Many important problems in mathematical physics can be modeled by means of elliptic partial differential equations or systems. Such equations arise in the study of, for example, steady-state heat conduction (the Laplace equation), acoustics (the Helmholtz equation), elasticity (the Lamé system), and electromagnetism (the Maxwell system).

An important tool for investigating boundary value problems associated with equations of this type is the boundary integral equation technique, which relies on the derivation of Fredholm or quasi-Fredholm integral equations over the boundary of the region of interest and leads to a very convenient representation of the solution. The kernels of the ensuing integral equations are expressed in terms of a two-point (scalar or matrix) function that is, in fact, a fundamental solution of the governing linear differential operator.

Boundary integral equation methods are extremely useful for a variety of reasons. First, they reduce the problem from one involving an unbounded partial differential operator to one with an integral operator, making it much more appealing from an analytic perspective; second, the methods are very general in that they can be applied to any linear second-order elliptic boundary value problem with constant coefficients; and third, the methods are attractive from a numerical point of view because they yield closed-form solutions and, therefore, lend themselves readily to boundary element treatment.

Boundary integral equation methods come in many versions. Thus, the classical indirect approach seeks the solution in an appropriate form that is chosen \textit{a priori}. This method is ‘indirect’ in the sense that the unknown function in the corresponding integral equation has no physical significance, being merely a convenient mathematical abstraction. By contrast, in the direct methods the unknown function in the integral formulation is an actual physical quantity. For example, in elasticity the solution of the integral equation may represent the displacement or the moment/stress on the boundary of the elastic body.

Another main class of boundary integral equation methods makes use of modified fundamental solutions. This approach was developed to address problems of existence of nonunique solutions to the integral equations derived by the classical
techniques. In certain instances, integral equations formulated for boundary value problems known to have at most one solution may themselves admit multiple solutions. Intuitively, this should not be the case. For this reason, we consider ways of modifying the standard fundamental solution so that it leads to uniquely solvable integral equations.

This book investigates an elliptic system of equations arising in the theory of elasticity which characterizes the stationary oscillations of thin elastic plates. The system is obtained by assuming a Mindlin-type form (also known as Kirchhoff’s kinematic hypothesis) for the displacements.

Approximate theories describing the bending of plates are important because they reduce the equations of classical three-dimensional elasticity to a system involving only two independent space variables, while highlighting the important bending characteristics of the elastic structure. Such theories have been used successfully in many practical engineering applications. The Mindlin-type model differs from the classical Kirchhoff theory in that it accounts for transverse shear deformation as well, thereby offering additional useful information to practitioners.

Boundary integral equation methods have been widely used in the study of various elliptic systems arising in the theory of elasticity and beyond, such as equilibrium and dynamic problems in the process of deformation of two- and three-dimensional elastic bodies, and the equilibrium and time-dependent bending of elastic and thermoelastic plates with transverse shear deformation.

Although the equations governing the stationary oscillations of Mindlin-type plates are related in a certain way to the equilibrium equations, the two systems display very different characteristics. The main difference is the presence in the oscillatory case of so-called eigenfrequencies. These are values of the oscillation frequency for which the main homogeneous boundary value problems in a bounded domain have nonzero solutions. The book will show how such difficulties can be resolved and how the problems in question can be reduced to uniquely solvable integral equations.

Here is a brief description of the contents.

Chapter 1 presents a derivation of the system of equations modeling the stationary oscillations of elastic plates with transverse shear deformation. A fundamental integral formula, analogous to Green’s second identity from potential theory, is also deduced.

The aim in Chapter 2 is to define the generalized single-layer and double-layer plate potentials and to describe their essential properties. All subsequent discussion of boundary value problems relies on the boundary properties of these integral functions. In order to construct the potentials, a suitable matrix of fundamental solutions is made available and its behavior, together with the behavior of a so-called matrix of singular solutions, near the boundary of the plate is investigated.

Chapter 3 deals with the setting up and smoothness properties of a particular solution to the inhomogeneous system obtained in Chapter 1.

In Chapter 4 we introduce the Dirichlet and Neumann problems associated with the governing system of equations. It is natural to discuss these two problems together because they are intrinsically linked in the analysis of their solvability. Radi-
ation conditions that ensure the uniqueness of the solutions of the exterior problems are given, which are then shown to be satisfied by the potential functions defined in Chapter 2. The bulk of the chapter is concerned with establishing integral representations for the regular solutions of the system, to be used later as a starting point in the direct boundary integral equation method.

The presence of eigenfrequencies in the interior problems, which makes the system of stationary oscillations so different from the corresponding equilibrium system, is investigated in Chapter 5. The proof of the existence of eigenfrequencies relies, however, on the relationship between the two systems.

Chapter 6 is concerned with the solvability of the boundary value problems mentioned above. This issue is approached through a classical indirect formulation that results in quasi-Fredholm integral equations of the second kind. Unfortunately, owing to the existence of eigenfrequencies in the interior problems, the solvability of the latter is not always guaranteed. Furthermore, the connection between the solvability of the Dirichlet and Neumann problems leads to difficulties regarding the unique solvability of the integral equations for the exterior problems as well, an effect that, given the available uniqueness results, is not expected.

The application of the direct boundary integral equation method is the subject of Chapter 7, where a coupled pair of equations for each problem—one a quasi-Fredholm second-kind equation and one an equation of the first kind—is obtained. Their analysis is simplified through the use of composition formulas relating various boundary integral operators of interest. It is shown that, as physically expected, each pair of equations for the exterior problems admits exactly one solution. A composite equation consisting of a linear combination of the first-kind and second-kind equations is also studied.

In Chapter 8 a theory of modified integral equations is developed. This is motivated by the need for uniquely solvable equations from which the solutions of the exterior boundary value problems can then be constructed. An indirect method is employed, where the solutions are postulated in the form of modified potentials that lead to quasi-Fredholm second-kind equations. Two different types of modification are considered, with existence and uniqueness results proved for each. The chapter concludes with a look at how uniquely solvable first-kind equations can be derived (again, by an indirect method).

The Robin boundary value problems are introduced in Chapter 9. After the question of uniqueness of solution has been investigated in three separate cases, integral equation methods analogous to those used in Chapters 6–8 are also constructed for these problems.

Chapter 10 considers a fourth type of fundamental boundary value problem associated with the stationary oscillations of thin elastic plates, namely, the transmission problem. The existence of the solution is proved by means of an indirect method after some regularization of the operators involved. A more refined method of solution is then described, based on a direct method in conjunction with a modified fundamental solution.

Finally, in Chapter 11 the null field method is examined. Though, strictly speaking, this is not an integral equation method, it is closely connected to much of the
work from the preceding chapters. Facts concerning the unique solvability of the
null field equations and the completeness of certain sets of functions are presented.

Brief announcements of the some of the results discussed in the book can be
found in [58]–[69].

The methodology and results presented in this monograph should prove useful to
applied mathematicians, scientists, and engineers engaged in the research of oscil-
latory phenomena and other similar models, as well as to graduate students in those
disciplines.

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