge-magic.*

Proof. For convenience, suppose the tail vertex is y and its point of attachment is z.

First, suppose n is odd. Denote by λ the edge-magic total labeling of C_n given in Theorem 2.18, with the vertices arranged so that $\lambda(z) = \frac{1}{2}(n+1)$. Define a labeling μ by $\mu(x) = \lambda(x) + 1$ whenever x is a vertex or edge of the cycle, $\mu(y) = 2n + 2$ and $\mu(y, z) = 1$. Then μ is an edge-magic total labeling with $k = \frac{1}{2}(5n + 9)$.

If n is even, λ is the edge-magic total labeling of Theorem 2.20, with $\lambda(z) = \frac{1}{2}(n + 2)$. Define a labeling μ by $\mu(x) = \lambda(x) + 1$ whenever x is a vertex or edge of the cycle, $\mu(y) = 2n + 2$ and $\mu(y, z) = 1$. Then μ is an edge-magic total labeling with $k = \frac{1}{2}(5n + 10)$. $\square$

Park et al. proved that an (n, t)-kite is super edge-magic if and only if n is even (see Sect. 2.9 below).

Research Problem 2.6 *Investigate the edge-magic properties of (n, t)-kites when n is odd, for general t.*

Exercise 2.10 *A* triangular book $B_{3,n}$ *consists of n triangles with a common edge. Prove that all triangular books are edge-magic.*

Research Problem 2.7 *The k-cycle book $B_{k,n}$ consists of n copies of C_k with a common edge. Are all k-cycle books edge-magic?*

Research Problem 2.8 *The books described in the preceding exercise and problem can be generalized by replacing the common edge by a path. Investigate the edge-magic properties of these graphs.*

2.5 Complete Bipartite Graphs

An edge-magic total labeling of a complete bipartite graph can be specified by giving two sets S_1 and S_2 of vertex labels.

Theorem 2.25 [54] *The complete bipartite graph $K_{m,n}$ is magic for any m and n.*

Proof. The sets $S_1 = \{n+1, 2n+2, \ldots, m(n+1)\}, S_2 = \{1, 2, \ldots, n\}$, define an edge-magic total labeling with $k = (m+2)(n+1)$. □

Research Problem 2.9 *Recall that the complete tripartite graph $K_{m,n,p}$ has three sets of vertices, of sizes m, n, p. Does $K_{m,n,p}$ always have an edge-magic total labeling?*

Research Problem 2.10 *Generalize Research Problem 2.9 to complete t-partite graphs (t parts).*

2.5.1 Small Cases

A computer search has been carried out for edge-magic total labelings of $K_{2,3}$. The usual considerations show that $14 \leq k \leq 22$, with cases $k = 19, 20, 21, 22$ being the duals of cases $k = 17, 16, 15, 14$. The solutions up to $k = 18$ are

$$\begin{aligned}
&k = 14, \text{ no solutions}\\
&k = 15,\ S_1 = \{1,2\},\quad S_2 = \{3,6,9\}\\
&k = 16,\ S_1 = \{1,2\},\quad S_2 = \{5,8,11\}\\
& S_1 = \{1,3\},\quad S_2 = \{5,6,11\}\\
& S_1 = \{4,6\},\quad S_2 = \{1,2,7\}\\
& S_1 = \{4,8\},\quad S_2 = \{1,2,3\}\\
&k = 17,\ S_1 = \{1,8\},\quad S_2 = \{5,6,7\}\\
& S_1 = \{5,6\},\quad S_2 = \{1,4,9\}\\
&k = 18,\ S_1 = \{1,5\},\quad S_2 = \{9,10,11\}\\
& S_1 = \{7,11\}, S_2 = \{1,2,3\}.
\end{aligned}$$

(The last two are of course duals.)

For $K_{3,3}$ one has $18 \leq k \leq 30$, and k must be even. Cases $k = 26, 28, 30$ are dual to cases $k = 22, 20, 18$. The solutions are

$$\begin{aligned}
&k = 18, \text{ no solutions}\\
&k = 20, \ S_1 = \{1,2,3\}, \quad S_2 = \{4,8,12\}\\
& \ S_1 = \{1,2,9\}, \quad S_2 = \{4,6,8\}\\
&k = 22, \ S_1 = \{1,2,3\}, \quad S_2 = \{7,11,15\}\\
& \ S_1 = \{1,3,5\}, \quad S_2 = \{7,8,15\}\\
& \ S_1 = \{1,5,12\}, S_2 = \{6,7,8\}\\
&k = 24, \text{ no solutions.}
\end{aligned}$$

2.5.2 *Stars*

Lemma 2.26 *In any edge-magic total labeling of a star, the center receives label* 1, $n + 1$ *or* $2n + 1$.

Proof. Suppose the center receives label x. Then

$$kn = \binom{2n+2}{2} + (n-1)x. \tag{2.20}$$

Reducing (2.20) modulo n we find

$$x \equiv (n+1)(2n+1) \equiv 1$$

and the result follows. □

Theorem 2.27 *There are* $3 \cdot 2^n$ *edge-magic total labelings of* $K_{1,n}$*, up to equivalence.*

Proof. Denote the center of a $K_{1,n}$ by c, the leaves by $v_1, v_2, \ldots, v_n$ and edge (c, v_i) by e_i. From Lemma 2.26 and (2.20), the possible cases for an edge-magic total labeling are $\lambda(c) = 1, k = 2n + 4$, $\lambda(c) = n + 1, k = 3n + 3$ and $\lambda(c) = 2n + 1, k = 4n + 2$. As the labeling is magic, the sums $\lambda(v_i) + \lambda(e_i)$ must all be equal to $M = k - \lambda(c)$ (so $M = 2n + 3, 2n + 2$ or $2n + 1$). Then in each case there is exactly one way to partition the $2n+1$ integers $1, 2, \ldots, 2n+1$ into $n + 1$ sets

$$\{\lambda(c)\}, \{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$$

where every $a_i + b_i = M$. For convenience, choose the labels so that $a_i < b_i$ for every i and $a_1 < a_2 < \cdots < a_n$. Then up to isomorphism, one can assume that $\{\lambda(v_i), \lambda(e_i)\} = \{a_i, b_i\}$. Each of these n equations provides two choices, according as $\lambda(v_i) = a_i$ or b_i, so each of the three values of $\lambda(c)$ gives 2^n edge-magic total labelings of $K_{1,n}$. □

Exercise 2.11 *A graph is derived from a star by adding a pendant edge to one of the vertices of degree* 1*. Prove every such graph is edge-magic.*

Double Stars

The *double star* $S_{m,n}$ has two adjacent central vertices x and y. There are m leaves $x_1, x_2, \ldots, x_m$ adjacent to x and n leaves $y_1, y_2, \ldots, y_n$ adjacent to y. An edge-magic total labeling of this graph can be specified by the list

$$(\{\lambda(x_1), \lambda(x_2), \ldots, \lambda(x_m)\}, \lambda(x), \lambda(y), \{\lambda(y_1), \lambda(y_2), \ldots, \lambda(y_n), \}).$$

One solution for $S_{2,2}$ is $(\{8, 11\}, 2, 5, \{4, 10\})$ with $k = 16$.

Research Problem 2.11 *For what m and n does $S_{m,n}$ have an edge-magic labeling?*

2.6 Wheels

Enomoto *et al.* [25] have checked all wheels up to $n = 29$ and found that the graph is magic if $n \not\equiv 3 \pmod 4$. We shall prove that W_n is magic in all other cases.

To describe a wheel, we refer to the central vertex as the *center*, the edges adjacent to the center as *spokes*, and the other vertex on a spoke as its *terminal*. The remaining edges are *arcs*. We write c for the label on the center; the spokes receive labels $a_1, a_2, \ldots, a_n$, the terminal of the spoke a_i gets label b_i, and the arc from b_i to b_{i+1} is labeled s_i. (This scheme is illustrated in Fig. 2.6.) It then follows that all the sums $a_i + b_i$ in an edge-magic total labeling are the same; they equal $k - c$, where k is the magic sum.

For the cases other than $n \equiv 6 \pmod 8$, our constructions (taken from [80]) use $k = 2c$. Then $a_i + b_i = c$ for every i. It follows that c is greater than each a_i and b_i; since labels are distinct and positive, $c \geq 2n + 1$. Also $s_i = a_i + a_{i+1}$. We then need to do the following:

1. Partition the integers from 1 to $2n$ into two classes A and B so that exactly one member of $\{i, 2n + 1 - i\}$ is in each set, for every i.
2. Order the elements of A in a cyclic sequence so that the n sums of consecutive pairs comprise the integers from $2n + 2$ to $3n + 1$ in some order (or, what is equivalent, the elements of B must be ordered so that the consecutive sums are a permutation of $\{n + 1, \ldots, 2n\}$).

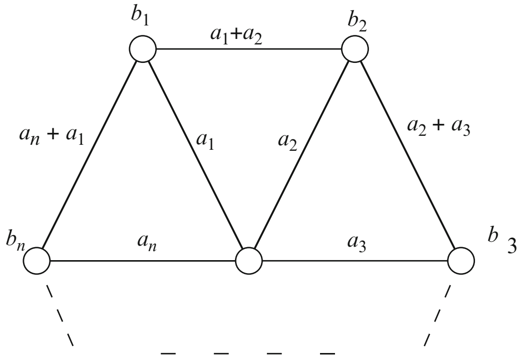


Fig. 2.6. Labeling a wheel

To visualize this process, it is perhaps easiest to consider a graph with vertices $1, 2, \ldots, 2n$, drawn in two sets so that i and $2n+1-i$ are vertical opposite pairs. Two vertices are connected if their sum lies between $n+1$ and $2n$. The graph for $n = 5$ is shown in Fig. 2.7. We need to find a cycle that contains exactly one member from each opposite pair, all of whose edges have different sums. From this representation it is clear, for example, that both 1 and 2 must be terminal labels, because neither $2n$ nor $2n-1$ have degree 2 or greater.

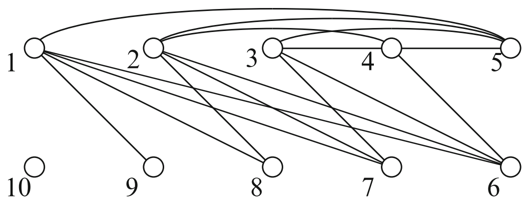


Fig. 2.7. The spoke-terminal graph

2.6.1 The Constructions for $n \not\equiv 2 \pmod 4$

We construct, for each $n \equiv 0, 1, 4 \text{ or } 5 \pmod 8$, a labeling of the terminal vertices of the wheel W_n. That is, we construct a sequence of n members of $\{1, 2, \ldots, 2n\}$ such that

(i) For each $i = 1, 2, \ldots, n$, exactly *one* of i and $2n+1-i$ is a member of the sequence, and

(ii) The set of sums of pairs of successive elements of the sequence (including last and first elements) is precisely $n+1, n+2, \ldots, 2n$.

In every case the sequence is defined in terms of several subsequences which are then joined.

It will be observed that the constructions do not provide solutions for n =5 or 13. There is no sequence for $n = 5$; for $n = 13$ one example is

$$(12\ 4\ 20\ 1\ 24\ 2\ 16\ 6\ 13\ 10\ 5\ 9\ 8).$$

Case $n \equiv 0 \pmod 8, n \geq 8$.

For each $i \equiv 1 \pmod 4$ with $1 \leq i \leq \frac{n}{2} - 3$ define

$$S_i = (i, 2n+1-3i, i+1, 2n+1-3i-4, i+3, 2n+1-3i-8),$$

and define

$$S = \left(\frac{n}{2} + j : j \equiv 0, 1 \pmod 4 \text{ and } 1 \leq j \leq \frac{n}{2}\right).$$

Then the desired sequence is

$$S_1 S_5 \ldots S_{\frac{n}{2}-3} S.$$

Case $n \equiv 4 \pmod 8, n \geq 4$.

For each $i \equiv 1 \pmod 4$ with $1 \leq i \leq \frac{n}{2} - 5$ define

$$S_i = (i+1, 2n+1-3i, i, 2n+1-3i-4, i+3, 2n+1-3i-8),$$

and define

$$S = \left(\frac{n}{2} - 2 + j : j \equiv 0, 1 \pmod 4 \text{ and } 1 \leq j \leq \frac{n}{2} + 2\right).$$

Then the desired sequence is

$$S_1 S_5 \ldots S_{\frac{n}{2}-5} \left(\frac{n}{2}, \frac{n}{2} + 4\right) S.$$

(Note that when $n = 4$, the sequence is simply $(\frac{n}{2}, \frac{n}{2} + 4)S = (2, 6, 1, 4)$.)

Case $n \equiv 1 \pmod 8, n \geq 9$.

For each $i \equiv 1 \pmod 4$ with $1 \leq i \leq \frac{n-1}{2} - 3$ define

$$S_i = \{i, 2n+1-3i, i+1, 2n+1-3i-4, i+3, 2n+1-3i-8\},$$

and define

$$S = \left\{\frac{n-1}{2} + j : j \equiv 1, 2 \pmod 4 \text{ and } 1 \leq j \leq \frac{n+1}{2}\right\}.$$

Then the desired sequence is

$$S_1 S_5 \dots S_{\frac{n-1}{2}-3} S.$$

Case $n \equiv 5 \pmod{24}, n \geq 29$.

Define

$$S = \left(\frac{n+10}{3}, n-5, \frac{n+1}{3}, n+3, \frac{n+7}{3}, n-1, 4, 2n+1-7, 1,\right.$$
$$\left. 2n+1-3, 2, 2n+1-11\right)$$

and

$$T = \left(\frac{n-2}{3}, n, n-3, n-4, n-7, \dots, \frac{n+5}{2}, \frac{n-1}{2}, \frac{n+3}{2}, \frac{n+11}{2}\right).$$

For each $i \equiv 6 \pmod 8, i \geq 6$, define

$$S_i = (i, 2n+1-(3i+1), i+2, 2n+1-(3i-3), i-1, 2n+1-(3i+5)),$$

and for each $i \equiv 1 \pmod 8, i \geq 9$, define

$$S_i = (i, 2n+1-(3i+4), i+3, 2n+1-3i, i+1, 2n+1-(3i+8)).$$

For each $j = 1, 2, \dots, \frac{n-29}{24}$ write

$$C_j = \left(\frac{n+10}{3} + 4j, n-5-12j, \frac{n+10}{3} + 4j - 3,\right.$$
$$\left. n-5-12j+4, \frac{n+10}{3} + 4j - 1, n-5-12j+8\right).$$

Then the desired sequence is

$$S S_6 S_9 \dots S_{\frac{n-35}{3}} S_{\frac{n-26}{3}} S_{\frac{n-11}{3}} T C_{\frac{n-29}{24}} \dots C_2 C_1.$$

(Note that when $n = 29$ this sequence is $S S_6 T$.)

Case $n \equiv 21 \pmod{24}, n \geq 21$.

Define

$$S = \left(\frac{n+6}{3}, n-1, 4, 2n+1-7, 1, 2n+1-3, 2, 2n+1-11\right)$$

and

$$T = \left(\frac{n-3}{3}, 2n+1-(n-2), \frac{n-3}{3} + 2, 2n+1-(n-6),\right.$$
$$\frac{n-3}{3} - 1, n, n-3, n-4, n-7, \dots,$$
$$\left.\frac{n+5}{2}, \frac{n-1}{2}, \frac{n+3}{2}, \frac{n+11}{2}\right).$$

For each $i \equiv 6 \pmod 8, i \geq 6$, and for each $i \equiv 1 \pmod 8, i \geq 9$, define S_i as in the $n \equiv 5 \pmod{24}$ case. Then, for each $j = 1, 2, \ldots, \frac{n-21}{24}$ write

$$C_j = \left(\frac{n+6}{3} + 4j, n-1-12j, \frac{n+6}{3} + 4j - 3, n-1-12j+4, \frac{n+6}{3} + 4j - 1, n-1-12j+8\right).$$

Then the desired sequence is

$$SS_6S_9 \ldots S_{\frac{n-27}{3}} S_{\frac{n-18}{3}} TC_{\frac{n-21}{24}} \ldots C_2C_1.$$

(Note that when $n = 21$ this sequence is ST.)

Case $n \equiv 13 \pmod{24}$.

Define

$$S = \left(n-1-8, \frac{n+5}{3} + 2, n-1-4, \frac{n+5}{3} + 6, \frac{2n+1}{3}, \frac{n+2}{3}, n-1, 4, 2n+1-7, 1, 2n+1-3, 2, 2n+1-11\right)$$

and

$$T = \left(\frac{n+5}{3}, n, n-3, n-4, n-7, \ldots, \frac{2n+1}{3} + 4, \frac{2n+1}{3} + 1, \frac{2n+1}{3} - 4, \frac{2n+1}{3} - 3, \ldots, \frac{n+5}{2}, \frac{n+7}{2}, \frac{n-3}{2}, \frac{n+11}{2}\right).$$

For each $i \equiv 6 \pmod 8, i \geq 6$, and for each $i \equiv 1 \pmod 8, i \geq 9$, define S_i as in the $n \equiv 5 \pmod{24}$ case. Now we must break into two subcases:

1. $n \equiv 13 \pmod{48}, n \geq 61$.

   Define

$$D = \left(\frac{n+5}{3} + 10, n-1-20, \frac{n+5}{3} + 4, n-1-16, \frac{n+5}{3} + 8, n-1-12, \frac{n+5}{3} + 3\right),$$

   and for each $j = 1, 2, \ldots, \dfrac{n-61}{48}$ write

$$C_j = \left(\frac{n+5}{3} + 10 + 8j, n-1-24j-20, \frac{n+5}{3} + 4 + 8j, n-1-24j-16, \frac{n+5}{3} + 8 + 8j, n-1-24j-12, \frac{n+5}{3} + 6 + 8j, n-1-24j-8, \frac{n+5}{3} - 1 + 8j, n-1-24j-4, \frac{n+5}{3} + 3 + 8j, n-1-24j\right).$$

Then the desired sequence is

$$SS_6S_9\dots S_{\frac{n-19}{3}}S_{\frac{n-10}{3}}TC_{\frac{n-61}{48}}\dots C_2C_1D.$$

(Note that when $n = 61$ this sequence is $SS_6S_9S_14S_1TD$.)

2. $n \equiv 37 \pmod{48}, n \geq 37$.

 Write $D = (\frac{n+5}{3} + 4)$, and for each $j = 1, 2, \dots, \frac{n-37}{48}$ define

$$C_j = \left(\frac{n+5}{3} + 6 + 8j, n - 1 - 24j - 8, \frac{n+5}{3} + 8j,\right.$$
$$n - 1 - 24j - 4, \frac{n+5}{3} + 4 + 8j, n - 1 - 24j,$$
$$\frac{n+5}{3} + 2 + 8j, n - 1 - 24j + 4, \frac{n+5}{3} - 5 + 8j,$$
$$\left. n - 1 - 24j + 8, \frac{n+5}{3} - 1 + 8j, n - 1 - 24j + 12\right).$$

 Then the desired sequence is

$$SS_6S_9\dots S_{\frac{n-19}{3}}S_{\frac{n-10}{3}}TC_{\frac{n-37}{48}}\dots C_2C_1D.$$

(Note that when $n = 37$ this sequence is $SS_6S_9S_14S_17TD$.)

Theorem 2.28 *Every wheel W_n with $n \equiv 0$ or $1 \pmod 4$ has an edge-magic total labeling.*

Proof. It will suffice to prove that the above constructions have the required properties (i) and (ii). We prove this for the hardest cases, the case $n \equiv 1 (\bmod\ 8)$ and $n \equiv 13 \pmod{48}, n \geq 61$; the proofs for $n \equiv 0$ and $4 \pmod 8$ are similar to the former, while the proofs in the other cases with $n \equiv 5 \pmod 8$ are similar to the latter.

$n \equiv 1 \pmod 8$

Property (i): Suppose $m \equiv 1$ or $2 \pmod 4, 1 \leq m \leq n$.If $1 \leq m \leq \frac{n-1}{2} - 2$ then m appears in one of the S_i (as an 'i' or '$i + 1$' term). If $\frac{n-1}{2} + 1 \leq m \leq n$, then m appears in S. Now suppose $m \equiv 0 (\bmod\ 4), 4 \leq m \leq 2n-2$. If $4 \leq m \leq \frac{n-1}{2}$, then m appears in one of the S_i (as an '$i+3$' term) while if $\frac{n-1}{2} + 4 \leq m \leq 2n-2$, then again m appears in one of the S_i, as follows. If $m \equiv \frac{n-1}{2} + 4 \bmod 12$, then m appears as a term of the form '$2n+1-3i-8$'; if $m \equiv \frac{n-1}{2} + 8 \bmod 12$, then m appears as a '$2n + 1 - 3i - 4$' term; if $m \equiv \frac{n-1}{2} \bmod 12$ (which means $m \equiv 2n - 2 \bmod 12$ since $n \equiv 1 \pmod 8$), then m appears as a '$2n + 1 - 3i$' term. Now since we have all $m \equiv 0 \pmod 4$, $4 \leq m \leq 2n - 2$, we also have represented all of their inverses mod $2n + 1$, which yields all of the $3 \pmod 4$ cases between 3 and $2n - 3$. Hence we have property (i).

Property (ii): First we note that those sums from $n+2$ to $2n-3$ inclusive which are congruent to $3 \pmod 4$ appear as successive sums of pairs in S. Then those sums from $n+4$ to $2n-1$ inclusive which are congruent to 1 mod 4 appear as the union of the sums of the first and second terms of each S_i, the last term in S_i with the first term in S_{i+4} and the last term in $S_{\frac{n-1}{2}-3}$ with the first term in S. The sums fron $n+1$ to $2n$ inclusive which are congruent to 2 mod 4 appear as the union of the sums of the second and third, and third and fourth terms in each S_i and the sum of the last term in S with the first term in S_1. Finally, those sums from $n+3$ to $2n-2$ inclusive which are congruent to 0 mod 4 appear as the union of the sums of the fourth and fifth, and fifth and sixth, terms in each S_i. Hence we have property (ii).

$n \equiv 13 \pmod{48}$

The sequence for the case $n = 13$ can be verified directly, so we assume $n \geq 61$.

Property (i): Suppose $m \equiv 1$ or $2 \pmod 4, 1 \leq m \leq n$. If $m = 1, 2, \frac{n+2}{3}$ or $\frac{2n+1}{3}$, then m appears in S. If $m = \frac{n+5}{3} + 3, \frac{n+5}{3} + 4$ or $\frac{n+5}{3} + 8$, then m appears in D. If $m = \frac{n+5}{3}$ or $\frac{n-3}{2}$, or if $\frac{n+5}{2} \leq m \leq n, m \neq \frac{2n+1}{3}$, then m appears in T. If $m \equiv 5$ or $6 \pmod 8$, $5 \leq m \leq \frac{n+5}{3} - 8$, then m appears in an S_i with $i \equiv 6 \pmod 8$, as the $i-1$ or i term, while if $m \equiv 1$ or 2 (mod $8), 9 \leq m \leq \frac{n+5}{3-4}$, then m appears in an S_i with $i \equiv 1 \pmod 8$, as the i or $i+1$ term. There remain those $m \equiv 1 \pmod 4$, $\frac{n+5}{3+7} \leq m \leq \frac{n-11}{2}$, and the $m \equiv 2 \pmod 4$, $\frac{n+5}{3+12} \leq m \leq \frac{n-1}{2}$. These terms appear in the C_js, the former as the $\frac{n+5}{3-1+8j}$ and $\frac{n+5}{3+3+8j}$ terms, $1 \leq j \leq \frac{n-61}{48}$, and the latter as the $\frac{n+5}{3+4+8j}$ and $\frac{n+5}{3+8+8j}$ terms, $1 \leq j \leq \frac{n-61}{48}$.

Now suppose $m \equiv 0 \pmod 4, 4 \leq m \leq 2n-2$. If $4 \leq m \leq \frac{n+5}{3+6}$ or if $n-9 \leq m \leq 2n-2$, then m appears in S or in one of the S_is. If $m = \frac{n+5}{3} + 10, n-21, n-17$ or $n-13$, then m appears in D. There remain the cases $m \equiv 0 \pmod 4$, $\frac{n+5}{3} + 14 \leq m \leq n-25$. Now the term $\frac{n+11}{2}$ appears in T. The terms $\frac{n+5}{3} + 14 \leq m \leq \frac{n+3}{2}$ appear in the C_js as the $\frac{n+5}{3} + 10 + 8j$ and $\frac{n+5}{3} + 6 + 8j$ terms, $1 \leq j \leq \frac{n-61}{48}$, while the terms $\frac{n+19}{2} \leq m \leq n-25$ appear in the C_js as the $n-1-24j-4k$ terms, $0 \leq k \leq 5, 1 \leq j \leq \frac{n-61}{48}$. Since we have all $0 \pmod 4$ terms between 4 and $2n-2$ we have also represented all $3 \pmod 4$ terms between 3 and $2n-3$ (i.e., the inverses modulo $(2n+1)$). This verifies property (i).

Property (ii): Sums from $n+2$ to $2n-3$ inclusive which are congruent to $3 \pmod 4$ appear as succesive sums of pairs of elements in T. Sums from $\frac{4n+11}{3}$ to $2n-1$ inclusive which are congruent to $1 \pmod 4$ appear as sums in S involving 1 (i.e., $2n-1$ and $42n-5$) and as sums of the fourth and fifth, and fifth and sixth terms in S_i, the sixth term in S_i with the first term in S_{i+3}, and the first and second terms in S_{i+3}, where $i \equiv 6 \pmod 8, 6 \leq i \leq \frac{n-19}{3}$. The sum

$\frac{4n-1}{3}$ appears as the sum of the sixth and seventh terms in S, while the sum $n+8$ appears as the sum of the fourth and fifth terms in S; $n+4$ appears as the sum of the last two terms in T. Now $\frac{4n-25}{3}$ and $\frac{4n-13}{3}$ appear as the sum of the last two terms in D and the sum of the last term in D with the first term in S, respectively. There remain the sums congruent to $1 \pmod 4$ between $n+12$ and $\frac{4n-37}{3}$ inclusive; these appear as successive sums over the eighth through twelfth terms in the C_js, $1 \le j \le \frac{n-61}{48}$. Sums from $\frac{4n+2}{3}$ to $2n$ inclusive which are congruent to $2 \pmod 4$ appear as sums of the eleventh and twelfth, and twelfth and last terms in S, the last term in S with the first term in S_6, and the first two terms in S_6; then as sums of the fourth and fifth, and fifth and sixth terms in S_i, the sixth term in S_i with the first term in S_i+5, and the first and second terms in S_i+5, where $i \equiv 1 \pmod 8), 9 \le i \le \frac{n-10}{3-8}$; then finally as sums of the fourth and fifth, and fifth and sixth terms in $S_{\frac{n-10}{3}}$, and the sixth term in $S_{\frac{n-10}{3}}$ with the first term in T. The sum $n+1$ appears as the sum of the fifth and sixth terms in S, while the sums $\frac{4n-10}{3-4k}, 3 \ge k \ge 0$, appear as successive sums over the second through sixth terms in D. There remain the sums congruent to $2 \pmod 4$ between $n+5$ and $\frac{4n-58}{3}$ inclusive; these appear as successive sums over the second through sixth terms in the C_js, $1 \le j \le \frac{n-61}{48}$. Sums from $\frac{4n+20}{3}$ to $2n-2$ inclusive which are congruent to $0 \pmod 4$ appear as sums of the eighth and ninth terms in S, and as sums of the second and third, and third and fourth terms in S_i and S_i+3, where $i \equiv 6 \pmod 8$ and $6 \le i \le \frac{n-19}{3}$. The sum $n+3$ appears as the sum of the seventh and eighth terms in S, while the sums $\frac{4n+20}{3-4k}, 4 \ge k \ge 1$, appear as the sum of the first two terms in D and as succesive sums over the first through fourth terms in S. There remain the sums congruent to $0 \pmod 4$ between $n+7$ and $\frac{4n-40}{3}$ inclusive. When $n=61$, this sum (namely 68) appears as the sum of the last term in T (namely $\frac{n+11}{2}=36$ and the first term in D ($\frac{n+5}{3}+10=32$); when $n>61$ these appear as the sum of the last term in T with the first term in $C_{\frac{n-61}{48}}$, the sums of the first and second, sixth and seventh, and seventh and eighth terms in each C_j, $\frac{n-61}{48} \ge j \ge 1$, the sum of the last term in C_j with the first term in C_j-1, $\frac{n-61}{48} \ge j \ge 2$, and the sum of the last term in C_1 with the first term in D. This completes the verification of property (ii). □

2.6.2 *The Constructions for* $n \equiv 2 \pmod 4$

Case $n \equiv 6 \pmod 8$.

If $n \equiv 6 (\mathrm{mod}\, 8)$, the following construction (taken from [85]) provides an edge-magic total labeling with $k = 5n+2$.

$c = 2n.$

For $i = 1, 3, \ldots, \frac{n}{2}$, $\begin{cases} a_i = \frac{1}{2}(i+1) \\ b_i = 3n+2-\frac{1}{2}(i+1) \\ s_i = n+i. \end{cases}$

For $i = 2, 4, \ldots, \frac{n-2}{2}$, $\begin{cases} a_i &= 2n + 1 + \frac{1}{2}i \\ b_i &= n + 1 - \frac{1}{2}i \\ s_i &= n + i. \end{cases}$

For $i = \frac{n+4}{2}, \frac{n+8}{2}, \ldots, \frac{3n-6}{4}$, $\begin{cases} a_i &= \frac{1}{2}(i + 3) \\ b_i &= 3n + 1 - \frac{1}{2}(i + 1) \\ s_i &= n + i + 1. \end{cases}$

For $i = \frac{n+2}{2}, \frac{n+8}{2}, \ldots, \frac{3n-2}{4}$, $\begin{cases} a_i &= 2n + 1 + \frac{1}{2}i \\ b_i &= n + 1 - \frac{1}{2}i \\ s_i &= n + i + 1. \end{cases}$

For $i = \frac{3n+2}{4}$, $\begin{cases} a_i &= \frac{1}{8}(5n + 2) \\ b_i &= \frac{1}{8}(19n + 14) \\ s_i &= 2n + 1. \end{cases}$

For $i = \frac{3n+6}{4}, \frac{3n+14}{4}, \ldots, n$, $\begin{cases} a_i &= 2n + 2 + \frac{1}{2}i \\ b_i &= n - \frac{1}{2}i \\ s_i &= n + i + 1. \end{cases}$

For $i = \frac{3n+10}{4}, \frac{3n+18}{4}, \ldots, n - 3$, $\begin{cases} a_i &= \frac{1}{2}(i + 5) \\ b_i &= 3n - \frac{1}{2}(i + 1) \\ s_i &= n + i + 1. \end{cases}$

For $i = n - 1$, $\begin{cases} a_i &= \frac{1}{4}(11n + 2) \\ b_i &= \frac{1}{4}(n + 6) \\ s_i &= \frac{1}{4}(7n + 6). \end{cases}$

with the exceptions $\begin{cases} s_{\frac{1}{4}(3n-2)} &= 2n - 1 \\ s_{n-2} &= \frac{1}{4}(7n + 2) \\ s_n &= \frac{1}{2}(3n + 2). \end{cases}$

Case $n \equiv 2 \pmod 8, n \geq 10$.

Research Problem 2.11 in our first edition was to show that wheels are edge-magic when $n \equiv 2 (\mathrm{mod}\, 8)$. The following construction was given in [30]:

Let $n = 8m+2, m \geq 1$. Thus, $v+e = 24m+7$. Let $c = 8m+6$ for all cases. In the case where $m = 1$ we have $(23, 2, 26, 3, 31, 4, 22, 5, 25, 8)$ as the vertex labels on the cycle with $k = 46$. When $m = 2$ we have $(38, 6, 39, 9, 40, 2, 41, 5, 42, 13, 444, 7, 45, 8, 46, 4, 55, 3)$ as the vertex labels on the cycle with $k = 78$. For $m \geq 3$ we have $k = 32m + 14$ and define

$$b_{2i-1} = \begin{cases} 16m + 5 + i & 1 \leq i \leq 3m - 1 \\ 16m + 6 + i & 3m \leq i \leq 4m \\ 24m + 7 & i = 4m + 1. \end{cases}$$

For the labels b_{2i} $(1 \leq i \leq 4m + 1)$, there are three subcases.

Subcase 1: $m \equiv 0(\bmod 3)$. Set $m = 3s$ ($s \geq 1$). Thus, $n = 24s + 2$ and $v + e = 72s + 7$. Define

$$b_{2i} = \begin{cases} i+5 & 1 \leq i \leq 6s \text{ and } m \not\equiv 0(\bmod 3) \\ i-1 & 1 \leq i \leq 6s \text{ and } m \equiv 0(\bmod 3) \\ 12s+1 & i = 6s+1 \\ i+4 & 6s+2 \leq i \leq 9s-2 \text{ and } m \not\equiv 1(\bmod 3),\ s \geq 2 \\ i-2 & 6s+2 \leq i \leq 9s-2 \text{ and } m \equiv 1(\bmod 3),\ s \geq 2 \\ 15s+3 & i = 9s-1 \\ 9s-1 & i = 9s \\ i+1 & 9s+1 \leq i \leq 12s-1 \\ 4 & i = 12s \\ 3 & i = 12s+1. \end{cases}$$

Subcase 2: $m \equiv 1(\bmod 3)$. Set $m = 3s + 1$ ($s \geq 1$). Thus, $n = 24s + 10$ and $v + e = 72s + 31$. Define

$$b_{2i} = \begin{cases} i+5 & 1 \leq i \leq 6s \text{ and } m \not\equiv 0(\bmod 3) \\ i-1 & 1 \leq i \leq 6s+3 \text{ and } m \equiv 0(\bmod 3) \\ 12s+5 & i = 6s+1 \\ 6s+5 & 6s+2i = 6s+2 \\ i+2 & 6s+4 \leq i \leq 9s+1 \\ 15s+8 & i = 9s+2 \\ i+1 & 9s+3 \leq i \leq 12s+3 \\ 4 & i = 12s+4 \\ 3 & i = 12s+5. \end{cases}$$

Subcase 3: $m \equiv 2(\bmod 3)$. Set $m = 3s + 2$ ($s \geq 1$). Thus, $n = 24s + 18$ and $v + e = 72s + 55$. Define

$$b_{2i} = \begin{cases} 6 & i = 1 \\ i+7 & 2 \leq i \leq 6s+4 \text{ and } i \equiv 2(\bmod 3) \\ 2 & i = 3 \\ i+1 & 4 \leq i \leq 6s+4 \text{ and } i \not\equiv 2(\bmod 3) \\ 12s+9 & i = 6s+5 \\ i+6 & 6s+6 \leq i \leq 9s+2 \text{ and } i \equiv 0(\bmod 3) \\ i & 6s+6 \leq i \leq 9s+4 \text{ and } i \not\equiv 0(\bmod 3) \\ 9s+7 & i = 9s+3 \\ 15s+13 & i = 9s+5 \\ 9s+5 & i = 9s+6 \\ i+1 & 9s+7 \leq i \leq 12s+7 \\ 4 & i = 12s+8 \\ 3 & i = 12s+9. \end{cases}$$

Exercise 2.12 *The* Petersen graph P *consists of two* 5*-cycles* $x_1x_2x_3x_4x_5$ *and* $y_1y_3y_5y_2y_4$*, together with the five edges* $x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5$*. What is the range of possible magic sums for an edge-magic total labeling of* P*? Prove that* P *is edge-magic.*

Exercise 2.13 *A* fan F_n *is constructed from a wheel* W_n *by deleting one arc. Prove that all fans are edge-magic.*

Research Problem 2.12 *A* helm H_n *is constructed from a wheel* W_n *by adding* n *vertices of degree 1, one adjacent to each terminal vertex. Which helms are edge-magic?*

Research Problem 2.13 *A* flower F_n *is constructed from a helm* H_n *by joining each vertex of degree 1 to the center. Which flowers are edge-magic?*

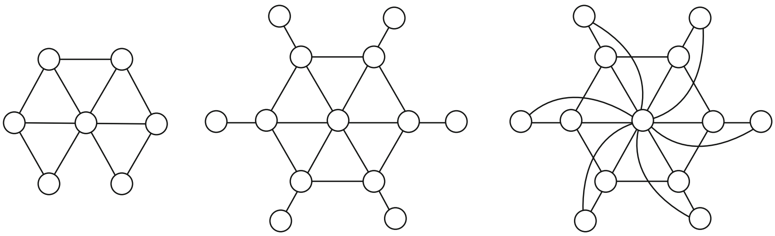


Fig. 2.8. A fan, a helm and a flower

2.7 Trees

It has been conjectured ([54], also [82]) that all trees are edge-magic. However, this seems to be a difficult problem.

We have already seen that stars and paths are edge-magic. A *caterpillar* is a graph derived from a path by hanging any number of leaves from the vertices of the path, so it can be seen as a sequence of stars where each star shares one edge with the next one. We shall show that all caterpillars are edge-magic.

The typical caterpillar is a graph $G = S^1 \cup S^2 \cup \ldots \cup S^n$, where S^i is a star with center c_i and e_i edges, and in every case S^i shares an edge with S^{i+1}. Then G has $e = \sum_1^n e_i - n + 1$ edges ($n - 1$ edges are shared by two stars), and $v = \sum_1^n e_i - n + 2$ vertices. c_i will be a leaf in S^{i-1} (unless $i = 1$) and in S^{i+1} (unless $i = n$), or equivalently the leaves of S^i will include c_{i-1} and c_{i+1}.

Theorem 2.29 [54] *All caterpillars are edge-magic.*

Proof. We describe an edge-magic total labeling λ of the caterpillar G described above. First the stars are ordered $S^1, S^3, S^5, \ldots, S^2, S^4, S^6, \ldots$ and then the leaves of the stars are labeled with the smallest positive integers, starting from 1 as a label on S^1 and ascending. When the leaves of S^i are labeled, c_{i-1} receives the smallest label (except when $i = 1$) and c_{i+1} the largest one. So the leaves of S^1 receive $1, 2, \ldots, e_1$,, with $\lambda(c_2) = e_1$, then the vertices of S_3 receive $v_1, v_1 + 1, \ldots, e_1 + e_3 - 1$, with $\lambda(c_2) = e_1$ and $\lambda(c_2) = e_1 + e_3 - 1$, and so on. This uses labels $1, 2, \ldots, \sum_1^n e_i - n + 2 = v$. Then the edges are labeled. The smallest available labels, namely $v+1, v+2, \ldots, v+e_n$, are applied to the edges of S^n, then the next e_{n-1} to the edges of S^{n-1}, and so on until the edges of S^1 are labeled. In each star, the smallest label is given to the edge whose perimeter vertex has the largest label, and so on. The labeling is illustrated in Fig. 2.9.

Verification that the labeling has the edge-magic property is an easy exercise. □

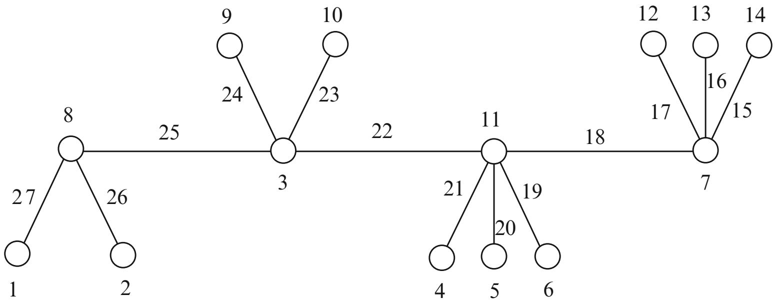


Fig. 2.9. Labeling a caterpillar

Exercise 2.14 *Verify that the labeling described in the proof of Theorem 2.9 is edge-magic.*

This exhausts our systematic knowledge of edge-magic total labelings of trees. However, Enomoto et al. [25] carried out a computer check and showed that all trees with fewer than 16 vertices are edge-magic. In fact, labelings were easy to find: they found that every tree on v vertices ($v \leq 16$) had a super edge-magic labeling.

Research Problem 2.14 *Are all trees edge-magic?*

2.8 Disconnected Graphs

2.8.1 Some Easy Cases

Kotzig and Rosa [54] showed that the one-factor F_{2n}, consisting of n independent edges, is edge-magic if and only if n is odd (see Exercise 2.15), and nK_4 is not edge-magic for n odd (see Exercise 2.1).

Exercise 2.15 *Prove that the one-factor F_{2n}, consisting of n independent edges, is edge-magic if and only if n is odd.* [54]

Another interesting type of disconnected graph is a cycle plus one disconnected edge ($K_2 \cup C_n$). In the first edition of this book, we proposed investigating whether such graphs are edge-magic. Park et al. proved something stronger:

Theorem 2.30 [50, 79] *$K_2 \cup C_n$ is super edge-magic if and only if n is even.*

The proof is lengthy, with separate analyses according to the value of n modulo 12.

Exercise 2.16 *Prove that $K_2 \cup C_3$ is not edge-magic, but $K_2 \cup C_4$ is edge-magic.*

Research Problem 2.15 *Is $K_2 \cup C_n$ edge-magic for n odd?*

2.8.2 Unions of Complete Graphs

In [70], edge-magic labelings of unions of triangles were discussed. Denote the vertices of the i-th triangle as x_i, y_i, z_i and the edges as X_i, Y_i, Z_i.

Exercise 2.17 *Show that for a component i of an edge-magic sC_3, $X_i - x_i = Y_i - y_i = Z_i - z_i$. This value will be referred to as the common difference.*

The following two theorems give constructions for edge-magic labelings of sC_3 for s odd and even, respectively.

Theorem 2.31 *Let s be an odd positive integer. There exists an EMTL of sC_3 such that the labels $1, 2, \ldots, s$ occur on the vertices of different components and such that the common difference of every component is $3s$. The magic sum of this EMTL is $k = \frac{1}{2}(21s + 3)$.*

Proof. Since every common difference is $3s$, we need to only declare the set $\{x_i, y_i, z_i\}$ of edge labels for each component $i = 1, 2, \ldots, s$. These triples must partition the set $\{1, 2, \ldots, 3s\}$ such that the sum of the elements in any triple is constant. An explicit formula is found by letting $s = 2q - 1$ and $x_i = i$, $y_i = 4q - 2i$ if $1 \leq q$ and $y_i = 6q - 1 - 2i$ if $q + 1 \leq i \leq s$. Let $z_i = 5q - 3 + i$ if $1 \leq i \leq q$ and $z_i = 3q - 2 + i$ if $q + 1 \leq i \leq s$. This can easily be verified to be edge-magic. $\square$

Theorem 2.32 *Let $s = 2t$ be a positive even integer. If $t \geq 3$ there exists an EMTL of sC_3 such that the labels $1, 2, \ldots, s-2$ occur on the vertices of different components and such that the common difference of each of these components is $3s$. The magic sum is $k = 21t - 1$.*

Proof. The two components without a common difference of $3s$ are labeled edge-vertex-edge-vertex-edge-vertex as follows: $[2t, 9t, 9t-1, 3t, 8t, 10t-1]$ and $[3t-1, 6t, 11t, 4t-1, 5t, 12t]$. For the remaining components, we will list only the set of edges for each component. This is sufficient as every common difference is $d = 3s$. We distinguish between two cases:

Case 1: t is odd. Let $t = 2r + 1$. The edge sets are:

$$\begin{array}{ll}
\{i, 5t+i, 4t-1-2i\}, & \text{for } 1 \leq i \leq r-1 \\
\{r, 4t, 4t+r\} \quad \text{and} & \{r+1, 4t-2, 4t+r+1\}, \\
\{i, 5t-2+i, 4t+1-2i\}, & \text{for } r+2 \leq i \leq t+1 \\
\{i, 3t-1+i, 6t-2i\}, & \text{for } t+2 \leq i \leq t+r \\
\{i, 3t+1+i, 6t-2-2i\}, & \text{for } t+r+1 \leq i \leq 2t-2.
\end{array}$$

Case 2: t is even. Let $t = 2r$. The edge sets are:

$$\begin{array}{ll}
\{i, 5t+i, 4t-1-2i\}, & \text{for } 1 \leq i \leq r-1 \\
\{r, 4t, 4t+r-1\} \quad \text{and} & \{r+1, 4t-2, 4t+r\}, \\
\{i, 5t-2+i, 4t+1-2i\}, & \text{for } r+2 \leq i \leq t+1 \\
\{i, 3t-1+i, 6t-2i\}, & \text{for } t+2 \leq i \leq t+r-1 \\
\{i, 3t+1+i, 6t-2-2i\}, & \text{for } t+r \leq i \leq 2t-2.
\end{array}$$

Since the common difference is $3s$, the edge-magic property can be verified by checking that the sum of the edge labels on each component is $9t - 1$. $\square$

Exercise 2.18 *Show that $2C_3$ does not have an edge-magic labeling.*

The following lemma can be used to give other possible values for the magic sum.

Lemma 2.33 *Let s be a positive integer and let λ_0 be an EMTL of sC_3 with magic sum of k. Assume that one of the components has edge labeled 1 and a common difference of $3s$. Then there exists another EMTL λ_1 of sC_3 with a magic sum of $k-3$.*

Proof. Let T be a component of sC_3 as in the statement of the lemma. Thus, λ_0 labels T as $[1, x+3s, y, 1+3s, x, y+3s]$ for some x, y. Define $\lambda_1(z) = \lambda_0 - 1$ for any vertex or edge z which is not in T. Clearly $wt_{\lambda_1(v)} = k-3$ for any vertex v not in T. Let λ_1 label T as $[6s, x-1, y+3s-1, 3s, x+3s-1, y-1]$. The fact that λ_1 is a total labeling is easily checked, noting that the label 1 was replaced by $6s$. Also, one can easily check the magic property holds. □

Using Lemma 2.33, other values for the magic sum can be found for sC_3 when s is both even and odd. In fact, when s is even, s other possible magic sums are found and when s is odd, $s-2$ other magic sums are found. In [70], the exact spectrum of sC_3 is found when $s = 2 \cdot 3^k$ with $k \geq 1$. Small cases of sC_3 have been checked by computer, and it was shown that $3C_3$ does not have an EMTL with magic constant 25, 26, 31, or 32. Furthermore, there is no EMTL for $4C_3$ with a magic constant of 32 or 43. But, an EMTL was found for each of the feasible values for $5C_3$. The spectrum of sC_3 has not been studied for $s \geq 6$, but the following conjecture was made.

Research Problem 2.16 *Prove or disprove the conjecture that for each $s \geq 5$, the spectrum of sC_3 coincides with its range of feasible values.*

Research Problem 2.17 *Prove or disprove that nK_4 is edge-magic when n is even.*

2.8.3 Unions of Stars

Suppose G is the union of two stars, $G = K_{1,m} \cup K_{1,n}$. Then $v(G) = m+n+2$ and $e(g) = m+n$. If m and n are both odd, then all the vertices have odd degree, $e(G)$ is even and $e(G)+v(G) = 2m+2n+2 \equiv 2 \pmod 4$, so by Corollary 2.3.1 G is not edge-magic. So let us assume n is even, say $n = 2t$.

Theorem 2.34 [82] *The graph $K_{1,m} \cup K_{1,2t}$ is edge-magic for any positive integers m and t.*

Proof. Denote the vertices of $K_{1,m}$ by x_0 (center), $x_1, x_2, \ldots, x_m$, and those of $K_{1,n}$ by y_0 (center), $y_1, y_2, \ldots, y_{2t}$. Then define a function λ by

$$\begin{array}{lll}
\lambda(x_0) & = 2 + 2m + 3t, & \\
\lambda(x_i) & = i & \text{for } 1 \le i \le m, \\
\lambda(y_0) & = 1 + m + t, & \\
\lambda(y_j) & = m + j & \text{for } 1 \le i \le t, \\
\lambda(y_j) & = 1 + m + j & \text{for } t + 1 \le i \le 2t, \\
\lambda(x_0 x_i) & = 2 + 2m + 2t - i & \text{for } 1 \le i \le m, \\
\lambda(y_0 y_j) & = 3 + 2m + 4t - j & \text{for } 1 \le j \le t, \\
\lambda(y_0 y_j) & = 2 + 2m + 4t - j & \text{for } t + 1 \le j \le 2t.
\end{array}$$

This is seen to be an edge-magic labeling with sum $4 + 4m + 5t$. □

(This construction is taken from [47].)

In fact, it is shown in [47] that $K_{1,m} \cup K_{1,n}$ is super-edge magic if and only if either m is a multiple of $n + 1$ or n is a multiple of $m + 1$.

It follows from Theorem 2.34 that the union $K_{1,1} \cup K_{1,2}$ of the two trivial stars P_2 and P_3 is edge-magic. We also know that the union of m copies of P_2 is edge-magic if and only if m is odd (see Exercise 2.15). More generally, we have

Theorem 2.35 [47] *Provided $n \ge 1$, $mP_2 + nP_3$ is always edge-magic.*

The proof consists of exhibiting labelings in several cases. It is further shown in [47] that all these graphs are super edge-magic, except for $2P_3$.

Exercise 2.19 *Verify that $2P_3$ has no super-magic edge labeling.*

2.8.4 Trichromatic Graphs

Graph colorings were defined and discussed briefly in Chap. 1. Formally, suppose $C = \{c_1, c_2, \ldots\}$ is a set of undefined objects called *colors*. A *C-coloring* (or C-vertex coloring) ξ of a graph G is a map

$$\xi : V(G) \to C.$$

The sets $V_i = \{x : \xi(x) = c_i\}$ are called *color classes*. A *proper coloring* of G is a coloring in which no two adjacent vertices belong to the same color class. In other words,

$$x \sim y \Rightarrow \xi(x) \ne \xi(y).$$

A proper coloring is called an *n-coloring* if C has n elements. If G has an n-coloring, then G is called *n-colorable*. 2-colorable and 3-colorable graphs are also called *bipartite* and *trichromatic*, respectively.

A *total coloring* is an assignment ξ of colors to the vertices and edges of a graph G. A total coloring is called *proper* if the colors on a vertex and all edges that touch it contain no repeats. We shall refer to a graph with a proper total coloring in three colors as *totally trichromatic*. Ivančo and Lučkaničová [47] subsequently redefined total 3-colorings, calling them *e-m-colorings*.

Lemma 2.36 *Every 3-colorable graph is totally trichromatic.*

Proof. Suppose G is a 3-colorable graph. Select a proper 3-coloring $\xi : V(G) \to \{1, 2, 3\}$. Then define a 3-total coloring $\eta : V \cup E \to \{1, 2, 3\}$ by

$$\text{if } x \in V, \eta(x) = \xi(x)$$
$$\text{if } x \sim y, \{\eta(x), \eta(y), \eta(xy)\} = \{1, 2, 3\}.$$

□

Theorem 2.37 [52] *Say G is a 3-colorable edge-magic graph and H is the union of t disjoint copies of G, t odd. Then H is edge-magic.*

Proof. Suppose G has a proper total 3-coloring η, as guaranteed by Lemma 2.36, and suppose λ is an edge-magic total labeling of G with magic sum k. Denote the copies of G by $G_0, G_1, \ldots, G_{2r}$, where $t = 2r + 1$, and write s for $v + e$. Write $A = a_{ij}$ for the matrix (1.2) that was constructed in Sect. 1.1.3. Then vertex x of G_i receives label

$$\lambda(x) + sa_{\eta(x),i}$$

and edge xy of G_i receives label

$$\lambda(xy) + sa_{\eta(xy),i}.$$

This is an edge-magic total labeling with magic sum $3sr + k$. □

The trichromatic graphs include all cycles and paths, so tC_v and tP_v are edge-magic for all odd t. (This was later independently proven by Wijaya and Baskoro [99], who gave a direct construction.)

The wheel W_n is trichromatic for n even. The union of an odd number of W_n's is not edge-magic when $n \equiv 3 (\bmod 4)$ (see Exercise 2.2), but the case $n \equiv 1 (\bmod 4)$ remains in doubt.

Research Problem 2.18 *Is tW_n edge-magic when $n \equiv 1 (\bmod 4)$ and t is odd?*

The single edge K_2 is of course trichromatic, so half of Exercise 2.15 follows from Theorem 2.37.

2.9 Super Edge-Magic Total Labelings

Recall that a $(V\text{-})$super edge-magic total labeling is one in which the vertex-labels are the integers $1, 2, \ldots, v$. As we noticed in Sect. 2.1.2, equation (2.3) can sometimes be used to show that a graph is not super edge-magic: if vertex x_i has degree d_i and is to receive label a_i, it is necessary to find an arrangement $\{a_i\}$ of the first v integers that makes $\sigma_0^{v+e} + \sum (d_i - 1)a_i$ divisible by e. We used this to show that the even cycles are not super edge-magic. However, Theorem 2.18 provides a super edge-magic labeling of every odd cycle. We have already observed (in Sect. 2.4.2) that all paths are super edge-magic.

Exercise 2.20 *Prove that all edge-magic one-factors are super edge-magic.*

Suppose G has a super edge-magic total labeling λ. Excluding the trivial case where G has no edges, some edge receives label $v + e$, so the weight of that edge is at least $1 + 2 + (v + e)$. On the other hand, the edge with label $v + 1$ will have weight at most $(v - 1) + v + (v + 1)$. So

$$1 + 2 + (v + e) \leq k \leq (v - 1) + v + (v + 1). \tag{2.21}$$

So we have

Lemma 2.38 [25] *Any super edge-magic graph other than K_1 satisfies*

$$e \leq 2v - 3. \tag{2.22}$$

This simple lemma has a number of consequences. For example, no wheel is super edge-magic, nor is any complete graph with more than three vertices.

Lemma 2.38 also implies that no regular graph of degree greater than 3 can be super edge-magic. Suppose G is a regular graph of degree 3 with an edge-magic labeling λ. Necessarily v is even, say $v = 2n$. Then $e = 3n$. From equation (2.5), $6kn = 4s + 5n(5n + 1)$ where s is the sum of the vertex labels. If the labeling is super, $s = n(2n + 1)$, so $6k = 4(2n + 1) + 5(5n + 1) = 33n + 9$, so n must be odd. The first case is $n = 3, v = 6$.

There are two cubic graphs on six vertices. One is the triangular prism, which is super edge-magic (one triangle receives labels $1, 2, 3$, and the other receives labels $5, 6, 4$ in the corresponding places). The other, $K_{3,3}$, has no super edge-magic total labeling, as was seen in the complete search in Sect. 2.5.1. In fact, this is an instance of a stronger result:

Theorem 2.39 [25] *The complete bipartite graph $K_{m,n}$ is super edge-magic if and only if $m = 1$ or $n = 1$.*

Proof. One of the labelings in Theorem 2.27 is super, so the "if" part is easy. Now assume $m \geq n > 1$. From Lemma 2.38, a super edge-magic $K_{m,n}$ must satisfy $mn \leq 2(m+n) - 3$, or $(m-2)(n-2) \leq 1$. So we only need to investigate cases $m = n = 3$ and $m = 2$.

We know from the complete enumerations that $K_{2,2}$ and $K_{3,3}$ are not super edge-magic. So assume $m = 2, n \geq 3$. (2.21) gives $3n + 5 \leq k \leq 3n + 6$, and these two cases are duals of each other. So let us assume that $K_{2,n}$ has a super edge-magic total labeling with $k = 3n + 6$. Denote the two vertex sets as U and W.

The largest (edge) label is $3n + 2$, so x_1x_2 cannot be an edge. Say x_1 and x_2 belong to U. To accommodate the edge with label $3n + 1$, x_1x_3 must be an edge, so $x_3 \in W$. Then x_2x_3 is the edge with label $3n + 1$, so x_1x_4 is not an edge, and $x_4 \in U$. At this stage we see that the only possible edge with label $3n$ is x_1x_5. But then $x_5 \in W$, and the $K_{2,n}$ contains two edges with label $3n - 1$, namely x_2x_5 and x_3x_4. □

Exercise 2.21 *Prove that a graph G is super edge-magic if and only if there is a map λ from $V(G)$ onto $\{1, 2, \ldots, v\}$ such that*

$$\{\lambda(x) + \lambda(y) \mid xy \in E(G)\} \tag{2.23}$$

is a set of consecutive integers. [29]

Research Problem 2.19 *Which unions of disjoint cycles are super edge-magic?*

Exercise 2.22 *Suppose G is a bipartite graph with vertex-sets V_1 and V_2, of sizes v_1 and v_2, respectively. An edge-magic total labeling of G is* super-strong *if the elements of V_1 receive labels $\{1, 2, \ldots, v_1\}$ and the elements of V_2 receive $\{v_1 + 1, v_1 + 2, \ldots, v\}$. Prove that every super-strongly edge-magic bipartite graph satisfies $e \leq v - 1$.* [78]

Recall that an $(n, 2)$-kite consists of a cycle of length n with a 2-edge path attached to one vertex. Consider such a kite with vertices $x_1, x_2, \ldots, x_n, y, z$, where $(x_1, x_2, \ldots, x_n)$ is the cycle and x_n, y, z is the tail.

Theorem 2.40 [79] *An $(n, 2)$-kite has a super edge-magic labeling if and only if n is even.*

Proof. Suppose λ is such a labeling, with magic sum k. If the sum of the vertex labels is V and the sum of the edge labels is E, then

$$\begin{aligned} V = \sum_{i=1}^{n+2} i &= \tfrac{1}{2}(n+2)(n+3), \\ E = \sum_{i=n+3}^{2n+4} i &= \tfrac{1}{2}(2n+4)(2n+5) - V \end{aligned}$$

and the sum of the weights of all the edges is

$$\begin{aligned} k(n+2) &= 2V + E + \lambda(x_n) - \lambda(z) \\ &= \tfrac{1}{2}(n+2)(n+3) + \tfrac{1}{2}(2n+4)(2n+5) + \lambda(x_n) - \lambda(z). \end{aligned}$$

So

$$\begin{aligned} k &= \frac{n+3}{2} + (n+2)(2n+5) + \frac{\lambda(x_n) - \lambda(z)}{n+2} \\ &= \text{an integer} + \frac{n+3}{2} + \frac{\lambda(x_n) - \lambda(z)}{n+2}. \end{aligned}$$

Now $\lambda(x_n) - \lambda(z)$ is nonzero and less than $n+2$, and k is an integer. The only possibility is that $\frac{\lambda(x_n)-\lambda(z)}{n+2} = \frac{1}{2}$, and that n is even.

Now assume n is even; put $n = 2m + 2$, where m is a nonnegative integer. If m is odd, define a labeling λ with vertex labels

$$\lambda(y) = (3m+7)/2,\ \lambda(z) = (3m+5)/2,\ \lambda(x_n) = (m+1)/2,$$

and

$$\lambda(x_i) = \begin{cases} (m-i+1)/2 & \text{for } i = 2, 4, \ldots, m-1 \\ (3m-i+4)/2 & \text{for } i = 1, 3, \ldots, m \\ (3m-i+3)/2 & \text{for } i = m+1, m+3, \ldots, 2m \\ (3m-i+10)/2 & \text{for } i = m+2, m+4, \ldots, 2m+1, \end{cases}$$

and edge labels

$$\lambda(x_n y) = 3m+7,\ \lambda(yz) = 2m+5,\ \lambda(x_1 x_n) = 3m+6,$$
$$\lambda(x_m x_{m+1}) = 3m+8,\ \lambda(x_n x_1) = 3m+9,$$

and

$$\lambda(x_i x_{i+1}) = \begin{cases} 3m+i+9 & \text{for } 1 \le i \le m-1 \\ m+i+5 & \text{for } m+1 \le i \le 2m. \end{cases}$$

For odd m, use

$$\lambda(y) = (3m+8)/2,\ \lambda(z) = (3m+4)/2,\ \lambda(x_n) = m/2,$$
$$\lambda(x_{m+1}) = m,\ \lambda(x_{n-1}) = (3m+6)/2,$$

and

$$\lambda(x_i) = \begin{cases} (m+i)/2 & \text{for } i = 2, 4, \dots, m-2 \\ (3m+i+9)/2 & \text{for } i = 1, 3, \dots, m-1 \\ (m+i+2)/2 & \text{for } i = m, m+2, \dots, 2m \\ (i-m-10)/2 & \text{for } i = m+3, m+5, \dots, 2m-1, \end{cases}$$

and edge labels

$$\lambda(x_n y) = 3m+7,\ \lambda(yz) = 2m+5,\ \lambda(x_1 x_n) = 3m+8,$$
$$\lambda(x_{m-1}x_m) = 2m+6,\ \lambda(x_n x_1) = 3m+6,\ \lambda(x_{n-2}x_{n-1}) = 2m+7,$$

and

$$\lambda(x_i x_{i+1}) = \begin{cases} 3m+i+6 & \text{for } 1 \le i \le m-2 \\ 4m-i+10 & \text{for } i = m, m+1 \\ 5m-i+10 & \text{for } m+2 \le i \le 2m. \end{cases}$$

In both cases λ is a super edge-magic labeling with sum $5m+11$. □

Finally, why have we restricted ourselves to the case of V-super edge-magic labelings, and ignored the existence of E-super edge-magic labelings? The answer is that the two problems are equivalent. Suppose λ is a V-super edge-magic labeling, with magic sum k, of a graph G with v vertices and e edges. For each vertex x, define $\mu(x) = \lambda(x) + e$, and for each edge xy, define $\mu(xy) = \lambda(xy) - v$. Then

$$\mu(x) + \mu(y) + \mu(xy) = \lambda(x) + \lambda(y) + \lambda(xy) + 2e - v,$$

and μ is an E-super edge-magic labeling of G with magic sum $k + 2e - v$. The two problems are not the same for vertex-magic, however; see Sect. 3.10.

2.10 A Cycle with a Chord

The graph constructed by adding a chord to a cycle always satisfies Lemma 2.38. We could ask wonder whether all such graphs are super edge-magic. The short answer is "no"—an exception is the graph formed from a six-cycle by joining two vertices of distance 2. But maybe this is the only exception. We shall provide solutions for all cases of an odd cycle with a chord, and the majority of cases of an even cycle with a chord. This work comes from [64].

We shall write C_v^t to mean the graph constructed from a C_v by joining two vertices whose distance in the cycle is t. In this notation, we have just remarked that C_6^2 is not super edge-magic.

Suppose C_v^t has a super edge-magic total labeling, and suppose the endpoints of the chord receive labels a and b. Then (2.3) is

$$(v+1)k = 2(1+2+\cdots+v)+a+b+((v+1)+(v+2)+\cdots+(2v+1)) \quad (2.24)$$

so $v+1$ must divide $a+b+\delta(v+1)$, where δ is 0 when v is even and $\frac{1}{2}$ when v is odd.

When v is even, say $v = 2n$, we have $a + b \equiv 0(\bmod 2n + 1)$, so $a + b = 0$ or $2n+1$ or $4n+2\ldots$ and the condition $0 \le a, b \le 2n$ implies $a+b = 2n+1$. From (2.24), $k = 2n + 1$ and the set of sums (2.23) is

$$\{n+1, n+2, \ldots, 2n+1, \ldots, 3n+1\}.$$

For odd $v = 2n + 1$ we get $a + b = n + 1$ or $3n + 3$ yielding sums

$$\{n+1, n+2, \ldots, 3n+2\} \text{ or } \{n+2, n+2, \ldots, 3n+3\},$$

respectively. In this case the vertex labels that induce a super edge-magic total labeling of any C_v^t will also induce such a labeling of C_v.

It may well happen that one vertex labeling of C_v provides super edge-magic total labelings of C_v^t for all possible chord lengths t. A labeling with this desirable property will be called *universal*.

2.10.1 Odd Cycles

A chord in C_v of length t is of course also a chord of length $v - t$. This means that we can restrict our attention to chords of length at most $\frac{1}{2}v$, or, in the case of odd cycles, to chords of odd length.

It is easy to see that the labeling of C_{2n+1} constructed in Theorem 2.18 provides chords of all lengths t where $t \equiv 3(\bmod 4)$, so C_v^t is super edge-magic for every $t \equiv 3(\bmod 4)$, when v is odd. However, we have constructed better labelings.

Theorem 2.41 *The following vertex labeling λ of C_{4m+3} is universal:*

i *values*	$\lambda(x_i)$
i *even,* $2 \le i \le 2m$	$2m+3+\frac{1}{2}i$
i *even,* $2m+2 \le i \le 4m+2$	$\frac{1}{2}(i+2)$
i *odd,* $1 \le i \le 2m+1$	$\frac{1}{2}(i+1)$
i *odd,* $2m+3 \le i \le 4m+1$	$2m+\frac{1}{2}(i+5)$
$i = 4m+3$	$2m+3$

Proof. It is easy to verify that λ yields edges with vertex-weights $\{2m+3, 2m+4, \ldots, 6m+5\}$. So it induces a super edge-magic labeling of C_{4m+3}. To prove that it is a super edge-magic labeling of C_{4m+3}^t it suffices to exhibit a chord of length t with vertex-weight $2m+2$ or $6m+6$.

For $1 \leq i \leq m$, $\lambda(x_{2i-1}) = i$ and $\lambda(x_{4m+2-2i}) = 2m + 2 - i$. So the chord $x_{2i-1}x_{4m+2-2i}$ of length $4(m - i) + 3$ has vertex-weight $2m + 2$. This covers chords of lengths $\{3, 7, \ldots, 4m - 1\}$.

For $1 \leq i \leq m - 1$, $\lambda(x_{2i}) = 2m + 3 + i$ and $\lambda(x_{4m+1-2i}) = 4m + 3 - i$. So the chord $x_{2i}x_{4m+1-2i}$ of length $4(m - i) + 1$ has vertex-weight $6m + 6$. This covers chords of lengths $\{5, 9, \ldots, 4m - 3\}$.

Finally, $x_{4m+1}x_{4m+3}$ is a chord of length $4m + 1$ whose vertex-weight is $\lambda(x_{4m+1}) + \lambda(x_{4m+3}) = (4m + 3) + (2m + 3) = 6m + 6$. So the construction covers all lengths. □

No one has yet found a universal labeling for all cycles C_{4m+1}. We present two theorems, one of which has chords with the appropriate weight of all odd lengths except 5 and 9 (and, consequently, all even lengths except $4m - 4$ and $4m - 8$), and another for chords of all lengths congruent to 1 (or 0) modulo 4 other than $4m - 3$ (or 4). So all odd cycles with chords are super edge-magic.

It should be noted that the constructions fail when $m < 3$. However, universal labelings of C_5 and C_9 exist; see Fig. 2.10.

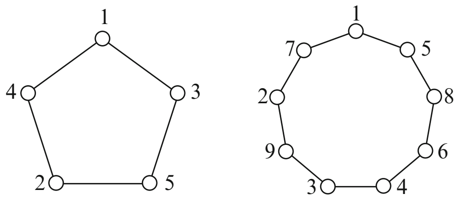


Fig. 2.10. Universal labelings of C_5 and C_9

Theorem 2.42 *The following vertex labeling λ induces a super edge-magic labeling of C_{4m+1}^t for every t except $t = 5, 9, 4m - 4, 4m - 8$, other than $m \geq 3$:*

i *values*	$\lambda(x_i)$
i *even*, $2 \leq i \leq 2m - 2$	$2m + 4 + \frac{1}{2}i$
i *even*, $2m \leq i \leq 4m$	$\frac{1}{2}i + 2$
i *odd*, $1 \leq i \leq 2m + 1$	$\frac{1}{2}(i + 1)$
i *odd*, $2m + 3 \leq i \leq 4m - 3$	$2m + \frac{1}{2}(i + 5)$
$i = 4m \pm 1$	$\frac{1}{2}(i + 7)$

Proof. Again it is easy to verify that λ induces a super edge-magic labeling of C_{4m+1}. To prove that it is a super edge-magic labeling of C_{4m+1}^t, it suffices to exhibit a chord of length t with vertex-weight $2m + 1$ or $6m + 3$.

For $1 \le i \le m-1$, $\lambda(x_{2i-1}) = i$ and $\lambda(x_{4m-2-2i}) = 2m+2-i$. So the chord $x_{2i-1}x_{4m-2-2i}$ of length $4(m-i)-1$ has vertex-weight $2m+1$. This covers chords of lengths $\{3, 7, \ldots, 4n-5\}$.

For $1 \le i \le m-5$, $\lambda(x_{2m+2i+1}) = 3m+3+i$ and $\lambda(x_{2m-8-2i}) = 3m-i$. So the chord $x_{2m+2i+1}x_{2m-8-2i}$ of length $4i+9$ has vertex-weight $6m+3$. This covers chords of lengths $\{13, 17, \ldots, 4m-11\}$.

Finally, we have the following chords: $x_{2m-1}x_{2m+1}$, length $4m-1$, vertex-weight $m+(m+1) = 2m+1$; $x_{4m-7}x_{4m+1}$, length $4m-7$, vertex-weight $(2m+4)+(4m-1) = 6m+3$; and $x_{4m-1}x_{4m-5}$, length $4m-3$, vertex-weight $(2m+3)+4m = 6m+3$.

So the construction covers all required lengths. □

Notice that the construction fails when $m < 3$. When $m = 1$ two values are assigned to $\lambda(x_3)$ and $\lambda(x_5) > 5$; when $m = 2$ the value assigned to $\lambda(x_{4m-5})$ comes from the third line of the table, not the fourth line, so chord length $4m-3$ does not arise.

Theorem 2.43 *The following vertex labeling λ induces a super edge-magic labeling of C^t_{4m+1} for every $t \equiv 1 (\mathrm{mod}\ 4)$ except $4m-3$:*

i values	$\lambda(x_i)$
i even, $2 \le i \le 2m-2$	$2m+3+\frac{1}{2}i$
$i = 2m$	$m+2$
i even, $2m+2 \le i \le 4m-2$	$2m+2+\frac{1}{2}i$
$i = 4m$	$2m+2$
i odd, $1 \le i \le 2m+1$	$\frac{1}{2}(i+1)$
i odd, $2m+3 \le i \le 4m-1$	$\frac{1}{2}(i+3)$
$i = 4m+1$	$\frac{1}{2}(i+5)$

Proof. It is easy to check that λ is an edge-magic labeling with vertex labels $\{1, 2, \ldots, 4m+1\}$ whose vertex-weights are $\{2m+2, 2m+3, \ldots, 6m+2\}$. for odd i, $1 \le i \le 2m-3$, the chord $x_i x_{4m-2-i}$ has vertex-weight $2m+1$ and length $4m-4-2i$, so the labeling induces a super edge-magic labeling of $C^{4m-4-2i}_{4m+1}$. As i ranges from 1 to $2m-3$, this produces labelings for chords of lengths $5, 9, \ldots, 4m-7$. □

2.10.2 Even Cycles

Suppose there is a super edge-magic labeling of C^t_{2n}. Suppose the sums of adjacent labels are $k, k+1, \ldots, k+2n$, where the sum of the labels on the chord is $k+i$. If we add the sums for all edges of the cycle, we add the label on each vertex twice, so

$$\sum_{j=0}^{2n}(k+j) = k+i+2\sum_{j=1}^{2n} j,$$

so

$$\begin{aligned}(2n+1)k+\frac{2n(2n+1)}{2} &= 2n(2n+1)+k+i,\\ i &= 2n\left(k-\frac{2n+1}{2}\right).\end{aligned}$$

As $1 \le i \le 2n$, and as k is an integer, the only possibility is for $k-\frac{2n+1}{2}$ to equal $\frac{1}{2}$, so that $i=n$ and $k=n+1$. If a suitable labeling of the vertices is found, we may take the chord to be any pair of vertices whose labels add to $2n+1$.

For example, consider the C_8 whose vertices are labeled (sequentially) $1,4,3,8,2,6,7,5$. A chord could be added joining the vertices labeled $1,8$, giving a super edge-magic labeling of C_8^3, or $2,7$, giving a super edge-magic labeling of C_8^2. This example does not provide a super edge-magic labeling of C_8^4, but such labelings are available (an example is $1,5,3,2,8,4,7,6$).

A complete search shows that the only examples for C_4 and C_6 are $1,3,4,2$ and $1,3,2,6,4,5$. These give super edge-magic labelings of C_4^2 and C_6^3. There is no such labeling of C_6^2.

2.10.3 Some General Constructions

Theorem 2.44 *The following vertex labeling λ induces a super edge-magic labeling of C_{4m}^t for all $t \equiv 2 \pmod 4$:*

i values	$\lambda(x_i)$
i even, $2 \le i \le 2m-2$	$2m+\frac{1}{2}(i+2)$
i even, $2m \le i \le 4m$	$\frac{1}{2}(i+2)$
i odd, $1 \le i \le 2m-1$	$\frac{1}{2}(i+1)$
i odd, $2m+1 \le i \le 4m-1$	$2m+\frac{1}{2}(i+1)$

Proof. There is no problem in seeing that the vertex labels run from 1 to $4m$ and the vertex-weights run from $2m+1$ to $6m+1$, omitting $4m+1$.

When i is odd and $1 \le i \le 2m-1$, the chord $x_i x_{4m-i}$, of length $4m-2$, has vertex-weight $4m+1$. So the construction induces super edge-magic labelings of C_{4m}^t in all the required cases. □

This construction gives no other cases: the other chords with the appropriate vertex-weight duplicate these lengths.

Theorem 2.45 *The following vertex labeling λ induces a super edge-magic labeling of $C^t_{4m+2}, m > 1,$ for all odd t other than 5, and for $t = 2, 6$:*

i *values*	$\lambda(x_i)$
$i = 2$	$2m + 4$
$i = 4$	$4m + 2$
$i = 2m$	$m + 2$
$i = 2m + 2$	$m + 3$
i *even*, $6 \le i \le 2m - 2$	$2m + 2 + \frac{1}{2}i$
i *even*, $2m + 4 \le i \le 4m + 2$	$2m + \frac{1}{2}i$
i *odd*, $1 \le i \le 2m + 1$	$\frac{1}{2}(i + 1)$
i *odd*, $2m + 3 \le i \le 4m + 1$	$\frac{1}{2}(i + 5)$

Proof. Again, the vertex labels have the required values, and the vertex-weights cover all possibilities once each, with $4m + 3$ missing. We see that the chords with vertex-weight $4m + 3$ are:

$x_{2j+1}x_{4m+4-2j}$, for $j = 1, 2, \ldots, m$, producing chords of all lengths $\equiv 3 \pmod 4$;

$x_{2m+2j+1}x_{4m+2j}$, for $j = 1, 2, \ldots, m - 5$, giving chords of all lengths $\equiv 1 \pmod 4$ other than 5;

$x_{2m-2}x_{2m}$ and $x_{2m-4}x_{2m+2}$, giving chords of lengths 2 and 6. $\square$

Complete searches have been made for small cases, up to 12 vertices; see [64]. For example, there are 308 labelings for 11 vertices, covering all possible chords, and 503 for 12.

Research Problem 2.20 *For which values of t does C^t_{2n} have a super edge-magic labeling?*

Research Problem 2.21 *For which n is there a universal vertex labeling of C_{2n}?*

2.11 Edge-Magic Injections

Recall that an edge-magic injection is like an edge-magic total labeling, except that the labels can be any positive integers. We define an $[m]$-edge-magic injection of G to be an edge-magic injection of G in which the largest label is m, and call

m the *size* of the injection. The *edge deficiency* $\mathrm{def_e}(G)$ of G to be the minimum value of $m - v(G) - e(G)$, such that an $[m]$-edge-magic injection of G exists.

Theorem 2.46 *Every graph has an edge-magic injection.*

Proof. Suppose G is a graph with v vertices and e edges. The empty graph is trivially edge-magic, so we assume that G has at least one edge. Let$(a_1, a_2, \ldots, a_v)$ be any Sidon sequence of length v with first element $a_1 = 1$. Define $k = a_{v-1} + 2a_v + 1$.

We now construct a labeling λ as follows. Select any edge of G and label its endpoints with a_{v-1} and a_v, and label the remaining vertices with the other members of the Sidon sequence in any order. If xy is any edge, define $\lambda(xy) = k - \lambda(x) - \lambda(y)$. Every edge weight will be equal to k. The smallest edge label will be $k - a_{v-1} - a_v = a_v + 1$, which is greater than any vertex label. If two edge labels were equal, say $\lambda(xy) = \lambda(zt)$, then $\lambda(x) + \lambda(y) = \lambda(z) + \lambda(t)$, and as the labels of vertices are members of a Sidon sequence this implies that $xy = zt$. The vertex labels are distinct by definition. So λ is an edge-magic injection. □

The proof of Theorem 2.46 gives us an upper bound on the deficiency:

Corollary 2.46.1 *If G is a graph with v vertices and $(a_1, a_2, \ldots, a_v)$ is any Sidon sequence of length v with $a_1 = 1$, then*

$$\mathrm{def_e}(G) \le a_{v-1} + 2a_v - a_2 - v - e(G).$$

Proof. In the above construction, no label can be greater than $k - 1 - a_2$. □

This upper bound will not usually be very good. For example, consider the graph constructed from C_5 by joining two inadjacent vertices. Using the Sidon sequence $(1, 2, 3, 5, 8)$, a labeling with $k = 22$ is obtained, and the best assignment of the sequence to the vertices gives largest label 17, and deficiency 6. However, the graph is actually edge-magic. See Fig. 2.11.

Exercise 2.23 *If G is an incomplete graph with v vertices and $(a_1, a_2, \ldots, a_v)$ is any Sidon sequence of length v with $a_1 = 1$, prove that*

$$\mathrm{def_e}(G) < a_{v-1} + 2a_v - a_2 - v - e(G).$$

In the case of complete graphs, we have essentially encountered the edge-magic deficiency already:

Theorem 2.47 *The edge-magic deficiency of K_v equals the magic number $M(v)$.*

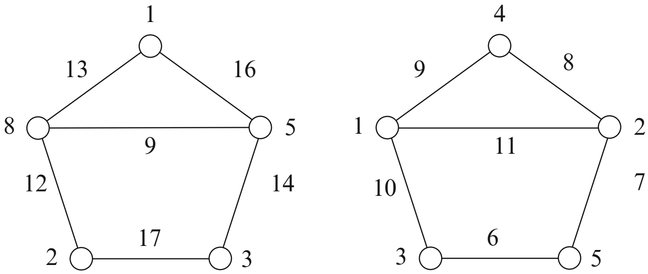


Fig. 2.11. Deficiency 6 on *left*; magic on *right*

Proof. Consider a edge-magic total labeling λ of $K_v \cup M(v)K_1$. This graph has $v + M(v)$ vertices and $e(K_v)$ edges, so the largest label is $v + M(v) + e(K_v)$, and clearly this label occurs on a vertex or edge of K_v. The labeling constructed by restricting λ to K_v is an $[v + M(v) + e(K_v)]$-edge-magic injection of K_v. Obviously any injection of size $v + m + e(K_v)$ gives rise to an edge-magic total labeling of $K_v \cup mK_1$ (apply the m unused labels to the extra vertices), so $v + M(v) + e(K_v)$ is the smallest possible size, and $def_e(K_v) = M(v)$. $\square$

From Theorem 2.47 it is clear that magic number and edge-magic deficiency are essentially equivalent, but the two concepts are rather different when applied to vertex-magic labelings.

Exercise 2.24 *Suppose G is a graph with v vertices. Prove that*

$$\mathrm{def_e}(G) \leq M(v) + \binom{v}{2} - e(G).$$

Both Corollary 2.46.1 and Exercise 2.24 give crude upper bounds for the $\mathrm{def_e}(G)$. Very little work has been done on finding good bounds for edge-magic deficiencies, and the only known families of graphs for which the exact values are known are the various families of edge-magic graphs (which of course have edge-magic deficiency 0). Further results on edge-magic injections can be found in [73].

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