Chapter 2

The topology of the non-singular level set and the variation operator of a singularity

2.1 The non-singular level set of a singularity

Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a singularity, that is the germ of a holomorphic function, with an isolated critical point at the origin. It follows from implicit function theorem that in a neighbourhood of the origin in the space \( \mathbb{C}^n \) the level set \( f^{-1}(\varepsilon) \) for \( \varepsilon \neq 0 \) is a non-singular analytic manifold and the level set \( f^{-1}(0) \) is a non-singular manifold away from the origin. At the point \( 0 \in \mathbb{C}^n \) the level set has a singular point.

Lemma 2.1. There exists \( \rho > 0 \), such that the sphere \( S_r \subset \mathbb{C}^n \) of radius \( r \leq \rho \) with centre at the origin intersects the level set \( f^{-1}(0) \) transversely.

Indeed, the function \( \|x\|^2 \) on the set \( f^{-1}(0) \) (in a neighbourhood of the point \( 0 \in \mathbb{C}^2 \)) can take only a finite number of critical values (for the case where \( f \) is a polynomial this assertion follows, for example, automatically from the ‘curve selection lemma’ of [256]; in the general case it can be derived from analogous reasoning). We choose as \( \rho \) a number such that its square is less than all critical values of the function \( \|x\|^2 \) on the manifold \( f^{-1}(0) \setminus \{0\} \). The fact that all critical values of the function \( \|x\|^2 \) on \( f^{-1}(0) \setminus \{0\} \) are greater than \( \rho^2 \) is equivalent to the fact that for \( r \leq \rho \) the sphere \( S_r \) of radius \( r \) with centre at the origin (which is a level manifold of the function \( \|x\|^2 \)) intersects the manifold \( f^{-1}(0) \setminus \{0\} \) transversely.

From Lemma 2.1 it follows that for sufficiently small \( \varepsilon_0 > 0 \) the level manifold \( f^{-1}(\varepsilon) \) is also transverse to the sphere \( S_{\varepsilon_0} \) for \( |\varepsilon| \leq \varepsilon_0 \). Thus the function \( f : \bar{B}_\rho \to \mathbb{C} \) satisfies conditions (i)–(iii) of §1.1 (with the ball \( \bar{B}_\rho \) of radius \( \rho \) with centre at the origin as \( M^n \), the disk \( \bar{D}_{\varepsilon_0} \) of radius \( \varepsilon_0 \) with centre at zero in the plane \( \mathbb{C} \) as \( U \) and the unique critical point \( 0 \)). We shall be interested in the topology of the level set \( f^{-1}(\varepsilon) \) in a neighbourhood of the origin.
**Definition.** The *non-singular level set* of the singularity \( f \) near the critical point 0 is the set

\[
V_\varepsilon = f^{-1}(\varepsilon) \cap \bar{B}_\varepsilon = \{ x \in \mathbb{C}^n : f(x) = \varepsilon, \| x \| \leq \varepsilon_0 \}
\]

for \( 0 < |\varepsilon| \leq \varepsilon_0 \), which is a complex manifold with boundary.

The manifold \( V_\varepsilon \) is defined uniquely up to diffeomorphism. It is known ([256]) that it has the homotopy type of a bouquet of spheres of dimension \((n - 1)\). The number \( \mu = \mu(f) \) of these spheres is called the *multiplicity* or *Milnor number* of the singularity \( f \). The homology group \( H_k(V_\varepsilon) \) of the non-singular level manifold is zero for \( k \neq (n - 1) \), \( H_{n-1}(V_\varepsilon) \cong \mathbb{Z}^\mu \) is a free abelian group with \( \mu \) generators. The assertions about the homology groups \( H_k(V_\varepsilon) \) of the non-singular level set we prove below (see Theorem 2.1). With a small addition (the proof of the simple connectedness of the manifold \( V_\varepsilon \), which arises from the same considerations as in Theorem 2.1), from this follows also the result on the homotopy type of the non-singular level set (for \( n > 2 \)). The fundamental group \( \pi_1(\bar{D}_{\varepsilon_0} \setminus 0) \) of the complement of the set of critical values is isomorphic to the group of integers and is generated by the class of the loop \( \gamma_0 \), which goes once round the critical value 0 in the positive direction (anticlockwise). We can, for example, set

\[
\gamma_0(t) = \varepsilon \cdot \exp(2\pi it) \quad (|\varepsilon| \leq \varepsilon_0, t \in [0, 1]).
\]

**Definition.** The *classical monodromy* \( h : V_\varepsilon \to V_\varepsilon \) of the singularity \( f \) is the monodromy \( h_{\gamma_0} \) of the loop \( \gamma_0 \). The *classical monodromy operator* of the singularity \( f \) is the automorphism \( h_\ast = h_{\gamma_0} \ast \) of the homology group \( H_{n-1}(V_\varepsilon) \) of the non-singular level set \( V_\varepsilon \). The *variation operator* of the singularity \( f \) is the variation operator

\[
\text{Var}_f = \text{var}_{\gamma_0} : H_{n-1}(V_\varepsilon, \partial V_\varepsilon) \to H_{n-1}(V_\varepsilon)
\]

of the loop \( \gamma_0 \).

A basis of the homology group \( H_{n-1}(V_\varepsilon) \cong \mathbb{Z}^\mu(f) \) of the non-singular level manifold \( V_\varepsilon \) of the singularity \( f \) can be constructed in the following manner. Let \( \widetilde{f} = f_\lambda \) be a perturbation of the function \( f \), defined in a neighbourhood of the ball \( \bar{B}_\varepsilon \) (we can, for example, take as \( \widetilde{f} \) the perturbation \( f_\lambda = f + \lambda g \), where \( g \) is a linear function: \( \mathbb{C}^n \to \mathbb{C} \)). For sufficiently small \( \lambda \) (\(|\lambda| \leq \lambda_0 \)) the level set \( \widetilde{f}^{-1}(\varepsilon) \) is transverse to the sphere \( S_\varepsilon \) for \( |\varepsilon| \leq \varepsilon_0 \) and the critical values of the function \( \widetilde{f} \) on the ball \( \bar{B}_\varepsilon \) are less than \( \varepsilon_0 \) in modulus. It is easy to show that the non-singular level set \( f^{-1}(\varepsilon) \cap \bar{B}_\varepsilon \) is diffeomorphic to the non-singular level set \( V_\varepsilon \) of the function \( f \) for \( |\varepsilon| \leq \varepsilon_0 \). From Sard's theorem it follows that almost all pertur-
bations $\tilde{f}$ of the function $f$ have in the ball $\mathcal{B}_q$ only non-degenerate critical points with distinct critical values (in the example this will take place for almost all linear functions $g$).

Let us prove, for example, that the function $\tilde{f} = f + g$ is Morse for almost all linear functions $g : \mathbb{C}^n \to \mathbb{C}$. For this we consider the mapping $df : \mathbb{C}^n \to \mathbb{C}^n$, given by the formula

$$df(x) = (\partial f / \partial x_1(x), \ldots, \partial f / \partial x_n(x))$$

($x = (x_1, \ldots, x_n) \in \mathbb{C}^n$). Almost all the values $(l_1, \ldots, l_n) \in \mathbb{C}^n$ are non-critical for this mapping (Sard's theorem). If $(l_1, \ldots, l_n) \in \mathbb{C}^n$ is a non-critical value of the mapping $df$, then the function $f - \sum_j l_j x_j$ has only non-degenerate critical points. Indeed the critical points of the function $f - \sum_j l_j x_j$ are those points at which $\partial f / \partial x_j - l_j = 0 (j = 1, \ldots, n)$, that is these are the preimages of the point $(l_1, \ldots, l_n)$ for the mapping $df$. Since the value $(l_1, \ldots, l_n)$ is non-critical for the mapping $df$, then at these points the matrix $(\partial^2 f / \partial x_j \partial x_k)$ has non-zero determinant, which means that the corresponding critical points of the function $\tilde{f} = f - \sum_j l_j x_j$ are non-degenerate. The set of non-critical values of the mapping $df$ is open. Therefore the addition to $\tilde{f}$ of a suitably small linear function does not remove it from the class of functions with non-degenerate critical points and allows us to obtain the fact that the critical values become pairwise distinct.

We again get the situation described in Chapter 1. As before, let

$$F_z = \tilde{f}^{-1}(z) \cap \mathcal{B}_q \ (|z| \leq \varepsilon_0),$$

the function $\tilde{f}$ has in the ball $\mathcal{B}_q$ several critical points $p_i$ with distinct critical values $z_i \ (|z_i| < \varepsilon_0), i = 1, \ldots, \mu$, and $\{u_i\}$ is a system of paths joining the critical values $z_i$ with the non-critical value $z_0 \ (|z_0| = \varepsilon_0)$ and defining in the homology group $H_{n-1}(F_{z_0})$ of the non-singular level set of the function $f$ a distinguished set of vanishing cycles $\{A_i\}$. Remember that the last condition means that the paths $u_i$ are not self-intersecting and pairwise do not have common points except the point $z_0$.

**Theorem 2.1.** The distinguished set of vanishing cycles $\{A_i\}$ forms a basis of the (free abelian) homology group $H_{n-1}(F_{z_0}) \cong H_{n-1}(V)$ of the non-singular level set of the singularity $f$. In particular the number of non-degenerate critical points of the function $\tilde{f}$ in $\mathcal{B}_q \cap \tilde{f}^{-1}(\mathcal{B}_{\varepsilon_0})$ (into which the critical point of the function $f$ decomposes) is equal to the multiplicity $\mu(f)$ of the singularity $f$. The group $H_k(F_{z_0})$ is zero for $k \neq (n - 1)$. 
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**Proof.** Let
\[ X = \overline{B}_q \cap f^{-1}(\overline{D}_{\varepsilon_0}) \]
\[ \tilde{X} = \overline{B}_q \cap \tilde{f}^{-1}(\overline{D}_{\varepsilon_0}), \]
where \( q > 0 \) and \( \varepsilon_0 > 0 \) as described above. We shall show that the space \( X \) is contractible. From the fact that the zero level set \( f^{-1}(0) \) of the function \( f \) is transverse to the spheres \( S_r \) of radius \( r \leq q \) with centre at the origin in the space \( \mathbb{C}^n \), it immediately follows that the set \( f^{-1}(0) \cap \overline{B}_q \) is homeomorphic to the cone over the manifold \( f^{-1}(0) \cap S_r \) and consequently contractible. The contraction of the set \( f^{-1}(0) \cap \overline{B}_q \) to the point 0, belonging to it, can be realised with the help of a vector field on it, orthogonal to the submanifolds \( f^{-1}(0) \cap S_r, r \leq q \) (remember that the set \( f^{-1}(0) \cap \overline{B}_q \) is a manifold everywhere except zero).

In its turn the space \( f^{-1}(0) \cap \overline{B}_q \) is a deformation retract of the space \( X = f^{-1}(\overline{D}_{\varepsilon_0}) \cap \overline{B}_q \). We can construct the required deformation retraction of the space \( X \), for example, in the following manner. Choose a sequence \( q = r_0 > r_1 > r_2 > \ldots > 0 \), monotonically decreasing to zero. Let \( \varepsilon_i \) be numbers such that \( \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \ldots > 0 \) and the level set \( f^{-1}(\varepsilon) \) is transverse to the sphere \( S_{r_i} \) of radius \( r_i \) with centre at zero for \( |\varepsilon| \leq \varepsilon_i \). The function \( f \) determines locally trivial, and hence also trivial, fibrations
\[ E_i = f^{-1}(\overline{D}_{\varepsilon_i}) \cap (\overline{B}_{\varepsilon_0} \setminus B_{r_i}) \rightarrow \overline{D}_{\varepsilon_i}. \]
Moreover trivialisations of these fibrations can be chosen so that they will coincide on the intersections
\[ E_i \cap E_{i-1} = f^{-1}(\overline{D}_{\varepsilon_i}) \cap (\overline{B}_{\varepsilon_0} \setminus B_{r_{i-1}}). \]

We consider the deformation \( g_t \) of the disk \( \overline{D}_{\varepsilon_0} \), defined for \( 0 \leq t \leq \varepsilon_0 \) and given by the formula
\[ g_t(x) = \begin{cases} 
  t \cdot x / \|x\| & \text{for } \|x\| \geq t, \\
  x & \text{for } \|x\| \leq t.
\end{cases} \]

The mapping \( g_t \) maps the disk \( \overline{D}_{\varepsilon_0} \) of radius \( \varepsilon_0 \) into the disk of radius \( t \), keeping the latter fixed. The mapping \( g_0 \) is a deformation retraction of the disk \( \overline{D}_{\varepsilon_0} \) into the point 0. Since the function \( f \) defines the locally trivial fibration
\[ f^{-1}(\overline{D}_{\varepsilon_0} \setminus 0) \cap \overline{B}_{\varepsilon_0} \rightarrow \overline{D}_{\varepsilon_0} \setminus 0, \]
there exists a family $G_t (0 < t \leq \epsilon_0)$ of mappings of the set $X = f^{-1}(\bar{D}_{\epsilon_0}) \cap \bar{B}_{r_o}$ into itself lifting the homotopy $g_t$. This family can be chosen in accordance with the structure of the direct product on the sets

$$E_i = f^{-1}(\bar{D}_{\epsilon_i}) \cap (\bar{B}_{r_0} \setminus B_i)$$

for $t \leq \epsilon_i$. It is not difficult to see that the family $G_t (0 < t \leq \epsilon_0)$ determines in a natural way the family $G_t$ with $0 \leq t \leq \epsilon_0$ in which the mapping $G_0$ is a deformation retraction of the set $X$ into the zero level set $f^{-1}(0) \cap \bar{B}_{r_0}$.

If $\tilde{f}$ is a sufficiently small perturbation of the function $f$, then the space $\tilde{X}$ is diffeomorphic to the space $X$ (as smooth manifolds with corners; indeed it is sufficient that $\tilde{X}$ is homotopy equivalent to $X$) and therefore is also contractible.

The function $\tilde{f}$ maps the space $\tilde{X}$ into the disk $\bar{D}_{\epsilon_0}$ and away from the critical points $z_1, z_2, \ldots, z_\mu$ is a locally trivial fibration with fibre $F_{z_0}$. We consider the union $\bigcup_{i=1}^{\mu} u_i(t) = V$ of images of paths $u_i$. It is a deformation retract of the disk $\bar{D}_{\epsilon_0}$. It is not difficult to see that a deformation retraction of the disk $\bar{D}_{\epsilon_0}$ onto the space $V$ can be lifted to a deformation retraction of the space $\tilde{X}$ onto the space $Y = \tilde{f}^{-1}(V)$ (analogous to the way that the deformation retraction of the disk $\bar{D}_{\epsilon_0}$ to the point 0 is lifted to a deformation retraction of the space $X$ to the zero level set $f^{-1}(0) \cap \bar{B}_{r_0}$). If the singular fibres $\tilde{f}^{-1}(z_i)$ are cut out from the space $Y$, then the remaining space $Y \setminus \bigcup_{i=1}^{\mu} \tilde{f}^{-1}(z_i)$ will be a fibration over the contractible space $V \setminus \{z_i | i = 1, \ldots, \mu\}$. Consequently, it is homeomorphic to the direct product of the fibre $F_{z_0}$ and the space $V \setminus \{z_i | i = 1, \ldots, \mu\}$ and therefore homotopy equivalent to the fibre $F_{z_0}$.

It is not difficult to show that up to homotopy type the space $Y$ is obtained from the fibre $F_{z_0}$ by gluing $n$-dimensional balls $B_i$ to the vanishing spheres $A_i$. Here we define the mapping in one direction, determining the homotopy equivalence of the considered spaces. Let

$$s_i(t) : S^{n-1}_{i} \rightarrow s_i(t) \subset F_{u_i(t)} \quad (0 \leq t \leq 1)$$

be a family of maps of the $(n-1)$-dimensional sphere (the index $i$ simply fixes the number of the copy), defining the vanishing cycle $A_i = s_i(1) (s_i(0) : S^{n-1}_{i} \rightarrow p_i)$. Let $B_i$ be the $n$-dimensional ball, which is the cone over the sphere $S^{n-1}_{i}$

$$(B_i = [0, 1] \times S^{n-1}_{i}/0 \times S^{n-1}_{i}).$$

The space

$$F_{z_0} \cup_{(A_i)} \{B_i\},$$
obtained from the fibre $F_{z_0}$ by gluing the $n$-dimensional balls $B_i$ to the vanishing cycles $A_i$ is the quotient space of the space

$$F_{z_0} \cup \bigcup_{i=1}^\mu B_i$$

by the equivalence relation

$$s_i(1)(a) \sim (1, a) \quad (a \in S^n, (1, a) \in B_i, i = 1, \ldots, \mu).$$

The map $\phi$ of it into the space $Y$, which is a homotopy equivalence, can be given in the following fashion:

$$\phi(x) = x \quad \text{for} \quad x \in F_{z_0} \subset Y,$$

$$\phi(t, a) = s_i(t)(a) \quad \text{for} \quad (t, a) \in B_i, \quad 0 \leq t \leq 1, \quad a \in S^n.$$

There is the exact homology sequence of the pair $(Y, Y - \bigcup_{i=1}^\mu \tilde{f}^{-1}(z_i))$:

$$\cdots \to H_{k+1}(Y) \to H_{k+1}(Y, Y - \bigcup_{i=1}^\mu \tilde{f}^{-1}(z_i)) \to H_k(Y - \bigcup_{i=1}^\mu \tilde{f}^{-1}(z_i)) \to \cdots$$

Here $H_k(Y) = 0$ (since the space $Y$ is homotopically equivalent to the contractible space $X$; remember that the homology is considered to be reduced modulo a point),

$$H_{k+1}(Y, Y - \bigcup_{i=1}^\mu \tilde{f}^{-1}(z_i)) = \bigoplus_{i=1}^\mu H_{k+1}(B_i, \partial B_i)$$

$$= \begin{cases} 0 & \text{for} \quad k \neq n - 1, \\ \mathbb{Z}^\mu & \text{for} \quad k = n - 1, \end{cases}$$

$$H_k(Y - \bigcup_{i=1}^\mu \tilde{f}^{-1}(z_i)) = H_k(F_{z_0}).$$

From the exactness of the sequence it follows that

$$H_k(F_{z_0}) \cong H_{k+1}(Y, Y - \bigcup_{i=1}^\mu \tilde{f}^{-1}(z_i))$$

$$= \begin{cases} 0 & \text{for} \quad k \neq n - 1, \\ \mathbb{Z}^\mu & \text{for} \quad k = n - 1, \end{cases}$$
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the generators of the group $H_n(B_i, \partial B_i)$ mapping into the vanishing cycles $\Lambda_i$. From this the result of the theorem follows.

It is not difficult to see that by considering the exact homotopy sequence of the pair $(Y, Y \setminus \bigcup_{i=1}^{\mu} \tilde{f}^{-1}(z_i))$ we can deduce the simple-connectedness of the space $Y \setminus \bigcup_{i=1}^{\mu} \tilde{f}^{-1}(z_i)$ or, which is the same thing, the simple-connectedness of the non-singular level set $F_{z_0}$ for $n > 2$.

From theorem 2.1 it follows that the multiplicity of the critical point of a singularity $f$ is equal to the number of non-degenerate critical points into which it decomposes under a perturbation of general form. This number is equal to the number of preimages (near zero) of a general point under the map

$$df : \mathbb{C}^n \to \mathbb{C}^n$$

$$(df(x_1, \ldots, x_n) = (\partial f/\partial x_1(x), \ldots, \partial f/\partial x_n(x)).$$

From this we can obtain the following formula for the multiplicity of an isolated critical point of a function $f$:

$$\mu(f) = \dim_{\mathbb{C}} \mathfrak{n}O/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n),$$

where $\mathfrak{n}O$ is the ring of germs at zero of holomorphic functions in $n$ variables, $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is the ideal in the ring $\mathfrak{n}O$, generated by the partial derivatives of the function $f$ (the Jacobian ideal of the germ $f$). This result was proved in Chapter 5 of Volume 1.

### 2.2 Vanishing Cycles and the Monodromy Group of a singularity

It was shown in §2.1 that almost all perturbations $\tilde{f}$ of the singularity $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are Morse, that is in a neighbourhood of zero in the space $\mathbb{C}^n$ they have only non-degenerate critical points, equal in number to the multiplicity of the singularity $f$, all critical values $z_1, \ldots, z_\mu$ of the function $\tilde{f}$ being different. The non-singular level set $V_\mu$ of the singularity $f$ is diffeomorphic to the non-singular level manifold $F_{z_0} = \tilde{f}^{-1}(z_0) \cap B_\epsilon$ of the function $\tilde{f}$. The presence of such a diffeomorphism allows us to introduce the following definition.

**Definition.** A *vanishing cycle* $\Delta$ in the homology group $H_{n-1}(V_\mu)$ of the non-singular level set of the singularity $f$ is an element of this group corresponding to a cycle in the homology group $H_{n-1}(F_{z_0})$ of the non-singular level set of the function $\tilde{f}$, vanishing along a path joining some critical value $z_i$ of the function $\tilde{f}$ with the non-critical value $z_0$.
Definition. A basis of the homology group \( H_{n-1}(V) \) of the non-singular level manifold consisting of a distinguished set of vanishing cycles \( \{ \mathcal{A}_i \} \) is called a distinguished basis. A basis consisting of a weakly distinguished set of vanishing cycles is called weakly distinguished.

Theorem 2.1 asserts that any distinguished set of vanishing cycles forms a basis. It will be shown later that any weakly distinguished set also forms a basis (see 2.6).

Remark. The terms “distinguished” and “weakly distinguished” were introduced by A.M. Gabrielov. In the work [205] a distinguished basis was called “geometrical”.

Definition. The monodromy group of the singularity \( f \) is the monodromy group of the (Morse) function \( \tilde{f} \).

It is not difficult to show that the set of vanishing cycles and the monodromy group of a singularity \( f \) do not depend on the choice of the Morse perturbation \( f = f_\lambda \) of the function \( f \). To see this we consider another such perturbation \( \tilde{f} = f'_\lambda \). The perturbations \( f_\lambda \) and \( f'_\lambda \) can be included in one two-parameter family of functions \( f_{\lambda, \nu} \) \( (f_{\lambda, 0} = f_\lambda, f_{0, \nu} = f'_\lambda) \). We can, for example, take \( f_{\lambda, \nu} = f_\lambda + f'_\lambda \) as this family. In the space \( \mathbb{C}^2 \) with coordinates \( (\lambda, \nu) \) the values of the parameters \( (\lambda, \nu) \) which correspond to non-Morse functions \( f_{\lambda, \nu} \) form (in a neighbourhood of the point \((0, 0) \in \mathbb{C}^2 \)) a set which is the image of an analytic set of complex dimension one. It does not, therefore, disconnect the space \( \mathbb{C}^2 \) of values of the parameters \( (\lambda, \nu) \). From this it follows that the perturbations \( \tilde{f} = f_\lambda \) and \( \tilde{f} = f'_\lambda \) can be joined by a continuous one-parameter family of Morse functions \( f_{\lambda(t), \nu(t)} \) \( (t \in [0, 1], f_{\lambda(0), \nu(0)} = \tilde{f}, f_{\lambda(1), \nu(1)} = \tilde{f}) \). It is easy to see that along such a family of Morse functions the set of vanishing cycles and the monodromy group do not change.

For the same reason the concepts of distinguished and weakly distinguished bases are independent of the choice of perturbation.

From the results of chapter 1 it follows that the monodromy group of a singularity \( f \) is generated by the Picard-Lefschetz operators \( h_i \) corresponding to the elements \( \mathcal{A}_i \) of a weakly distinguished basis in the homology of a non-singular level set of the function \( f \) near the critical point. If the number of variables \( n \) is odd, this operator is the reflection in a hyperplane, orthogonal (in the sense of intersection forms) to the vanishing cycle \( \mathcal{A}_i \). When, therefore, the number of variables is odd, the monodromy group of a singularity is a group generated by reflections.
Examples. As examples we can consider the functions \( f(x) = x^3 \) and \( f(x, y) = x^3 + y^2 \), which have the singularity type \( A_2 \) in the sense of volume 1. Their Morse perturbations can be chosen in the form \( \tilde{f}(x) = x^3 \pm 3\lambda x \) and \( \tilde{f}(x, y) = x^3 - 3\lambda x + y^2 \) respectively, where \( \lambda \) is a small positive number. The distinguished bases in the homology of the non-singular level manifolds and the monodromy groups of the Morse functions \( \tilde{f}(x) \) and \( \tilde{f}(x, y) \) (coinciding with the distinguished bases and monodromy groups of the singularities \( f(x) \) and \( f(x, y) \)) were considered in the examples of § 1.2.

2.3 The Variation Operator and Seifert Form of a Singularity

In § 2.1 the concept of the variation operator of a singularity was introduced. In order to study the properties of this operator we give another interpretation of it ([101], [217]).

As above let \( f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a singularity, that is the germ of a holomorphic function, with an isolated critical point at zero, let \( q \) be a sufficiently small positive number, and let \( S^{2n-1}_q \) be a sphere of radius \( q \) with centre at the origin in the space \( \mathbb{C}^n \). Put \( K = f^{-1}(0) \cap S^{2n-1}_q \). From the fact that the level set \( f^{-1}(0) \) intersects the sphere \( S^{2n-1}_q \) transversely, it follows that \( K \) is smooth submanifold of the sphere \( S^{2n-1}_q \) of codimension two. We denote by \( T \) a sufficiently small open tubular neighbourhood of the manifold \( K \) in the sphere \( S^{2n-1}_q \). We define the mapping \( \Phi: S^{2n-1}_q \setminus T \to S^1 \subset \mathbb{C} \) from the complement of the tubular neighbourhood of the manifold \( K \) to the circle by the formula \( \Phi(x) = f(x)/|f(x)| = \exp (i \arg f(x)) \). In [256] (§ 4) it is shown that the mapping \( \Phi \) is a smooth fibration. Moreover the restriction of the mapping \( \Phi \) to the boundary \( \partial (S^{2n-1}_q \setminus T) = \partial T \) has a natural structure of a trivial fibration \( K \times S^1 \to S^1 \).

The restriction of the function \( f \) to \( f^{-1}(S^1_{\varepsilon_0}) \cap \bar{B}_\varepsilon \) defines a fibration over the circle \( S^1_{\varepsilon_0} \) of radius \( \varepsilon_0 \), lying in the complex line \( \mathbb{C} \), the fibre of which is the non-singular level manifold \( V_{\varepsilon_0} = f^{-1}(\varepsilon_0) \cap \bar{B}_\varepsilon \) of the singularity \( f \). As we explained above, the restriction of the function \( f \) to the boundary \( f^{-1}(S^1_{\varepsilon_0}) \cap S^1_q \) of the manifold \( f^{-1}(S^1_{\varepsilon_0}) \cap \bar{B}_\varepsilon \) also has the structure of a trivial fibration. The classical monodromy and variation operators of the singularity are defined by way of the fibration

\[
 f^{-1}(S^1_{\varepsilon_0}) \cap \bar{B}_\varepsilon \to S^1_{\varepsilon_0}.
\]

Lemma 2.2 (see [256] § 5). The two fibrations over the circles \( S^1 \) and \( S^1_{\varepsilon_0} \) described above are equivalent (relative to the isomorphism of the circles given by
multiplication by \( \epsilon_0 \). In particular, the fibre \( \Phi^{-1}(z) \) of the fibration \( \Phi \) is diffeomorphic to the non-singular level set of the singularity \( f \) near the critical point.

In this way we can use the fibration \( \Phi \) to define the variation operator \( \text{Var}_f \) of the singularity \( f \). As before, we shall denote by

\[
\Gamma_t : \Phi^{-1}(1) \to \Phi^{-1}(\exp(2\pi it))
\]

the family of diffeomorphisms which lifts the homotopy

\[
t \mapsto \exp(2\pi it) \quad (\Gamma_0 = id, t \in [0, 1])
\]

and agree with the structure of the direct product on the boundary.

We digress a little to recall some definitions.

Let \( M \) be a (real) oriented \( n \)-dimensional manifold with boundary \( \partial M \), let \( e_1, \ldots, e_{n-1} \) be a frame in the tangent space to the boundary \( \partial M \) at some point, and let \( e_0 \) be the outward normal to the boundary \( \partial M \) in the manifold \( M \) at the same point. We say that the frame \( e_1, \ldots, e_{n-1} \) defines the orientation of the boundary \( \partial M \), if the frame \( e_0, e_1, \ldots, e_{n-1} \) is a positively oriented frame in the tangent space of the manifold \( M \). There is an analogous convention for chains and their boundaries.

Let \( a \) and \( b \) be non-intersecting \((n-1)\)-dimensional cycles in the \((2n-1)\)-dimensional sphere \( S^{2n-1} \). When \( n = 1 \) we shall suppose additionally that the cycles \( a \) and \( b \) are homotopic to zero. When \( n > 1 \) this condition is satisfied automatically. We choose in the sphere \( S^{2n-1} \) an \( n \)-dimensional chain \( A \), the boundary of which coincides with the cycle \( a \). It is easy to see that the intersection number \( (A \circ b) \) of the chains \( A \) and \( b \) in the sphere \( S^{2n-1} \) (which is well-defined, since the boundary of the chain \( A \), which is equal to \( a \), does not intersect the cycle \( b \)) does not depend on the choice of the chain \( A \). Indeed if \( A' \) is another such chain then the difference \( (A - A') \) will be an absolute \( n \)-dimensional cycle in the sphere \( S^{2n-1} \), from which it follows that \(( (A - A') \circ b) = 0 \), that is that \( (A \circ b) = (A' \circ b) \). The intersection number \( (A \circ b) \) of the chains \( A \) and \( b \) is called the linking number of the cycles \( a \) and \( b \) and denoted \( l(a, b) \).

Another method of calculating the linking number goes as follows: Let \( D^{2n} \) be the ball, the boundary of which is the sphere \( S^{2n-1} \). We choose two \( n \)-dimensional chains \( \bar{A} \) and \( \bar{B} \) in the ball \( D^{2n} \), the boundaries of which coincide with the cycles \( a \) and \( b \) respectively and which lie wholly inside the ball \( D^{2n} \), with the exception of their boundaries. In this case we can make sense of the
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intersection number \((\tilde{A} \circ \tilde{B})_D\) of the chains \(\tilde{A}\) and \(\tilde{B}\) in the ball \(D^{2n}\) and

\[
l(a, b) = (A \circ b)_S
= (-1)^n(\tilde{A} \circ \tilde{B})_D
= (\tilde{B} \circ \tilde{A})_D
= (-1)^n l(b, a).
\]

In order to prove this result we must remark that the intersection number \((\tilde{A} \circ \tilde{B})_D\) is well-defined, that is it does not depend on the concrete choice of the chains \(\tilde{A}\) and \(\tilde{B}\) for which \(\partial \tilde{A} = a\) and \(\partial \tilde{B} = b\). We can consider the ball \(D^{2n}\) as a cone over the sphere \(S^{2n-1}\), that is as the quotient space obtained from the product \([0, 1] \times S^{2n-1}\) of the interval \([0, 1]\) and the sphere \(S^{2n-1}\) by factoring out the subspace \(\{0\} \times S^{2n-1}\) (the slices \(\{t\} \times S^{2n-1}\) \((0 \leq t \leq 1)\) corresponding in the ball to concentric spheres of radius \(t\)). Then as the chain \(\tilde{B}\) we can take the cone over the cycle \(b\) with vertex at the centre of the ball \(D^{2n}\) \((\tilde{B} = [0, 1] \times b/\{0\} \times b)\), and as the chain \(\tilde{A}\) we can take the union of the cylinder \([1/2, 1] \times a\) over the cycle \(a\) and the chain \(\{1/2\} \times A\), lying in the sphere \(\{1/2\} \times S^{2n-1}\) of radius \(1/2\) \((A \subset S^{2n-1}, \partial A = a); for n = 1 see figure 17\). In this case the chains \(\tilde{A}\) and \(\tilde{B}\) will intersect at points of the form \((1/2, x)\), where \(x\) is an intersection point of the chain \(A\) with the cycle \(b\). The sign, which differs the corresponding intersection numbers can be calculated without difficulty.

![Figure 17](image)

We return to our consideration of the singularity \(f\). Let \(a\) and \(b\) be \((n-1)\)-dimensional cycles in the fibre \(\Phi^{-1}(1)\) of the fibration

\[
\Phi : S_q^{2n-1} \setminus T \to S^1.
\]

The cycle \(\Gamma_{1/2 \ast} b\) lies in the fibre \(\Phi^{-1}(-1)\) and therefore does not intersect the cycle \(a\). Consequently, it makes sense to talk about the linking number of the cycles \(a\) and \(\Gamma_{1/2 \ast} b\) as cycles lying in the \((2n-1)\)-dimensional sphere.
**Definition.** The Seifert Form of the singularity $f$ is the bilinear form $L$ on the homology group $H_{n-1}(\Phi^{-1}(1)) \cong H_{n-1}(V_f)$, defined by the formula

$$L(a, b) = l(a, \Gamma_{1/2\ast} b),$$

where $a, b \in H_{n-1}(\Phi^{-1}(1))$.

The theorem of Alexander duality asserts that the linking number defines a duality between the homology groups $H_{n-1}(\Phi^{-1}(1))$ and $H_{n-1}(S^{2n-1} \setminus \Phi^{-1}(1))$.

It is not difficult to see that the fibre $\Phi^{-1}(-1)$ is a deformation retract of the space $S^{2n-1} \setminus \Phi^{-1}(1)$. Consequently, the homology group $H_{n-1}(S^{2n-1} \setminus \Phi^{-1}(1))$ is isomorphic to the group $H_{n-1}(\Phi^{-1}(-1))$.

Since the transformation $\Gamma_{1/2\ast}$ is an isomorphism between the groups $H_{n-1}(\Phi^{-1}(1))$ and $H_{n-1}(\Phi^{-1}(-1))$, then the Seifert form defines a duality between the homology group $H_{n-1}(\Phi^{-1}(1))$ and itself, that is it is a non-degenerate integral bilinear form with determinant equal to $(\pm 1)$. We remark that the Seifert form $L$, generally speaking, does not possess the property of symmetry.

Let $b \in H_{n-1}(\Phi^{-1}(1))$ be an absolute homology class and $a \in H_{n-1}(\Phi^{-1}(1), \partial \Phi^{-1}(1))$ be a relative homology class modulo the boundary.

**Lemma 2.3.** $L(\text{Var}_f a, b) = (a \circ b)$.

**Proof.** Let us choose a relative $(n-1)$-cycle in the pair $(\Phi^{-1}(1), \partial \Phi^{-1}(1))$ which is a representative of the homology class $a$ (we shall denote it also by $a$). Let us consider the mapping $[0, 1] \times a \to S^{2n-1}$ from the cylinder over the cycle $a$ into the sphere, mapping $(t, c) \in [0, 1] \times a$ to $\Gamma_i(c) \in S^{2n-1}$. Under this mapping the lower end $\{0\} \times a$ of the cylinder $[0, 1] \times a$ maps to the chain $a$, the upper end $\{1\} \times a$ to the chain $\Gamma_1 a$, $[0, 1] \times \partial a$ maps to the boundary $\partial T$ of the tubular neighbourhood of the manifold $K$. Therefore this mapping defines an $n$-chain in the sphere $S^{2n-1}$ (its image), the boundary of which consists of two parts: the variation $\text{Var}_f a = \Gamma_1 a - a$ of the cycle $a$ (lying in the fibre $\Phi^{-1}(1))$ and a cycle lying on $\partial T$. Contracting the second part of its boundary inside the tubular neighbourhood $T$ along radii, we obtain a chain $A$ in the sphere $S^{2n-1}$, the boundary of which lies in the fibre $\Phi^{-1}(1) \subset S^{2n-1}$ and is equal to $\text{Var}_f(a)$. The intersection of the chain $A$ with the cycle $\Gamma_{1/2\ast} b$ is the same as the intersection of the cycles $\Gamma_{1/2\ast} a$ and $\Gamma_{1/2\ast} b$.
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in the fibre $\Phi^{-1}(-1)$. Therefore

$$L(\text{Var}_f a, b) = l(\text{Var}_f a, \Gamma_{1/2} b) = (A \circ \Gamma_{1/2} b)_{(1)} = (\Gamma_{1/2} a \circ \Gamma_{1/2} b)_{\Phi^{-1}(-1)} = (a \circ b)_{\Phi^{-1}(1)},$$

which is what we were trying to prove.

Since the Seifert form $L$ defines a duality between the homology group $H_{n-1}(\Phi^{-1}(1))$ and itself, and the intersection number defines a duality between the groups $H_{n-1}(\Phi^{-1}(1))$ and $H_{n-1}(\Phi^{-1}(1), \partial \Phi^{-1}(1))$, then we have

**Theorem 2.2.** The variation operator $\text{Var}_f$ of the singularity $f$ is an isomorphism of the homology groups

$$H_{n-1}(\Phi^{-1}(1), \partial \Phi^{-1}(1)) \cong H_{n-1}(\Phi^{-1}(1))$$

or, which is the same thing, of the groups

$$H_{n-1}(V, \partial V) \cong H_{n-1}(V).$$

**Remark.** If we already had a proof of the Picard-Lefschetz theorem in the general case, then this result could be obtained by assigning the matrix of the operator $\text{Var}_f$ in a distinguished basis of the homology group $H_{n-1}(V)$ and the basis of the group $H_{n-1}(V, \partial V)$ dual to it (see §2.5).

From this theorem and Lemma 2.3 follows

**Theorem 2.3.** If $a, b \in H_{n-1}(V)$, then

$$L(a, b) = (\text{Var}_f^{-1} a \circ b).$$

**Remark.** The definition of the linking number and the Seifert form sometimes differs from that given here either in sign or by a permutation of the arguments (for example in [101]).

The Seifert form is very useful for studying the topological structure of singularities. In particular, it can be shown that the Seifert form (or the variation operator $(H_{n-1}(V))^* \rightarrow H_{n-1}(V)$) determines the intersection form on the homology group $H_{n-1}(V)$ of the non-singular level manifold.

**Theorem 2.4.** For $a, b \in H_{n-1}(V)$

$$(a \circ b) = -L(a, b) + (-1)^n L(b, a).$$
Proof. Since the variation operator of a singularity is an isomorphism, there exist relative cycles \( a', b' \in H_{n-1}(V_\varepsilon, \partial V_\varepsilon) \) such that \( a = \text{Var}_f a' \) and \( b = \text{Var}_f b' \). It remains to apply the result of lemma 1.1 to the cycles \( a' \) and \( b' \).

In addition to the intersection form the variation operator also determines the action of the classical monodromy operator of a singularity. The inverse of the variation operator, acts from the homology group \( H_{n-1}(V_\varepsilon) \) of the non-singular level manifold to the group \( H_{n-1}(V_\varepsilon, \partial V_\varepsilon) \) dual to it. To it corresponds the operator

\[
(\text{Var}_f^{-1})^T : H_{n-1}(V_\varepsilon) \rightarrow H_{n-1}(V_\varepsilon, \partial V_\varepsilon)
\]

defined so that

\[
((\text{Var}_f^{-1})^T a \circ b) = L(b, a) = (\text{Var}_f^{-1} b \circ a)
\]

for \( a, b \in H_{n-1}(V_\varepsilon) \). In matrix form it means that the matrix of the operator \((\text{Var}_f^{-1})^T\) is obtained from the matrix of the operator \(\text{Var}_f^{-1}\) by transposition.

**Theorem 2.5** ([199]). The classical monodromy operator \( h_* \) of a singularity can be expressed in terms of its variation operator \( \text{Var}_f \) by the formula

\[
h_* = (-1)^n \text{Var}_f (\text{Var}_f^{-1})^T.
\]

Proof. We have the equality \((x \circ y) = (i_* x \circ y)\), where \( x, y \in H_{n-1}(V_\varepsilon) \), \( i_* \) is the homomorphism \( H_{n-1}(V_\varepsilon) \rightarrow H_{n-1}(V_\varepsilon, \partial V_\varepsilon) \), induced by the inclusion \( V_\varepsilon \hookrightarrow (V_\varepsilon, \partial V_\varepsilon) \). Together with Theorem 2.4 it gives

\[
i_* = -\text{Var}_f^{-1} + (-1)^n (\text{Var}_f^{-1})^T.
\]

For the classical monodromy operator of the singularity we have

\[
h_* = \text{id} + \text{Var}_f i_*
= \text{id} - \text{Var}_f \text{Var}_f^{-1} + (-1)^n \text{Var}_f (\text{Var}_f^{-1})^T
= (-1)^n \text{Var}_f (\text{Var}_f^{-1})^T
\]

which is what we had to prove.

There is an analogous result for the action of the classical monodromy in the relative homology group.
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Theorem 2.6.

\[ h_\varphi^n = (-1)^n (\text{Var}^{-1})^T \text{Var}. \]

2.4 Proof of the Picard-Lefschetz theorem

We shall use here the notation of §1.3.

From the fact that the variation operator

\[ \text{var}_\varphi : H_{n-1}(\bar{F}_1, \partial \bar{F}_1) \to H_{n-1}(\bar{F}_1), \]

being the variation operator of the singularity

\[ f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2, \]

is an isomorphism (Theorem 2.2), it follows that \( \text{var}_\varphi(V) = \pm \Delta \). To determine the sign in this formula we use Theorem 2.3. In the definition of the fibration \( \Phi \) (for the critical point \( 0 \) of the function \( x_1^2 + \ldots + x_n^2 \)) we can suppose that \( q = 1 \).

The fibration \( \Phi : S^{2n-1} \setminus T \to S^1 \) is given by the formula

\[ \Phi(x_1, \ldots, x_n) = (x_1^2 + \ldots + x_n^2)/|x_1^2 + \ldots + x_n^2| \]

\((|x_1|^2 + \ldots + |x_n|^2 = 1)\). The fibre \( \Phi^{-1}(1) \) of this fibration is diffeomorphic to the level manifold \( \bar{F}_1 \). The vanishing cycle \( \Delta \) in the manifold \( \bar{F}_1 \) corresponds in the fibre \( \Phi^{-1}(1) \) to the cycle defined by the equations

\[ x_1^2 + \ldots + x_n^2 = 1, \quad \text{Im} \ x_j = 0. \]

We shall denote this cycle by \( \Delta \) also.

We have

\[ (\text{Var}^{-1} \Delta \circ \Delta) = L(\Delta, \Delta) \]

\[ = l(\Delta, \Gamma_{1/2} \ast \Delta) \]

\[ = (-1)^n (\tilde{A} \circ \tilde{B})_D, \]

where \( \tilde{A} \) and \( \tilde{B} \) are \( n \)-dimensional chains in the ball \( D = D^{2n} \), the boundaries of which lie on the sphere \( S^{2n-1} \) and are equal to \( \Delta \) and \( \Gamma_{1/2} \ast \Delta \) respectively. It is not difficult to see that in order to calculate the linking number \( l(\Delta, \Gamma_{1/2} \ast \Delta) \) it is...
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possible to use the family of diffeomorphisms

\[ \Gamma_t : \Phi^{-1}(1) \to \Phi^{-1}(\exp(2\pi it)), \]

which do not necessarily agree with the structure of the direct product on the boundary. We can take for this the family defined by the formula

\[ \Gamma_t(x_1, \ldots, x_n) = (\exp(\pi it)x_1, \ldots, \exp(\pi it)x_n). \]

Then the cycle \( \Gamma_{1/2\pi} A \) will be determined by the equations

\[ x_1^2 + \ldots + x_n^2 = -1 \quad \text{and} \quad \text{Re } x_j = 0. \]

We can take as the chains \( \tilde{A} \) and \( \tilde{B} \) the chains in the ball \( D^{2n} \), given by the equations \( \{ \text{Im } x_j = 0 \} \) and \( \{ \text{Re } x_j = 0 \} \) respectively. The orientations of the chains \( \tilde{A} \) and \( \tilde{B} \) are in agreement with the help of a mapping from \( \tilde{A} \) to \( \tilde{B} \), which is multiplication by \( i \). If a positively oriented system of coordinates on the disc \( \tilde{A} \) is the set \( u_1, \ldots, u_n \) \( \times_j = u_j + iv_j \) then a positively oriented system of coordinates on the disc \( \tilde{B} \) will be \( v_1, \ldots, v_n \). The chains \( \tilde{A} \) and \( \tilde{B} \) are smooth manifolds (\( n \)-dimensional discs) and intersect transversely at the point 0. From this it follows that

\[ (\tilde{A} \circ \tilde{B})_D = (-1)^{n(n-1)/2}. \]

Therefore

\[ (\text{Var}^{-1} A \circ A) = (-1)^{n(n+1)/2}, \]

that is

\[ \text{Var}^{-1} A = (-1)^{n(n+1)/2} V, \]
\[ \text{Var} V = (-1)^{n(n+1)/2} A, \]

which is what we had to prove.

2.5 The intersection matrix of a singularity

As we have already said, the monodromy group of a singularity is generated by the Picard-Lefschetz operators \( h_i \), corresponding to the elements \( A_i \) of a weakly distinguished basis in the homology of the non-singular level manifold of the
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singularity $f$ near a critical point. By the Picard-Lefschetz theorem we have

$$h_i(a) = a + (-1)^{n+1/2} (a \circ \Delta_i) \Delta_i.$$ 

Thus the matrix of pairwise intersections of elements of a weakly distinguished basis determines the monodromy group of the singularity.

**Definition.** The matrix $S = (\Delta_i \circ \Delta_j)$ is called the *intersection matrix* of the singularity $f$ (with respect to the basis $\{\Delta_i\}$).

**Remark.** Here we use $i$ for the number of the column, and $j$ for the number of the row. This way of writing down the matrix of the bilinear form coincides with the way of writing it down as the matrix of an operator (in this case $i_\bullet$) from the homology space $H_{n-1}(\{\})$ to its dual space $H_{n-1}(\{\}, \partial \{\})$ with bases $\{\Delta_i\}$ and its dual $((\Delta_i \circ \Delta_j) = (i_\bullet \Delta_i \circ \Delta_j))$.

**Definition.** The *bilinear form associated with the singularity* $f$ is an integral bilinear form defined on the homology group $H_{n-1}(\{\})$ of the non-singular level manifold of the singularity $f$ by the intersection number.

The bilinear form associated with the singularity is symmetric for an odd number of variables $n$ and antisymmetric for an even number of variables. The intersection matrix of the singularity is the matrix of the form with respect to the basis $\{\Delta_j\}$. The diagonal elements of the intersection matrix are determined in Lemma 2.4 of §2.3 and are equal to 0 for even $n$ and $\pm 2$ for odd $n$.

If $\tilde{f}$ is a perturbation of the function $f$, and $\{\Delta_i\}$ is a distinguished basis of vanishing cycles, defined by a system of paths $u_1, \ldots, u_n$, then the loop $\tau'$, which goes in a positive direction round all the critical values into which the zero critical

![Fig. 18.](image-url)
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value of the function $f$ decomposes, is homotopic to the product $\tau_1 \cdots \tau_1$ of simple loops, corresponding to the paths $u_{\mu}, \ldots, u_1$ (figure 18).

From this follows

**Lemma 2.4.** The classical monodromy operator $h_\ast$ of the singularity $f$ is equal to the product $h_1 \cdots h_\mu$ of Picard-Lefschetz operators, corresponding to the elements $\{A_i\}$ of a distinguished basis in the homology of the non-singular level manifold.

The action of the variation operator of the singularity $f$ can be defined by the formulae

$$\text{Var}_f = \text{var}_{\tau_1 \cdots \tau_1}$$

$$= \sum_{r=1}^\mu \sum_{i_1 < i_2 < \cdots < i_r} \text{var}_{\tau_{i_1}} \cdot i_\ast \cdot \text{var}_{\tau_{i_2}} \cdot i_\ast \cdot \cdots \cdot i_\ast \cdot \text{var}_{\tau_{i_r}}, \quad (*)$$

$$\text{var}_{\tau_i}(a) = (-1)^{n(n+1)/2} (a \circ A_i) \Delta_i \quad (a \in H_{n-1}(V_\varepsilon, \partial V_\varepsilon)).$$

Choose in the group $H_{n-1}(V_\varepsilon, \partial V_\varepsilon)$, which is the dual of the group $H_{n-1}(V_\varepsilon)$, the basis $\{V_i\}$, dual to the basis $\{A_i\}$, that is such that

$$(V_i \circ A_j) = \delta_{ij}.$$

From the formula $(*)$ it follows that

$$\text{Var}_f (V_i) = (-1)^{n(n+1)/2} A_i + \sum_{j < i} c_i^j A_j,$$

where $c_i^j$ are certain integers. So we have proved

**Lemma 2.5.** With respect to a distinguished basis the matrix of the variation operator $\text{Var}_f$ of a singularity $f$ is an upper triangular matrix with diagonal entries equal to $(-1)^{n(n+1)/2}$.

The same properties are possessed by the matrix of the operator $\text{Var}_f^{-1}$, which by theorem 2.3 coincides with the matrix of the Seifert form $L$ of the singularity $f$ (see the remark at the beginning of the section, defining the matrix entries of a bilinear form).

Let $S$ be the intersection matrix of the singularity $f$ with respect to any basis, $L$ be the matrix of the Seifert form (or of the operator $\text{Var}_f^{-1}$) of this singularity with respect to the same basis, $H$ be the matrix of the classical monodromy
operator $h_\#$, $H^{(r)}$ be the matrix of the operator $h_\#^{(r)}$ with respect to the dual basis. Theorems 2.4, 2.5 and 2.6 show that

$$S = -L + (-1)^n L^T,$$
$$H = (-1)^n L^{-1} L^T,$$
$$H^{(r)} = (-1)^n L^T L^{-1}$$

(the symbol $^T$ means the transpose of the matrix). If $\{A_i\}$ is a distinguished basis of vanishing cycles, then the matrix $L$ with respect to it is upper triangular and the matrix $L^T$ is lower triangular. Thus the intersection matrix with respect to a distinguished basis has an invariant decomposition into the sum of an upper triangular and a lower triangular matrix.

It was stated above that the intersection matrix of a singularity with respect to a distinguished basis determines its classical monodromy operator (with respect to the same basis). The converse is also true. Before proving this we formulate one useful general result.

**Lemma 2.6.** Let $A$ and $B$ be upper triangular matrices with ones on the diagonal, and let $C = AB^T$. Then the matrices $A$ and $B$ can be reconstructed from the matrix $C$.

The following formulation of this result is equivalent to the previous one.

**Lemma 2.7.** Let $A$ and $B$ be upper triangular matrices with ones on the diagonal. If $AB^T$ is the identity matrix then $A$ and $B$ are also identity matrices.

The proof of this lemma does not present any difficulty.

**Theorem 2.7** ([205]). The matrix of the classical monodromy operator of a singularity with respect to a distinguished basis determines its variation operator and its intersection matrix.

The proof applies Lemma 2.6 to the identity

$$H = (-1)^n \tilde{L}^{-1} \tilde{L}^T,$$

where

$$\tilde{L} = (-1)^{n(n+1)/2} L,$$

in which $\tilde{L}$ and $\tilde{L}^{-1}$ are upper triangular matrices with ones on the diagonal.
2.6 Change of basis

The system of paths \( \{u_i\} \), defining a distinguished or weakly distinguished basis, can be chosen in more than one way. If we change the initial system of paths, we can get different bases of vanishing cycles in the homology group \( H_{n-1}(V) \) of the non-singular level set of the singularity near the critical point. We describe several elementary operations of change of basis, preserving its distinguished or weakly distinguished character. Let \( \{u_i\} \) be a system of paths, defining the distinguished basis \( \{A_i\} \) in the homology group

\[
H_{n-1}(F_{z_0}) \cong H_{n-1}(V)
\]

of the non-singular level manifold. This means that the \( u_i \) are non-self-intersecting paths, joining the critical values \( z_i \) of the perturbation \( f \) of the function \( f \) with the non-critical value \( z_0 \) and intersecting each other only at the point \( z_0 \). Let \( \tau_i \) be a simple loop corresponding to the path \( u_i \).

**Definition** of the operation \( \alpha_m \) \((1 \leq m < \mu)\). We define a new system of paths \( \{\tilde{u}_i\} \) in the following manner:

\[
\tilde{u}_i = u_i \quad \text{for} \quad i \neq m, m+1;
\]

\[
\tilde{u}_{m+1} = u_m;
\]

\[
\tilde{u}_m = u_{m+1} \tau_m.
\]

Here by \( u_{m+1} \tau_m \) we understand the path obtained by traversing the path \( u_{m+1} \) followed by the loop \( \tau_m \). It is clear (see below) that the system of paths \( \{\tilde{u}_i\} \) defines a weakly distinguished set of vanishing cycles \( \{\tilde{A}_i\} \). It is not difficult to see that the system of paths \( \{\tilde{u}_i\} \) can be deformed a little so that it satisfies the conditions of the definition of a distinguished basis (figure 19). Therefore the basis \( \{\tilde{A}_i\} \) is distinguished. The basis \( \{\tilde{A}_i\} \) is related to the basis \( \{A_i\} \) by the following formulae:

\[
\tilde{A}_i = A_i \quad \text{for} \quad i \neq m, m+1;
\]

\[
\tilde{A}_{m+1} = A_m;
\]

\[
\tilde{A}_m = h_m(A_{m+1}) = A_{m+1} + (-1)^{n(n+1)/2} \cdot (A_{m+1} \circ A_m) A_m
\]

(the Picard-Lefschetz transformation). The operation of transferring from the
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distinguished basis \( \{ \mathcal{A}_i \} \) to the distinguished basis \( \{ \mathcal{A}_i' \} \), described by these formulae, is denoted by \( \alpha_m \).

Fig. 19.

Fig. 20.

**Definition** of the operation \( \beta_{m+1} \) \((1 \leq m < \mu)\). Let the system of paths \( \{ \tilde{u}_i \} \) be defined in the following manner:

\[
\begin{align*}
\tilde{u}_i &= u_i \quad \text{for} \quad i \neq m, m+1; \\
\tilde{u}_m &= u_{m+1}; \\
\tilde{u}_{m+1} &= u_m \tau_{m+1}
\end{align*}
\]

(figure 20). This system of paths defines a distinguished basis \( \{ \mathcal{A}_i' \} \), related to the basis \( \{ \mathcal{A}_i \} \) by the formulae:

\[
\begin{align*}
\mathcal{A}_i &= \Delta_i \quad \text{for} \quad i \neq m, m+1; \\
\mathcal{A}_m' &= \Delta_{m+1}; \\
\mathcal{A}_{m+1}' &= h_{m+1}^{-1}(\Delta_m) = \Delta_{m+1} - (-1)^{n+1/2} \cdot (\Delta_{m+1} \circ \Delta_m) \Delta_{m+1}
\end{align*}
\]

(the inverse Picard-Lefschetz transformation). The operation of transferring from the distinguished basis \( \{ \mathcal{A}_i \} \) to the distinguished basis \( \{ \mathcal{A}_i' \} \), described by these formulae, is denoted by \( \beta_{m+1} \).

It is not difficult to see that the operation \( \beta_{m+1} \) is the inverse of the operation \( \alpha_m \) in the sense that the successive application of these in either order brings one back to the initial basis. We consider the free group generated by the elements \( \alpha_m \) \((m = 1, \ldots, \mu - 1)\). To each element of this group (a word in the symbols \( \alpha_m \) and \( \alpha_m^{-1} \)) there corresponds an operation of change of distinguished basis (taking into consideration the fact that the action of \( \alpha_m^{-1} \) on the basis coincides with the action of the operation \( \beta_{m+1} \)). It is clear that the actions of the operations \( \alpha_m \alpha_m^{-1} \) and
\( \alpha_m \alpha_m \) are the same when \(|m - m'| \geq 2\). In addition, the actions of the operations \( \alpha_m \alpha_{m+1} \alpha_m \) and \( \alpha_{m+1} \alpha_m \alpha_{m+1} \) are the same for any \( m \) from 1 to \((\mu - 2)\). The "proof" of this fact is given in figure 21.

In this way we get an action, on the set of distinguished bases of the homology group of the non-singular level set near a critical point, of the quotient group of the free group on the \((\mu - 1)\) generators \( \alpha_m \) \((m = 1, \ldots, \mu - 1)\) by the relations

\[
\alpha_{m+1} \alpha_m \alpha_{m+1} = \alpha_m \alpha_{m+1} \alpha_m \quad \text{for} \quad 1 \leq m < \mu - 1,
\]

\[
\alpha_m \alpha_{m'} = \alpha_{m'} \alpha_m \quad \text{for} \quad |m - m'| \geq 2.
\]

This group is the braid group with \( \mu \) strands (see, for example [57]; see also Section 3.3).

We consider an operation which preserves the property of being weakly distinguished for a set of vanishing cycles. As a preliminary we show that any such set forms a basis in the homology of the non-singular level manifold.

**Theorem 2.8.** Any weakly distinguished set of vanishing cycles forms a basis of the homology group \( H_{n-1}(V_\ell) \) of the non-singular level manifold.
Let \( \{ \Delta_i \} \) be a weakly distinguished set of vanishing cycles defined by the system of paths \( \{ u_j \} \), and let \( \tau_i \) be the corresponding simple loops. The set of loops \( \{ \tau_i \} \) is a system of free generators of the fundamental group \( \pi_1 (U \setminus \{ z_i \}; z_0) \) of the complement of the set of critical values. In order to prove that the set of vanishing cycles \( \{ \Delta_i \} \) forms a basis of the group \( H_{n-1} (V_\varepsilon) \), it is sufficient to prove that any vanishing cycle \( \Delta \) (defined with the help of the path \( v \) joining a critical value \( z_j \) with the non-critical value \( z_0 \)) is linearly dependent on the cycles \( \Delta_1, \ldots, \Delta_\mu \) with integer coefficients. We can suppose that the paths \( u_j \) and \( v \) coincide near the critical value \( z_j \). In this case the loop \( \gamma = u_j v^{-1} \) can be considered as an element of the fundamental group \( \pi_1 (U \setminus \{ z_i \}; z_0) \) of the complement of the set of critical values. We have \( \Delta = \pm h_\gamma \Delta_j \) (the sign depending on the orientation of the vanishing cycles \( \Delta \) and \( \Delta_j \)). In the group \( \pi_1 (U \setminus \{ z_i \}; z_0) \) the loop \( \gamma \) can be expressed in terms of the generators \( \tau_1, \ldots, \tau_\mu \). Consequently the vanishing cycle \( \Delta \) can be obtained from the cycle \( \Delta_j \) by the successive application of some Picard-Lefschetz operators \( h_i \) and their inverses, and is therefore linearly dependent on the cycles \( \Delta_1, \ldots, \Delta_\mu \) with integer coefficients.

**Definition** of the operations \( \alpha_m (m') \) and \( \beta_m (m') \) of change of weakly distinguished basis. Let \( \{ u_i \} \) be a system of paths, defining the weakly distinguished basis \( \{ \Delta_i \} \) of the homology group \( H_{n-1} (V_\varepsilon) \) of the non-singular level manifold. For \( m \neq m' \) we define the operation of change of basis \( \alpha_m (m') \) \( [\beta_m (m')] \), corresponding to the change of the path \( u_{m'} \) to the path \( u_{m'} \tau_m [u_{m'} \tau_m^{-1}] \), that is transforming the weakly distinguished basis \( \{ \Delta_i \} \) into the basis \( \{ \tilde{\Delta}_i \} \) defined by the formulae

\[
\tilde{\Delta}_i = \Delta_i \quad \text{for} \quad i \neq m',
\]

\[
\tilde{\Delta}_{m'} = h_m (\Delta_{m'}) = \Delta_{m'} + (-1)^{n(n+1)/2} (\Delta_{m'} \circ \Delta_m) \Delta_m
\]

\[
[\tilde{\Delta}_{m'} = h_m^{-1} (\Delta_{m'}) = \Delta_{m'} + (-1)^{n(n+1)/2} (\Delta_m \circ \Delta_m) \Delta_m].
\]

The action of the operations \( \alpha_m (m') \) and \( \beta_m (m') \) on the system of simple loops \( \{ \tau_i \} \) consists of changing the loop \( \tau_{m'} \) into a conjugate of it in the fundamental group \( \pi_1 (U \setminus \{ z_i \}; z_0) \) of the complement of the set of critical values (that is \( \tau_{m'}^{-1} \tau_m \tau_m \) for \( \alpha_m (m') \) and \( \tau_m \tau_m \tau_{m'}^{-1} \) for \( \beta_m (m') \)). For this reason, if the initial system of simple loops is a system of free generators of the group \( \pi_1 (U \setminus \{ z_i \}; z_0) \), then the same property will be possessed also by the system of simple loops, obtained after application of the operation \( \alpha_m (m') \) or \( \beta_m (m') \). Therefore the operations \( \alpha_m (m') \) and \( \beta_m (m') \) preserve the property of a basis being weakly distinguished.
It is easy to see that the operations $\alpha_m(m')$ and $\beta_m(m')$ are inverses of each other. When the number of variables $n$ is odd, these operations coincide. If a distinguished basis is considered as weakly distinguished, and in particular we forget the order of the vanishing cycles, then the action of the operation $\alpha_m$ coincides with the action of the operation $\alpha_m(m+1)$, and $\beta_{m+1}$ with $\beta_{m+1}(m)$.

It can be shown (see [150]) that any two distinguished bases can be obtained one from the other by iterations of operations $\alpha_m$ and $\beta_m$ and a change of orientations of some elements. It was proved that any two weakly distinguished bases can be obtained one from the other in an analogous way with the help of operations $\alpha_m(m')$ and $\beta_m(m')$ (Humphries, S. P. [170] and, apparently, already Whitehead J. H. C., 1936).

2.7 The Variation Operator and the Intersection Matrix of a "Direct Sum" of Singularities

**Definition.** The direct sum of the singularities $f: (\mathbb{C}^n, 0)\rightarrow(\mathbb{C}, 0)$ and $g: (\mathbb{C}^m, 0)\rightarrow(\mathbb{C}, 0)$ of functions of $n$ and $m$ variables respectively is the singularity of the function $f \oplus g: (\mathbb{C}^{n+m}, 0)\rightarrow(\mathbb{C}, 0)$ of $(n+m)$ variables defined by the formula

$$f \oplus g(x, y) = f(x) + g(y)$$

$(x \in \mathbb{C}^n, y \in \mathbb{C}^m, (x, y) \in \mathbb{C}^{n+m} \approx \mathbb{C}^n \oplus \mathbb{C}^m)$.

**Lemma 2.8.** The multiplicity $\mu(f \oplus g)$ of the direct sum of the singularities $f$ and $g$ is equal to the product $\mu(f) \mu(g)$ of their multiplicities.

Indeed if $\tilde{f}(x)$ is a perturbation of the singularity $f$, with $\mu(f)$ non-degenerate critical points $p_i$, and $\tilde{g}(y)$ is a perturbation of the singularity $g$, with $\mu(g)$ non-degenerate critical points $q_j$, then $\tilde{f}(x) + \tilde{g}(y)$ is a perturbation of the singularity $f \oplus g$ with $\mu(f) \mu(g)$ non-degenerate critical points $(p_i, q_j)$ ($i=1, \ldots, \mu(f)$; $j=1, \ldots, \mu(g)$).

M. Sebastiani and R. Thom ([322]) proved that the classical monodromy operator of the singularity $f \oplus g$ is equal to the tensor product of the classical monodromy operators of the singularities $f$ and $g$. A. M. Gabrielov ([116]) obtained a description of the intersection matrix of the singularity $f \oplus g$ under the condition that the intersection matrices of the singularities $f$ and $g$, with respect to distinguished bases, are known. We give an account of these results in a form somewhat different from that found in [322] and [116].

We need one topological concept.
**Definition.** The join $X \star Y$ of the topological spaces $X$ and $Y$ is the quotient space of the direct product $X \times I \times Y$ ($I = [0, 1]$) by the equivalence relation:

\[(x, 0, y_1) \sim (x, 0, y_2) \quad \text{for any} \quad y_1, y_2 \in Y, \ x \in X; \]
\[(x_1, 1, y) \sim (x_2, 1, y) \quad \text{for any} \quad x_1, x_2 \in X, \ y \in Y.\]

We can consider that the spaces $X$ and $Y$ lie in their join $X \star Y$ as the lower and upper bases respectively ($\{(x, 0, y)\}$ and $\{(x, 1, y)\}$). Therefore the join $X \star Y$ can be represented as the space swept out by non-intersecting segments joining every point of the space $X$ to every point of the space $Y$. If we consider the projection $(x, t, y) \mapsto t$ of the join $X \star Y$ to the line segment $I = [0, 1]$, then the preimage of the point 0 coincides with the space $X$, the preimage of the point 1 coincides with the space $Y$ and that of a point $t \in (0, 1)$ with the product $X \times Y$.

If $Y$ is the space consisting of one point, then the join $X \star Y$ coincides with the cone over the space $X$. If $Y$ is the space consisting of two points, then the join $X \star Y$ is homeomorphic to the suspension of the space $X$ (the quotient space of the cylinder $[-1, 1] \times X$ over the space $X$ by the equivalence relations

\[(-1, x_1) \sim (-1, x_2), \ (1, x_1) \sim (1, x_2)\]

for all $x_1, x_2 \in X$). If the space $X$ is homeomorphic to the $k$-dimensional sphere $S^k$, and $Y$ to the $l$-dimensional sphere $S^l$, then the join $X \star Y$ is homeomorphic to the $(k+l+1)$-dimensional sphere $S^{k+l+1}$.

**Lemma 2.9.** Let the homology groups of the spaces $X$ and $Y$ either not have torsion or be considered with coefficients in a field. Then the homology group $H_n(X \star Y)$ of the join of the spaces $X$ and $Y$ is isomorphic to

\[\oplus_{0 \leq k \leq n-1} H_k(X) \otimes H_{n-k-1}(Y).\]

In other words $H_*(X \star Y) = H_*(X) \otimes H_*(Y)$ if we consider that $\dim (a \otimes b) = \dim a + \dim b + 1$ for $a \in H_*(X), \ b \in H_*(Y)$. If $\alpha$ is a cycle in the space $X$ and $\beta$ is a cycle in the space $Y$, then the cycle corresponding to $\alpha \otimes \beta$ in the space $X \star Y$ is the join of the cycles $\alpha$ and $\beta$. Here it is essential that the homology groups are supposed reduced modulo a point.

The embedding

\[H_k(X) \otimes H_{n-k-1}(Y) \hookrightarrow H_n(X \star Y),\]
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generally speaking, is defined only up to multiplication by \((\pm 1)\). It's choice is determined by a method of orientating the join of cycles. We can, for example, suppose that the orientation of the join \(a \star b\) is induced from the usual orientation of the direct product \(a \times I \times b\). We note, however, that the results formulated below do not depend on this choice.

Let \(f\) be a singularity \((\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\), let \(V_e\) be a non-singular level manifold of the singularity \(f\) near the critical point \((V_e = f^{-1}(\epsilon) \cap \overline{B}_e)\), and let \(u\) be a path joining the non-critical value \(\epsilon\) with the critical value 0.

**Lemma 2.10.** There exists a continuous family of mappings

\[ H_t : V_e \to f^{-1}(u(t)) \cap \overline{B}_e \quad (t \in [0, 1]), \]

such that

1) \(H_0 = id : V_e \to V_e\);
2) \(H_t\) is the inclusion \(V_e \to V_{u(t)}\) for \(0 \leq t < 1\);
3) \(H_1\) maps \(V_e\) into the point \(0 \in \mathbb{C}^n\).

The proof of this result can be constructed in an analogous way to the way that, in Theorem 2.1, it was shown that the space \(f^{-1}(0) \cap \overline{B}_e\) is a deformation retract of the space \(f^{-1}(\overline{D}_{\epsilon_0}) \cap \overline{B}_e\).

The family of mappings \(H_t\) is determined uniquely up to isotopy. It gives an embedding of the cone over the non-singular level manifold \(V_e\) into the space \(\mathbb{C}^n\) \((x, t) \to H_t(x)\) for \(0 \leq t \leq 1\).

Now let \(f\) and \(g\) be two singularities in \(n\) and \(m\) variables respectively, let

\[ V_e(f) = f^{-1}(\epsilon) \cap \overline{B}_{e_1} \]

and

\[ V_e(g) = g^{-1}(\epsilon) \cap \overline{B}_{e_2} \]

be the non-singular level manifolds of the singularities \(f\) and \(g\) respectively and let \(u\) be a non-self-intersecting path in the target plane of the function \(f\), joining \(\epsilon\) with zero (without loss of generality, we can suppose that \(u(t) = (1 - t)\epsilon\)). We define a path \(v\), joining \(\epsilon\) with zero in the target plane of the function \(g\), by the formula \(v(t) = \epsilon - u(1 - t)\). Let \(H_f(f) : V_e(f) \to V_{u(t)}(f)\) and \(H_f(g) : V_e(g) \to V_{v(t)}(g)\) be the families of functions described in Lemma 2.10. We define the inclusion \(j\) of the join \(V_e(f) \star V_e(g)\) of the non-singular level manifolds \(V_e(f)\) and \(V_e(g)\) into the level set

\[ (f \oplus g)^{-1}(\epsilon) \subset \mathbb{C}^{n+m} \]
by the formula

\[ j(x, t, y) = (H_t(f)x, H_{1-t}(g)y) \]

for \( x \in V_\varepsilon(f), y \in V_\varepsilon(g), t \in [0, 1] \). If we impose natural limits on the radii \( q_1, q_2 \) and \( q \) (for example,

\[ q_1 \leq \varepsilon/\sqrt{2}, \quad q_2 \leq \varepsilon/\sqrt{2} \]

then \( j \) is an embedding of the join \( V_\varepsilon(f) \ast V_\varepsilon(g) \) into the level manifold

\[ V_\varepsilon(f \oplus g) = (f \oplus g)^{-1}(\varepsilon) \cap B_\varepsilon \]

of the singularity of \( f \oplus g \) near the critical point.

The mapping

\[ j : V_\varepsilon(f) \ast V_\varepsilon(g) \rightarrow V_\varepsilon(f \oplus g) \]

together with the isomorphism

\[ H_{n+m-1}(V_\varepsilon(f) \ast V_\varepsilon(g)) \cong H_{n-1}(V_\varepsilon(f)) \otimes H_{m-1}(V_\varepsilon(g)) \]

defines the homomorphism

\[ j_* : H_{n-1}(V_\varepsilon(f)) \otimes H_{m-1}(V_\varepsilon(g)) \rightarrow H_{n+m-1}(V_\varepsilon(f \oplus g)). \]

In the work [322] was proved

**Theorem 2.9.** The homomorphism \( j_* \) is an isomorphism and the inclusion

\[ j : V_\varepsilon(f) \ast V_\varepsilon(g) \rightarrow V_\varepsilon(f \oplus g) \]

is a homotopy equivalence.

The fact that the non-singular level manifold \( V_\varepsilon(f \oplus g) \) of the singularity \( f \oplus g \) is homotopically equivalent to the join \( V_\varepsilon(f) \ast V_\varepsilon(g) \) can be explained in the following manner. We consider the function \( f \) on the manifold \( V_\varepsilon(f \oplus g) \) (more precisely we consider the function \( f \circ \pi_1 \), where

\[ \pi_1 : V_\varepsilon(f \oplus g) \subset \mathbb{C}^{n+m} \cong \mathbb{C}^n \oplus \mathbb{C}^m \rightarrow \mathbb{C}^n \]
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is the projection on the first factor. The preimage \((f \circ \pi_1)^{-1}(z)\) of the point \(z \in \mathbb{C}\) consists of points \((x, y) \in \mathbb{C}^n \oplus \mathbb{C}^m\), for which \(f(x) = z, g(y) = \varepsilon - z\). Therefore (if we ignore the details connected with the radii of the balls in which we consider the non-singular level manifolds of the functions)

\[(f \circ \pi_1)^{-1}(z) = f^{-1}(z) \times g^{-1}(\varepsilon - z).\]

The mapping

\[
(f \circ \pi_1): V_\varepsilon(f \oplus g) \to \mathbb{C}
\]

is non-degenerate outside the preimages of the points 0 and \(\varepsilon\). We consider in the plane \(\mathbb{C}\) the segment \(J = u([0, 1])\) which is the image of the path \(u\). It joins the points 0 and \(\varepsilon\) \((u(0) = \varepsilon, u(1) = 0)\). Over the complement of the segment \(J\) the mapping \(f \circ \pi_1\) defines a locally trivial fibration. The segment \(J\) is a deformation retract of the plane \(\mathbb{C}\). From this it follows that the space \((f \circ \pi_1)^{-1}(J)\) is a deformation retract of the space \(V_\varepsilon(f \oplus g)\) and is therefore homotopy equivalent to it. The space \((f \circ \pi_1)^{-1}(u(t))\) for \(t \in (0, 1)\), is diffeomorphic to the product \(V_{\varepsilon_1}(f) \times V_{\varepsilon_2}(g)\) of the non-singular level manifolds of the singularities \(f\) and \(g\).

The space \((f \circ \pi_1)^{-1}(u(0))\) is diffeomorphic to \(f^{-1}(\varepsilon) \times g^{-1}(0)\). The space \(g^{-1}(0)\) is contractible to a point. Therefore \((f \circ \pi_1)^{-1}(u(0))\) is contractible to a space diffeomorphic to the non-singular level manifold \(V_\varepsilon(f)\). Similarly the space

\[
(f \circ \pi_1)^{-1}(u(1)) = f^{-1}(0) \times g^{-1}(\varepsilon)
\]

is contractible to a space diffeomorphic to the non-singular level manifold \(V_\varepsilon(g)\).

This description of the fibres of the mapping

\[
(f \circ \pi_1): (f \circ \pi_1)^{-1}(J) \to J
\]

over the points of the segment \(J\) coincides with the description of the preimage of the points \(t \in I = [0, 1]\) under the projection \(V_\varepsilon(f) \ast V_\varepsilon(g) \to I\) (see the definition of the join). Therefore the space \((f \circ \pi_1)^{-1}(J)\) is homotopy equivalent to the join \(V_\varepsilon(f) \ast V_\varepsilon(g)\). A little more accurate reasoning allows one to turn this explanation into a proof.

From now on we shall identify the homology group \(H_{n+m-1}(V_\varepsilon(f \oplus g))\) of the non-singular level manifold of the singularity \(f \oplus g\) with the tensor product of the groups \(H_{n-1}(V_\varepsilon(f))\) and \(H_{m-1}(V_\varepsilon(g))\). This identification also determines an
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identification of the relative homology group

\[ H_{n+m-1}(V_\epsilon(f \oplus g), \partial V_\epsilon(f \oplus g)) \]

(which is the dual of the group \( H_{n+m-1}(V_\epsilon(f \oplus g)) \)) with the tensor product of the groups \( H_{n-1}(V_\epsilon(f), \partial V_\epsilon(f)) \) and \( H_{m-1}(V_\epsilon(g), \partial V_\epsilon(g)) \).

Theorem 2.10 (P. Deligne, see [94]).

\[ \text{Var}_{f \oplus g} = (-1)^m \text{Var}_f \otimes \text{Var}_g. \]

For the proof it is sufficient to show that for any homology classes

\[ a_1, a_2 \in H_{n-1}(V_\epsilon(f)), b_1, b_2 \in H_{m-1}(V_\epsilon(g)) \]

we have the equality

\[ ([\text{Var}_f^{-1}(a_1 \otimes b_1)] \circ [a_2 \otimes b_2]) = (-1)^m(\text{Var}_f^{-1}a_1 \circ a_2) \cdot (\text{Var}_g^{-1}b_1 \circ b_2). \]

We shall not carry out this proof in detail, but only outline its main steps (though it would not be difficult to reconstruct the whole proof).

Let

\[ H_t(f): V_\epsilon(f) \to V_{(1-t)\epsilon}(f) = f^{-1}((1-t)\epsilon) \cap \overline{B_\epsilon} \]

be the family of mappings described in Lemma 2.10 (for the sake of definiteness we suppose that \( u(t) = (1-t)\epsilon \)). As we have already said, the family \( H_t(f) \) defines an embedding of the cone over the level manifold \( V_\epsilon(f) \) into the space \( \mathbb{C}^n \). Let \( A_1 \)

be the cone over the cycle \( a_1 \), determined by the family \( H_t(f) \). Then \( A_1 \) is an \( n \)-dimensional chain, the boundary of which lies in the non-singular level manifold \( V_\epsilon(f) \) and coincides with the cycle \( a_1 \). Let

\[ \Gamma_t(f): V_\epsilon(f) \to V_{\text{exp}(2\pi it)\epsilon}(f) \]

be the family of mappings obtained by lifting the homotopy

\[ \epsilon \mapsto \text{exp}(2\pi it)\epsilon \quad (0 \leq t \leq 1), \]
and let

\[ \tilde{a}_2 = \Gamma_{1/2}(f)(a_2) \]

be the \((n-1)\)-dimensional cycle in the level manifold \(V_{-\varepsilon}(f)\). Let \(\tilde{A}_2\) be the cone over the cycle \(\tilde{a}_2\), by an analogous construction. From the considerations of §2.3 it follows that

\[ (\text{Var}^{-1}_f a_1 \circ a_2) = (-1)^n (A_1 \circ \tilde{A}_2), \]

the chains \(A_1\) and \(\tilde{A}_2\) intersecting only at zero. The chains \(B_1\) and \(\tilde{B}_2\) are defined in the same way, with

\[ (\text{Var}^{-1}_g b_1 \circ b_2) = (-1)^m (B_1 \circ \tilde{B}_2). \]

In order to define

\[ ([\text{Var}^{-1}_f \otimes_g (a_1 \otimes b_1)] \circ [a_2 \otimes b_2]) \]

by the same method it is necessary to construct cones \(C_1\) and \(\tilde{C}_2\) over the cycles \(a_1 \otimes b_1\) and \(a_2 \otimes \tilde{b}_2 = \Gamma_{1/2}(f \oplus g)(a_2 \otimes b_2)\). It is not difficult to see that we can take

\[ (A_1 \times B_1) \cap \{(x, y): (f(x) + g(y))/\varepsilon \leq 1\} \]

as \(C_1\). We have an analogous result for \(\tilde{C}_2\): we can take

\[ C_2 = \tilde{A}_2 \times \tilde{B}_2 \cap \{(x, y): (f(x) + g(y))/(-\varepsilon) \leq 1\}. \]

(Here we use the fact that \(\Gamma_i(f \oplus g)(a_2 \otimes b_2) = \Gamma_i(f)(a_2) \otimes \Gamma_i(g)(b_2)\).) It follows that

\[ ([\text{Var}^{-1}_f \otimes_g (a_1 \otimes b_1)] \circ [a_2 \otimes b_2]) = (-1)^{n+m} (C_1 \circ \tilde{C}_2) \]

\[ = (-1)^{n+m} ([A_1 \times B_1] \circ [\tilde{A}_2 \times \tilde{B}_2]) \]

\[ = (-1)^{n+m+m} (A_1 \circ \tilde{A}_2)(B_1 \circ \tilde{B}_2) \]

\[ = (-1)^m (\text{Var}^{-1}_f a_1 \circ a_2)(\text{Var}^{-1}_g b_1 \circ b_2), \]

which is what we were trying to prove.
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Let \{\Delta_i\} (i=1, \ldots, \mu(f)) be a distinguished basis in the homology group \(H_{n-1}(V_\varepsilon(f))\) of the non-singular level manifold of the singularity \(f\), let \{\Delta_j\} (j=1, \ldots, \mu(g)) be a distinguished basis in the homology group \(H_{m-1}(V_\varepsilon(g))\). From Theorem 2.9 it follows that the elements

\[
\tilde{\Delta}_{ij} = j_*(\Delta_i \otimes \Delta_j)
\]

form a basis of the homology group

\(H_{n+m-1}(V_\varepsilon(f \oplus g))\)

of the non-singular level manifold of the singularity \(f \oplus g\). The intersection matrix \(S\) of the singularity \(f \oplus g\) with respect to this basis can be obtained with the help of Theorem 2.10 from the formula

\[
S = -L + (-1)^{n+m}L^T
\]

where \(L\) is the matrix of the operator \(\text{Var}_{\varepsilon} f \oplus g\) (or the Seifert form). From this follows

**Theorem 2.11.** The intersection numbers of the cycles \(\tilde{\Delta}_{ij}\) are given by the following formulae:

\[
(\tilde{\Delta}_{ij} \circ \tilde{\Delta}_{ij}) = \text{sgn} (j_2 - j_1)^n (-1)^{nm + m(n-1)/2} (\Delta_{ij} \circ \Delta_{ij}) \quad \text{for} \quad j_1 \neq j_2,
\]

\[
(\tilde{\Delta}_{ij} \circ \tilde{\Delta}_{ij}) = \text{sgn} (i_2 - i_1)^m (-1)^{nm + n(m-1)/2} (\Delta_{ij} \circ \Delta_{ij}) \quad \text{for} \quad i_1 \neq i_2,
\]

\[
(\tilde{\Delta}_{ij} \circ \tilde{\Delta}_{ij}) = 0 \quad \text{for} \quad (i_2 - i_1)(j_2 - j_1) < 0,
\]

\[
(\tilde{\Delta}_{ij} \circ \tilde{\Delta}_{ij}) = \text{sgn} (i_2 - i_1)(-1)^{nm} (\Delta_{ij} \circ \Delta_{ij}) (\Delta_{ij} \circ \Delta_{ij})
\]

\[
\text{for} \quad (i_2 - i_1)(j_2 - j_1) > 0.
\]

This result was obtained by A. M. Gabrielov in [116]. In addition the following result was proved.

**Theorem 2.12.** The cycles \(\tilde{\Delta}_{ij}\) are vanishing cycles and form a distinguished basis of the homology group \(H_{n+m-1}(V_\varepsilon(f \oplus g))\) of the non-singular level manifold of the singularity \(f \oplus g\). It is implied that they are ordered lexicographically, that is that the cycle \(\tilde{\Delta}_{i_1,j_1}\) precedes the cycle \(\tilde{\Delta}_{i_2,j_2}\) if

\[i_1 < i_2, \quad \text{or} \quad i_1 = i_2, \quad j_1 < j_2.\]
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Theorem 2.10 is a generalisation of the above-mentioned results of M. Sebastiani and R. Thom ([322]), describing the classical monodromy operator $h_{*}(f \oplus g)$ of the singularity $f \oplus g$.

**Theorem 2.13.**

$$h_{*}(f \oplus g) = h_{*f} \otimes h_{*g}.$$  

This theorem is an immediate corollary of Theorem 2.10 and the relation $h_{*} = (-1)^n \text{Var}(\text{Var}^{-1})^T$ (Theorem 2.5). Conversely, Theorem 2.10 follows from Theorems 2.13, 2.12, the relation $h_{*} = (-1)^n \text{Var}(\text{Var}^{-1})^T$ and Theorem 2.7, confirming that the matrix of the classical monodromy operator of a singularity with respect to a distinguished basis determines the matrix of its variation operator.

Theorems 2.11 and 2.12 give the following description of the Dynkin diagram of the singularity $f \oplus g$ (for the definition see the following section). Its vertex set coincides with the direct product of the vertex sets of the diagrams corresponding to the singularities $f$ and $g$. Two vertices $(i_1, j_1)$ and $(i_2, j_2)$ are joined to each other

(i) by an edge of the same multiplicity as that joining the vertices $j_1$ and $j_2$ in the second diagram, if $i_1 = i_2$;

(ii) by an edge of the same multiplicity as that joining the vertices $i_1$ and $i_2$ in the first diagram, if $j_1 = j_2$;

(iii) by an edge of multiplicity equal to minus the product of the multiplicities of the edges, joining the vertices $i_1$ and $i_2$ in the first diagram and $j_1$ and $j_2$ in the second diagram, if $(i_2 - i_1)(j_2 - j_1) > 0$.

If $(i_2 - i_1)(j_2 - j_1) < 0$, then the vertices $(i_1, j_1)$ and $(i_2, j_2)$ are not joined to each other.

**2.8 The stabilisation of a singularity**

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated critical point at the origin.

**Definition.** The germ of the function

$$f(x) + \sum_{j=1}^{m} y_j^2 : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}, 0)$$

is called a stabilisation of the germ $f$. 
The multiplicity of the singularity is equal to the multiplicity of its stabilisation. Indeed, if $\tilde{f}$ is a perturbation of the singularity $f$, decomposing the critical point at zero into $\mu$ non-degenerate ones, then $\tilde{f}(x) + \Sigma_{j=1}^m y_j^2$ is the perturbation of its stabilisation, possessing the same property. Moreover, the functions $\tilde{f}(x)$ and $\tilde{f}(x) + \Sigma_{j=1}^m y_j^2$ have the same set of critical values. The connection between the intersection matrix of a singularity and the intersection matrix of its stabilisation is given by the following theorem, which is a special case of Theorem 2.11.

**Theorem 2.14.** Let $\{A_i\}$ be a distinguished basis of vanishing cycles in the homology of the non-singular level manifolds of the singularity $f(x)$. Then there exists a distinguished basis $\{\tilde{A}_i\}$ of the singularity $f(x) + \Sigma_{j=1}^m y_j^2$, such that the intersection matrix of its elements is defined by the relation

$$(\tilde{A}_i \circ \tilde{A}_j) = [\text{sgn} (j-i)]^m (-1)^{nm + m(m-1)/2} (A_i \circ A_j) \quad \text{for } i \neq j.$$

Moreover the distinguished bases $\{A_i\}$ and $\{\tilde{A}_i\}$ correspond to identical sets of paths joining the critical values of the perturbations $\tilde{f}(x)$ and $\tilde{f}(x) + \Sigma_{j=1}^m y_j^2$ with the non-critical value.

From Theorem 2.14 it follows that the intersection matrices of the stabilisations of the singularities determine each other. In addition for $m \equiv 0 \pmod{4}$ the intersection numbers $(\tilde{A}_i \circ \tilde{A}_j)$ and $(A_i \circ A_j)$ are equal for all $i$ and $j$, for $m \equiv 2 \pmod{4}$ they differ by a sign. In this way we associate with each singularity two symmetric and two antisymmetric bilinear forms (the intersection forms of its stabilisation). Moreover the symmetric (and antisymmetric) forms differ only by a sign. With each singularity we associate also two groups of transformations of the integral lattice $\mathbb{Z}^n$ (the monodromy groups of its stabilisations). The classical monodromy operator of the singularity $f(x)$ coincides with the classical monodromy operator of its stabilisation $f(x) + \Sigma_{j=1}^m y_j^2$ for even $m$ and differs from it by a sign for odd $m$.

Theorem 2.14 allows us to formulate results on intersection matrices of singularities restricted to cases the dimensions of which have fixed residue modulo four. In the majority of cases it will be convenient to suppose that the number of variables is conjugate to three modulo four.

**Definition.** The quadratic form of a singularity is the quadratic form, defined by the intersection numbers in the homology of the non-singular level manifolds of its stabilisation with number of variables $N \equiv 3 \pmod{4}$.
For this stabilisation the self-intersection number of the vanishing cycles \( (\Delta_i \circ \Delta_j) \) is equal to \(-2\), the Picard-Lefschetz operator acts on the homology group of the non-singular level manifold according to the formula \( h_i(a) = a + (a \circ \Delta_i) \Delta_i \). From this it is clear that \( h_i(\Delta_i) = -\Delta_i \) and that the transformation \( h_i \) is reflection in the hyperplane orthogonal to the vector \( \Delta_i \). The orthogonality is in terms of the scalar product, defined by the quadratic form of the singularity. Thus we can see that the corresponding monodromy groups are groups generated by reflections. Such groups (or the corresponding quadratic forms which determine them) are most conveniently described with the help of certain graphs.

**Definition.** The *Dynkin diagram* (or *D-diagram*) of a singularity is a graph defined as follows:

(i) its vertices are in one-to-one correspondence with the elements \( \Delta_i \) of a weakly distinguished basis of the homology of the non-singular level manifold of the stabilisation of the singularity with number of variables \( N \equiv 3 \mod 4 \);

(ii) the \( i \)th and the \( j \)th vertices of the graph are joined by an edge of multiplicity \((\Delta_i \circ \Delta_j)\) (The edges of negative multiplicity are depicted by dashed lines).

The *D*-diagram of a singularity determines its monodromy group (although an effective description of the latter is obviously quite a hard problem). If the *D*-diagram of a singularity (with a known number of variables) is given relative to a distinguished basis and with its vertices numbered in the same order, then from it we can determine the bilinear form of the singularity, and also its variation operator, its classical monodromy operator, etc.

### 2.9 Example

We consider the singularity \( f(x)=x^{k+1} \) (a singularity of type \( A_k \) in the terminology of part II of volume 1). The level manifold \( V_\varepsilon \) consists of \( k+1 \) points, the \( (k+1) \)th roots of \( \varepsilon \). The multiplicity of this singularity is equal to \( k \), and the homology group \( H_0(V_\varepsilon) \) (reduced modulo a point) is isomorphic to \( \mathbb{Z}^k \).

The function \( \tilde{f}(x)=x^{k+1}-\lambda x \ (\lambda \neq 0) \) is a Morse perturbation of the singularity \( f \). We shall suppose that \( \lambda \) is real and greater than zero. The zero level manifold \( \tilde{f}^{-1}(0) \) of the function \( \tilde{f} \) also consists of \( k+1 \) points: \( x_0=0 \), \( x_m=\frac{1}{k} \lambda \xi_m \) \((m=1, \ldots, k)\). Here \( \xi_m \) are the \( k \)th roots of unity, enumerated clockwise: \( \xi_m=\exp\left(-2\pi im/k\right) \). The critical points of the function \( \tilde{f} \) are determined by the equation \( \tilde{f}'(x)=(k+1)x^k-\lambda=0 \). Therefore \( \tilde{f} \) has \( k \) critical points

\[
p_m=\sqrt[k]{(\lambda/(k+1))} \xi_m
\]
with critical values

\[ z_m = -\left(\frac{\lambda k}{(k+1)}\right)^{\frac{1}{\lambda/(k+1)}} \xi_m \quad (m=1, \ldots, k). \]

We choose as the non-critical value \( z_0 \) a negative number of large modulus

\[ \left( |z_0| \gg \left(\frac{\lambda k}{(k+1)}\right)^{\frac{1}{\lambda/(k+1)}} \right). \]

Let \( u_m \) be the path joining the critical value \( z_m \) of the function \( \tilde{f} \) with zero along the radius \( (u_m(t) = (1-t)z_m, t \in [0,1]) \), and let \( v \) be the path going from zero to \( z_0 \) along the negative real axis and going round the critical value

\[ z_k = -\left(\frac{\lambda k}{(k+1)}\right)^{\frac{1}{\lambda/(k+1)}} \xi_k \]

in the positive direction (anticlockwise). See figure 22.

![Fig. 22.](image)

It is easy to see that the system of paths \( \{u_m v\} \) define a distinguished basis of vanishing cycles \( \{\Delta_m\} \) in the homology group \( H_0(\tilde{f}^{-1}(z_0)) \) (because by a small perturbation it can be reduced to a system of paths, satisfying the definition of a distinguished basis). In order to calculate the intersection numbers \( (\Delta_m \circ \Delta_m') \) of the vanishing cycles in the homology group \( H_0(\tilde{f}^{-1}(z_0)) \) it is convenient to homotop the point \( z_0 \) along the path \( v \) to zero. In this way we reduce the problem to the calculation of the intersection numbers of the vanishing cycles, defined in the group \( H_0(\tilde{f}^{-1}(0)) \) by the system of paths \( \{u_m\} \) (we shall denote these cycles by \( \Delta_m \) too).

It is easy to show that the cycles \( \Delta_m = x_m - x_0 \) vanish along the paths \( u_m \) (that is as we move in the target plane of the function \( \tilde{f} \) along the path \( u_m \) from zero to the critical value \( z_m \) the points \( x_m \) and \( x_0 \) merge). Consequently we have \( (\Delta_m \circ \Delta_m) = 2 \), \( (\Delta_m \circ \Delta_m') = 1 \) for any \( m \neq m' \). For the stabilisation \( f(x) + y_1^2 + y_2^2 \) the appropriate
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The formula is \((A_m \circ A_m) = -2\), \((A_m \circ A_{m'}) = -1\) for any \(m' \neq m\). We obtain the \(D\)-diagram of the singularity \(f\) in the following form: there are \(k\) vertices; every pair of which are joined by a dashed line (that is by an edge of multiplicity \(-1\)).

We simplify this diagram with the help of operations which change the distinguished basis. The operation \(\alpha_{k-1} (A_{k-1}' = A_k - A_{k-1}', A_k' = A_{k-1})\) reduces the diagram to the following form: all the vertices except the \((k-1)\)th are joined pairwise by dashed lines (edges of multiplicity \(-1\)), the \((k-1)\)th vertex is joined only to the \(k\)th by a line of multiplicity \(+1\). The operations \(\alpha_{k-2} (A_{k-2}' = A_{k-1}' = A_k - A_{k-2}, A_{k-1}' = A_{k-2})\), \(\alpha_{k-3}, \ldots, \alpha_1\) do not change the form of the diagram, but lead only to the renumbering of the vertices. The application of the following sequence of operators

\[
\alpha_{k-1}, \alpha_{k-2}, \ldots, \alpha_2, \alpha_{k-1}, \ldots, \alpha_3, \ldots, \alpha_{k-1}, \alpha_{k-2}, \alpha_{k-1}
\]

reduces the diagram to the classical Dynkin diagram \(A_k\) (figure 23). The basis of vanishing cycles we get can be described by the formulae

\[
\begin{align*}
A_0 &= (x_k - x_{k-1}), & A_0' &= (x_{k-1} - x_{k-2}), & \ldots, \\
A_{k-1} &= (x_2 - x_1), & A_k &= (x_1 - x_0).
\end{align*}
\]

As we move in the target plane of the function \(\hat{f}\) along the path \(v\) from zero to the non-critical value \(z_0\), the points \(x_m (m = 0, 1, \ldots, k)\) move in the complex plane \(\mathbb{C}\) tending to the rays

\[
\arg x = \pi(2s + 1)/(k + 1)
\]

(as \(z_0 \to -\infty\)). The points \(x_0\) and \(x_k\) approach each other along the real axis, do a quarter rotation, anticlockwise, round the critical point \(p_k\), and go apart again. It is easy to show that on the ray

\[
\arg x = \pi(2s + 1)/(k + 1)
\]

(that is \(x = t \exp(\pi i(2s + 1)/(k + 1))\), \(t > 0\)) the function \(\hat{f}(x) = x^{k+1} - \lambda x\) does not
take negative real values except in the case when \( k \) is even and \( s = k/2 \). Indeed,

\[
    \tilde{f}(x) = -t^{k+1} - \lambda t \exp (\pi i(2s + 1)/(k + 1)),
\]

where the second term is real only when \( k \) is even and \( s = k/2 \). In this case the point \( x_{k/2} \) moves along the negative real axis. From this it follows that as we move in the target plane of the function \( \tilde{f} \) along the path \( v \) from zero to \( z_0 \) the points \( x_m \) approach the points

\[
    \tilde{x}_m = \frac{k+1}{\sqrt{(-z_0)}} \exp \left( -\pi i(2m + 1)/(k + 1) \right)
\]

\((m = 0, 1, \ldots, k)\). If in addition we travel in the target plane of the function in the negative direction (clockwise) from the point \( z_0 \) to the point \( z'_0 = -z_0 \), then the points \( \tilde{x}_m \) will cross over to the points

\[
    \tilde{\tilde{x}}_m = \frac{k+1}{\sqrt{(-z_0)}} \exp \left( -2\pi i(m + 1)/(k + 1) \right)
\]

\((m = 0, 1, \ldots, k)\).

We arrive at the following result:

**Theorem 2.15.** On the level manifold \( V_1 = \{ x : x^{k+1} = 1 \} \) of the singularity \( f(x) = x^{k+1} \) the distinguished basis is formed by the vanishing cycles

\[
    A_1 = \zeta_1 - \zeta_2, \ A_2 = \zeta_2 - \zeta_3, \ldots, A_k = \zeta_k - \zeta_{k+1},
\]

where

\[
    \zeta_j = \exp \left( 2\pi i(j - 1)/(k + 1) \right)
\]

are the \((k+1)\)th roots of unity \((j = 1, \ldots, (k+1))\). The intersection numbers of these cycles are given by the formulae

\[
    (A_j \circ A_j) = 2, \\
    (A_j \circ A_{j+1}) = -1, \\
    (A_j \circ A_{j'}) = 0 \quad \text{for} \quad |j - j'| \geq 2.
\]

The first calculation of the intersection forms and the classical monodromy operator for functions of several variables was given by F. Pham ([284]) for singularities of the form \( f(x) = \sum_{k=1}^n x_k^{a_k} \) \((a_k \geq 2)\). The multiplicity of this
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singularity is equal to \( \Pi_k^{a_k-1} \). F. Pham proved that in the homology group \( H_{n-1}(V) \) of the non-singular level set of the function \( f \) there exists a basis \( e_{i_1} \ldots e_{i_n} \) \((0 \leq i_k \leq a_k-2)\) (in the notation of F. Pham \( e_{i_1} \ldots e_{i_n} = (\Pi_k^{\omega_k})e \)), such that

\[
(e_{i_1} \ldots e_{i_n} \circ e_{i_1} \ldots e_{i_n}) = (-1)^{\frac{n(n-1)}{2}} (1 + (-1)^{n-1});
\]

\[
(e_{i_1} \ldots e_{i_n} \circ e_{j_1} \ldots e_{j_n}) = (-1)^{\frac{n(n-1)}{2}} (-1)^{k} \sum (i_k - i_n),
\]

if \( i_k \leq j_k \leq i_k + 1 \) for all \( k \). In the remaining cases (except those arising from the previous ones by a permutation of cycles)

\[
(e_{i_1} \ldots e_{i_n} \cdot e_{j_1} \ldots e_{j_n}) = 0.
\]

The result of F. Pham can be obtained from Theorem 2.11 (§2.7). Applying it to the singularity \( f(x) = \sum a_k x^{a_k} \) gives the same intersection matrix as that of F. Pham, if as a distinguished basis of the singularity \( f_k(x_k) = x_k^{a_k} \) we use the basis described in theorem 2.15. For the singularity \( f_k(x_k) = x_k^{a_k} \) we put

\[
\varepsilon = 1, \; u(t) = (1-t), \; H(t)x_k = \sqrt{(1-t)} x_k.
\]

The application in sequence of the constructions described in §2.7 to the distinguished basis of the singularity \( f_k(x_k) \) given by Theorem 2.15 reduces it, as it is not difficult to convince oneself, to the basis constructed by F. Pham in [284]. We obtain the following result.

**Assertion.** The basis of F. Pham is distinguished relative to the lexicographic ordering of its elements.

This means that the \( D \)-diagram of the singularity of F. Pham has the form depicted in figure 24 \((n = 2, a_1 = 6, a_2 = 5)\).

Fig. 24.
Singularity of Differentiable Maps, Volume 2
Monodromy and Asymptotics of Integrals
Arnold, E.; Gusein-Zade, S.M.; Varchenko, A.N.
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