Chapter 2

Trigonometry

Hyperbolic trigonometry is a basic tool in various studies, and various approaches are known (cf. e.g. Beardon [1], Fenchel [1], Meschkowski [1], Perron [1], Rees [1], Thurston [1]). This chapter gives an account based on the hyperboloid model which has grown out of discussions with Patrick Eberlein and Klaus-Dieter Semmler. In the first part we use the isometry group acting on the hyperboloid model to obtain the trigonometric formulae of the triangle by comparing matrix elements. In the second part, beginning with Section 4, we generalize this to hexagons and similar configurations. Then we briefly sidestep to variable curvature. In the final part, Section 6, we describe a variant of the approach which is from Semmler [1] and uses a vector product and quaternions.

The results of Sections 2.1 - 2.4 will be widely used in this book. Sections 2.5 and 2.6 are appendices and may be skipped in a first reading.

2.1 The Hyperboloid Model

The starting point is the bilinear form

\[ h(X, Y) = x_1y_1 + x_2y_2 - x_3y_3, \quad X, Y \in \mathbb{R}^3, \]

where \( x_i, y_i \) are the coordinates of \( X, Y \). The linear mappings \( L_{\sigma}, M_{\rho} \in \text{GL}(3, \mathbb{R}) \) given by

\[
L_{\sigma} = \begin{pmatrix}
\cos \sigma & -\sin \sigma & 0 \\
\sin \sigma & \cos \sigma & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad M_{\rho} = \begin{pmatrix}
\cosh \rho & 0 & \sinh \rho \\
0 & 1 & 0 \\
\sinh \rho & 0 & \cosh \rho
\end{pmatrix}
\]
(acting on column vectors) leave $h$ and also the corresponding differential form

$$h^1 = dx_1^2 + dx_2^2 - dx_3^2$$

invariant. We let $\Omega \subset \text{GL}(3, \mathbb{R})$ be the subgroup generated by all $L_\sigma$, and $M_\rho; \sigma, \rho \in \mathbb{R}$. Now consider the hyperboloid

$$(2.1.3) \quad H = \{ X \in \mathbb{R}^3 \mid h(X, X) = -1, x_3 > 0 \}$$

and let $g$ be the quadratic differential form on $H$ which is obtained by restricting $h^1$ to the tangent vectors of $H$. Then $H$ and $g$ are invariant under $\Omega$. Moreover, we have

**2.1.4 Lemma.** $\Omega$ acts twice transitively on $H$.

Since $g$ is positive definite on the tangent plane of the point

$$p_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in H,$$

the two-fold transitivity implies that $H = (H, g)$ is a complete two-point homogeneous Riemannian manifold, and $\Omega = \text{Is}^+(H)$. We could now use the constant curvature of $H$ to prove that (possibly up to a scaling factor) $H$ is a model of the hyperbolic plane, but it is easy to give a direct proof. The proof runs as follows. For fixed $\sigma$, the curve

$$\rho \mapsto L_\sigma M_\rho(p_0), \quad \rho \geq 0,$$

has unit speed on $H$, and for every fixed $\rho > 0$ the curve

$$\sigma \mapsto L_\sigma M_\rho(p_0), \quad \sigma \in \mathbb{R},$$

has constant speed $\sinh \rho$ and intersects orthogonally the curves of the first type. If we introduce therefore $(\rho, \sigma)$ as the pair of coordinates of the point $p = L_\sigma M_\rho(p_0)$, for $0 < \rho < \infty$ and $-\pi \leq \sigma < \pi$, then the coordinate system obtained in this way covers $H - \{ p_0 \}$, and the tensor $g$ has the expression $g = ds^2 = d\rho^2 + \sinh^2 \rho d\sigma^2$. This proves that $H$ is isometric to $\mathbb{H}$ and that $(\rho, \sigma)$ is the pair of geodesic polar coordinates centered at $p_0$. For later application we shall denote by $\mu$ the geodesic

$$(2.1.5) \quad r \mapsto \mu(r) := M_\rho(p_0), \quad -\infty < r < \infty.$$ 

Observe that $\mu$ is invariant under $M_\rho$ for all $\rho \in \mathbb{R}$.

At some places we shall be obliged to calculate in $\Omega$. Since every $A \in \Omega$ leaves the quadratic form $h$ invariant, we have that
The Hyperboloid Model

(2.1.6) \[ A^{-1} = S A' S, \quad A \in \Omega, \]
where \( A' \) is the transpose of \( A \), and \( S \) denotes the diagonal matrix with diagonal elements 1, 1, -1.

2.2 Triangles

For geodesic triangles in the hyperbolic plane, we denote by \( a, b, c \) the sides and by \( \alpha, \beta, \gamma \) the corresponding opposite angles. These letters also denote the side-length and the angular measure. If the triangle is right-angled, we usually let \( \gamma \) be the right angle.

2.2.1 Theorem. (Ordinary triangles). The following formulae hold for geodesic triangles in the hyperbolic plane.

(i) \[ \cosh c = - \sinh a \sinh b \cos \gamma + \cosh a \cosh b, \]
(ii) \[ \cos \gamma = \sin \alpha \sin \beta \cosh c - \cos \alpha \cos \beta, \]
(iii) \[ \sinh a : \sin \alpha = \sinh b : \sin \beta = \sinh c : \sin \gamma. \]

![Figure 2.2.1](image)

Proof. Let the triangle \( T \), say, be placed in \( H \) such that \( \beta \) is at \( p_0 \), the "origin" (cf. Section 2.1), \( \alpha \) is at \( M_c(p_0) \) and \( \gamma \) is at \( L_{\pi-\beta}(M_a(p_0)) \), as shown in Fig. 2.2.1. The isometry \( L_{\pi-a}M_c \) first parallel translates \( T \) along \( \mu \), bringing \( \alpha \) to \( p_0 \), and then rotates \( T \) about \( p_0 \) by an angle \( \pi - \alpha \). Hence, \( L_{\pi-a}M_c \) brings \( T \) back into a position like that of Fig. 2.2.1, but now with side \( b \), rather than side \( c \), on \( \mu \), and with \( \alpha \), rather than \( \beta \), at \( p_0 \). Clearly, the product \( L_{\pi-\beta}M_aL_{\pi-\gamma}M_bL_{\pi-\alpha}M_c \) brings \( T \) back into its original position. Hence, this product is the identity, and the following relationship holds:

(1) \[ M_aL_{\pi-\gamma}M_b = L_{\pi-\beta}M_cL_{\alpha-\pi}. \]

Computing the components in (1) we obtain nine identities, four of which are those of the theorem. \[ \diamond \]
2.2.2 Theorem. (Right-angled triangles). For any hyperbolic triangle with right angle \( \gamma \), the following hold.

(i) \( \cosh c = \cosh a \cosh b \),
(ii) \( \cosh c = \cot \alpha \cot \beta \),
(iii) \( \sinh a = \sin \alpha \sinh c \),
(iv) \( \sinh a = \cot \beta \tanh b \),
(v) \( \cos \alpha = \cosh a \sin \beta \),
(vi) \( \cos \alpha = \tanh b \coth c \).

Proof. The first three identities are the restatement of Theorem 2.2.1 in the case \( \gamma = \pi/2 \). The remaining three identities are obtained via cyclic permutation and elementary computations.

In hyperbolic geometry, angles and perpendiculars are, in some sense, the same thing, and there are various configurations whose trigonometry is similar to that of the triangle. The right-angled hexagon of Lemma 1.7.1 is an example. In the remainder of this section we give a general definition of this configuration - the generalized triangle - and prove a relation which will later imply all trigonometric formulae. For another configuration which unites triangles and hexagons we refer to the book of Fenchel [1]. The uniformizing configuration in Fenchel's book is the general right-angled geodesic hexagon in hyperbolic three space (any pair of consecutive sides is orthogonal, but the sides of the hexagon are in general not coplanar).

In what follows, a polygon \( P \) is a piecewise geodesic oriented closed curve, possibly with self-intersections. The geodesic arcs of \( P \) are the sides. We do not admit sides of length zero.

Let \( (u, w) \) be an ordered pair of consecutive sides of \( P \) with respect to the given orientation, and let \( p \) be the common vertex of \( u \) and \( w \). The angle of \( P \) at \( p \) is defined to be the rotation \( v \in \text{Is}^+(\mathbb{H}) \) which fixes \( p \) and rotates \( w \) to \( u \). We shall say that \( v \) is the subsequent angle of side \( u \), and \( w \) is the subsequent side of angle \( v \).

With \( v \) we also denote the corresponding angle of rotation. To allow addition and subtraction, angles of rotation are considered elements of the

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**Figure 2.2.2**
group $S^1 = \mathbb{R}/[s \mapsto s + 2\pi]$. Our next aim is to define a concept which unites the various configurations of Fig. 2.2.3.

2.2.3 Definition. Let $x$ and $y$ belong to the set of sides and angles of polygon $P$. The ordered pair $(x, y)$ is said to be of angle type if one of the following conditions holds:

(i) $y$ is the subsequent angle of side $x$,

(ii) $(x, y)$ is a pair of consecutive sides, and $y$ is orthogonal to $x$,

(iii) $y$ is the subsequent side of angle $x$.

In Fig. 2.2.3 all pairs $(x, y)$ and $(y, z)$ are of angle type. To each pair $(x, y)$ of angle type we associate an isometry $N_y \in \mathbb{I}_s^+(H)$ as follows ($L$ and $M$ are as in (2.1.2)).

2.2.4 Definition. (i) If $y$ is the subsequent angle of side $x$, then we define

$$ N_y := L_{x-y} = \begin{pmatrix} -\cos y & -\sin y & 0 \\ \sin y & -\cos y & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

(ii) If $(x, y)$ is a pair of consecutive sides with angle $\varepsilon \pi/2$, where $\varepsilon = \pm 1$, then we define

$$ N_y := L_{\varepsilon \pi/2}M_y = \begin{pmatrix} 0 & -\varepsilon & 0 \\ \varepsilon \cosh y & 0 & \varepsilon \sinh y \\ \sinh y & 0 & \cosh y \end{pmatrix}. $$

(iii) If $y$ is the subsequent side of angle $x$, then we define

$$ N_y := M_y = \begin{pmatrix} \cosh y & 0 & \sinh y \\ 0 & 1 & 0 \\ \sinh y & 0 & \cosh y \end{pmatrix}. $$
2.2.5 Definition. A generalized triangle is a closed oriented hyperbolic geodesic polygon $P$ together with a cycle $a, \gamma, b, \alpha, c, \beta$ of consecutive sides and angles of $P$ in which all pairs $(a, \gamma), (\gamma, b), \ldots, (c, \beta), (\beta, a)$ are of angle type.

The polygons in Figures 2.2.1, 2.2.4, 2.3.1, 2.4.1, 2.4.3 and 2.6.2 are generalized triangles.

We adopt the convention that all sides of $P$ have positive length, independently of the orientation, whereas the angles have values in $S^1$. If angular measures with values in $[0, \pi]$ are preferred, as e.g. in Section 2.6, then the signs in the formulae which follow must be adjusted accordingly.

If $P$ is the boundary of a convex domain in $H$, we shall always orient $P$ positively, so that the angles of rotation will be the interior angles of $P$.

2.2.6 Theorem. For every generalized triangle $a, \gamma, b, \alpha, c, \beta$ we have

$$N_aN_\gamma N_b = (N_{\alpha}N_cN_\beta)^{-1}.$$

Proof. A pair $(x, y)$ of angle type in the generalized triangle $T$, say, is said to have standard position if it is placed as in Fig. 2.2.3. That is, if $y$ is an angle, then the subsequent side $z$ lies on the geodesic $\mu$ (cf. (2.1.5)) with initial point at $p_0$ and endpoint at $M_\gamma(p_0)$; if $y$ is a side, then $y$ lies on $\mu$ with initial point at $M_(y)(p_0)$ and endpoint at $p_0$. One checks with Definition 2.2.4 that if $(x, y)$ has standard position and if $N_y$ is applied to $T$, then the preceding pair, say $(w, x)$, moves into standard position. Therefore, if $T$ is placed such that $(\alpha, c)$ has standard position (as in Fig. 2.2.4 and Fig. 2.2.1) and if we successively apply $N_c$, $N_{\alpha}$, $\ldots$, $N_\beta$, then $T$ returns to its original position. Hence, $N_\beta N_{\alpha} N_\gamma N_b N_{\alpha} N_c = id$.

2.2.7 Example. In the generalized triangles of Fig. 2.2.4, we have in both cases $N_aN_\gamma N_b = L_{\pi/2}M_aL_{\pi-\gamma}M_b$. Notice the difference between the two examples for the product $N_aN_cN_\beta$.

![Figure 2.2.4](image-url)
In the first example,
\[ N_\alpha N_\beta N_\gamma = L_{\pi/2}M_\alpha L_{\pi/2}M_\beta, \]
while in the second example,
\[ N_\alpha N_\beta N_\gamma = L_{\pi/2}M_\alpha L_{\pi/2}M_\beta L_{\pi/2}M_\beta. \]

Theorem 2.2.6 yields for the first example

(i) \[ \cos c = -\cosh a \cosh b \cos \gamma + \sinh a \sinh b, \]
(ii) \[ \cosh a : \cosh \alpha = \cosh b : \cosh \beta = \sin c : \sin \gamma. \]

The corresponding identities for the second example are

(iii) \[ \cosh c = -\cosh a \cosh b \cos \gamma + \sinh a \sinh b, \]
(iv) \[ \cosh a : \sinh \alpha = \cosh b : \sinh \beta = \sin c : \sin \gamma, \]
(v) \[ \cos \gamma = \sinh \alpha \sinh \beta \cosh c - \cosh \alpha \cosh \beta. \]

### 2.3 Trirectangles and Pentagons

Consider two perpendicular geodesic arcs, \(a\) and \(b\), in \(D\) (the disk model of the hyperbolic plane) with a common vertex at the origin as in Fig. 2.3.1. Let \(\alpha\) and \(\beta\) be the perpendicular geodesics at the endpoints of \(b\) and \(a\). If \(a\) and \(b\) are small, then \(\alpha\) and \(\beta\) intersect each other at an acute angle \(\varphi\) and we obtain a geodesic quadrilateral with three right angles. Following Coxeter [1] we call this configuration a trirectangle. By abuse of notation we also denote by \(\alpha\) and \(\beta\) the sides of the trirectangle.

Tryrectangles have played a decisive role in the discovery of non-Euclidean
geometry (see for instance the very early approach of Saccheri [1]). They occur in many geometric constructions and are useful for computations. The following formulae, similar to those of the right-angled triangle in Theorem 2.2.2, hold.

2.3.1 Theorem. (Trirectangles). For every trirectangle with sides labelled as in Fig. 2.3.1 the following relations are true:

(i) \( \cos \phi = \sinh a \sinh b \),
(ii) \( \cos \phi = \tanh \alpha \tanh \beta \),
(iii) \( \cosh a = \cosh \alpha \sin \phi \),
(iv) \( \cosh a = \tanh \beta \coth b \),
(v) \( \sinh \alpha = \sinh a \cosh \beta \),
(vi) \( \sinh \alpha = \coth b \cot \phi \).

Proof. The first three identities are the restatement of formulae (i) and (ii) of Example 2.2.7 for the case \( \gamma = \pi/2 \). The remaining ones are obtained via cyclic permutation and elementary computations.

As an example we prove the distance formula for Fermi coordinates. We let \((\rho, t)\) denote the Fermi coordinates with respect to a fixed base line \( \eta \) in \( \mathbf{H} \), (cf. Section 1.1), and let \( p_1 = (\rho_1, t_1) \) and \( p_2 = (\rho_2, t_2) \) be points in \( \mathbf{H} \). Then the following formula holds. Note that in this formula the quantities \( \rho_1 \) and \( \rho_2 \) are oriented lengths and have opposite signs when the corresponding sides lie on opposite sides of \( \eta \).

\[
(2.3.2) \quad \cosh \text{dist}(p_1, p_2) = \cosh \rho_1 \cosh \rho_2 \cosh(t_2 - t_1) - \sinh \rho_1 \sinh \rho_2.
\]

**Figure 2.3.2**

Proof. We give a proof which is based on a decomposition into trirectangles and right-angled triangles. Another proof would consist of understanding the figure as a generalized triangle and to apply Theorem 2.2.6 as in Example 2.2.7.
We may assume that $\rho_1 \geq 0$. Dropping the perpendicular $r$ from $p_1$ to that geodesic through $p_2$ which is orthogonal to $\eta$, we obtain a trirectangle with sides $\rho_1$, $|t_2 - t_1|$, $s$, say, and $r$ and also a right-angled triangle with sides $r$, $|\rho_2 - s|$ and $c$, whose hypothenuse $c$ has length dist$(p_1, p_2)$. The trirectangle is again decomposed into two right-angled triangles with common hypothenuse. Formula (i) of Theorem 2.2.2 and formula (v) of Theorem 2.3.1 yield the relations $\cosh r \cosh s = \cosh \rho_1 \cosh(t_2 - t_1)$, and $\cosh r \sinh s = \sinh \rho_1$. Hence,

$$\cosh \text{dist}(p_1, p_2) = \cosh r \cosh(\rho_2 - s)$$

$$= \cosh r \cosh \rho_2 \cosh s - \cosh r \sinh \rho_2 \sinh s$$

$$= \cosh \rho_1 \cosh \rho_2 \cosh(t_2 - t_1) - \sinh \rho_1 \sinh \rho_2. \quad \diamond$$

We return to the trirectangle of Fig. 2.3.1. If side $a$ grows continuously, the vertex at angle $\varphi$ moves towards the endpoint at infinity of geodesic $\alpha$. Let the limiting position be obtained for $a = a_\varphi$. As $a \to a_\varphi$, $\varphi \to 0$. This follows from Theorem 2.3.1(vi) or by a glance at Fig. 2.3.1. From Theorem 2.3.1(i) we obtain, by continuity,

$$(2.3.3) \quad \sinh a_\varphi \sinh b = 1.$$  

If $a$ grows beyond $a_\varphi$, the angle $\varphi$ disappears and is replaced by $c$, the common perpendicular of $\alpha$ and $\beta$. We obtain a right-angled pentagon.

2.3.4 Theorem. (Right-angled pentagons). For any right-angled pentagon with consecutive sides $a$, $b$, $\alpha$, $c$, $\beta$ we have:

(i) $\cosh c = \sinh a \sinh b$, 
(ii) $\cosh c = \coth \alpha \coth \beta$.

Proof. This is Example 2.2.7(iii) - (v) in the particular case $\gamma = \pi/2$. \quad \diamond

2.3.5 Lemma. Let $a$ and $b$ be any positive real numbers satisfying $\sinh a \sinh b > 1$. Then there exists a unique right-angled geodesic pentagon with two consecutive sides of lengths $a$ and $b$.

Proof. $a$ and $b$ are the consecutive orthogonal sides of either a trirectangle, a limiting trirectangle with a vertex at infinity, or a right-angled pentagon. By Theorem 2.3.1(i) and by (2.3.3), the first two cases are excluded. The uniqueness follows from Theorem 2.3.4 (or Theorem 1.1.6). \quad \diamond
2.4 Hexagons

If we paste two right-angled pentagons together along a common side \( r \), we obtain a right-angled hexagon. Every compact Riemann surface of genus greater than one can be obtained by pasting together such hexagons (Chapter 3). The boundary of a right-angled hexagon is a generalized triangle with consecutive sides \( a, \gamma, b, \alpha, c, \beta \). We use the word *hexagon* for both the domain and its boundary. The hexagon is called *convex* if it is convex as a domain. Fig. 2.4.3 shows a right-angled hexagon which is not the boundary of a convex domain.

![Hexagon Diagram](image)

**Figure 2.4.1**

The pairs \((a, \gamma), \ldots, (\beta, a)\) are all of type (ii) from Definition 2.2.3, with the associated matrices \( N_a = L_{\pi/2} M_a \), \( N_{\gamma} = L_{\pi/2} M_{\gamma} \), \ldots (cf. (ii) in Definition 2.2.4). The next theorem follows from Theorem 2.2.6.

2.4.1 Theorem. (Right-angled hexagons). For any convex right-angled geodesic hexagon with consecutive sides \( a, \gamma, b, \alpha, c, \beta \), the following are true:

(i) \[ \cosh c = \sinh a \sinh b \cosh \gamma - \cosh a \cosh b, \]
(ii) \[ \sinh a : \sinh \alpha = \sinh b : \sinh \beta = \sinh c : \sinh \gamma, \]
(iii) \[ \coth \alpha \sinh \gamma = \cosh \gamma \cosh b - \coth a \sinh b. \]

An alternative proof would be to use the decomposition into pentagons along \( r \) and to proceed as in the proof of (2.3.2), but now with the pentagon formulæ.

2.4.2 Theorem. Let \( x, y \) and \( z \) be any positive real numbers. Then there exists a unique convex right-angled geodesic hexagon \( a, \gamma, b, \alpha, c, \beta \) such that \( x = a, y = b \) and \( z = c \).
Proof. Uniqueness follows from Theorem 2.4.1. A construction has been described in Section 1.7. We also may paste together two pentagons as in Fig. 2.4.1. For this we first determine $s \in ]0, b[$ by the equation

$$\frac{\sinh(b - s)}{\sinh s} = \frac{\cosh c}{\cosh a'}$$

where $a, b, c = x, y, z$. After that we let $r > 0$ be defined by the equation $\sinh r \sinh s = \cosh a$. Then $\sinh r \sinh(b - s) = \cosh c$. By Lemma 2.3.5 and Theorem 2.3.4, the pentagons of Fig. 2.4.1 exist for our values of $r, s, a, b$ and $c$.

The trigonometric formulae show the strong analogy between triangles and right-angled hexagons (Theorems 2.2.1 and 2.4.1). Another analogy is the following.

2.4.3 Theorem. In any convex right-angled geodesic hexagon the three altitudes are concurrent.

Proof. By definition, an altitude is the common perpendicular of two opposite sides. By Theorem 1.5.3(iii), the three altitudes exist and are contained in the hexagon. We prove their concurrence as an application of trigonometry.

![Figure 2.4.2](image)

With the notation as in Fig. 2.4.2, we let $p$ be the intersection of altitudes $a\alpha$ and $b\beta$. We drop the perpendiculars $p\gamma$ from $p$ to $\gamma$ and $pc$ from $p$ to $c$. By Theorem 2.3.1(ii), the acute angles satisfy the relation

$$\cos u \cos v \cos w = \cos u' \cos v' \cos w',$$

where $u = u'$. Hence, if $v < v'$, then $w > w'$ and vice versa. But $v + w' = v' + w$. This implies that $v = v'$ and $w = w'$. Hence $p\gamma$ and $pc$ together form
the remaining altitude $c\gamma$. This proves the theorem.

While convex hexagons are important for the construction of Riemann surfaces, self-intersecting hexagons like that of Fig. 2.4.3 are important for the twist parameters (cf. Section 3.3).

**2.4.4 Theorem.** *In a right-angled geodesic hexagon $a, \gamma, b, \alpha, c, \beta$ with intersecting sides $c$ and $\gamma$, we have*

$$\cosh c = \sinh a \sinh b \cosh \gamma + \cosh a \cosh b.$$  

**Proof.** This follows from Theorem 2.2.6. Another proof, based on pentagons and trirectangles is as follows.

Assume first that the geodesic extensions of $\beta$ and $b$ have a common perpendicular $r$. Since $c$ and $r$ are orthogonal to $\beta$, they do not intersect. Hence, we have two right-angled pentagons, one with sides $a, \gamma, s, r$ and $t$ and the other with sides $c, \alpha, (b + s), r$ and $(t - \beta)$. Theorem 2.3.4(i) (right-angled pentagons) yields:

$$\cosh c = \sinh r \sinh(b + s)$$
$$= \sinh r \sinh b \cosh s + \sinh r \cosh b \sinh s$$
$$= \sinh a \sinh b \cosh \gamma + \cosh a \cosh b.$$

Assume next that the extensions of $\beta$ and $b$ intersect. The pentagons are now replaced by two trirectangles, one with sides $a, \gamma, s$ and $t$, the other with sides $c, \alpha, (b + s)$ and $(t - \beta)$. The proof is the same, now with Theorem 2.3.1(iii) and (v) (trirectangles). The remaining case, where $b$ and $\beta$ meet at infinity, is filled-in by a continuity argument in the proof.  

\[\Box\]
2.5 Variable Curvature

This section belongs to a small subculture of the book consisting of Sections 2.5, 4.3 and 5.4. In these sections we extend some of the results to variable curvature. The results, in this form, will not be used in other parts of the book. We use two comparison theorems. The first one is Sturm's theorem, where we use Klingenberg [1, 2], Lemma 6.5.5 as a reference, the second one is Toponogov's theorem, for which we refer to Cheeger-Ebin [1].

Pentagons and hexagons are also useful on surfaces of variable curvature. In this section we show that they satisfy trigonometric inequalities. We derive them from the constant curvature case with Sturm's comparison theorem.

Throughout the section (with the exception of the final lemma) we make the following assumption. $M$ is a complete simply connected two dimensional Riemannian manifold of negative curvature $K$ with the following bounds

$$-\kappa^2 \leq K \leq -\omega^2 < 0,$$

where $\kappa$ and $\omega$ are positive constants.

Using Hadamard's theorem (Klingenberg [1, 2], Theorem 6.6.4), we introduce polar coordinates $(\rho, \sigma)$ centered at some given point $p_0 \in M$. These coordinates are valid on $M - \{p_0\}$. We consider $\sigma$ to be an element of $S^1$. The metric tensor has the following form

$$ds^2 = d\rho^2 + f^2(\rho, \sigma) \, d\sigma^2$$

with a smooth positive function $f: ]0, \infty[ \times S^1 \mapsto \mathbb{R}$. Sturm's comparison theorem now states that

$$\frac{1}{\kappa} \sinh \kappa \rho \geq f(\rho, \sigma) \geq \frac{1}{\omega} \sinh \omega \rho.$$

This suggests introducing the comparison metrics

$$ds^2_\tau = d\rho^2 + \frac{1}{\tau^2} \sinh^2 \tau \rho \, d\sigma^2$$

of constant curvature $-\tau^2$ for $\tau = \kappa$ and $\tau = \omega$. The three lengths of a smooth curve $c$ on $M$ with respect to $ds^2_\kappa$, $ds^2$ and $ds^2_\omega$ will be denoted respectively by $\ell_\kappa(c)$, $\ell(c)$ and $\ell_\omega(c)$. By (2.5.3) we have the following inequalities, for any such curve,

$$\ell_\kappa(c) \geq \ell(c) \geq \ell_\omega(c).$$

Note that for each $\sigma$ the straight line

$$\rho \mapsto (\rho, \sigma), \quad \rho \in ]0, \infty[, \quad \sigma \in S^1,$$
is a unit speed geodesic with respect to all three metrics. This will be used tacitly below.

In order to translate a trigonometric identity from curvature \(-1\) to curvature \(-\tau^2\), we replace every length \(x\) by \(\tau x\) (curvature \times length^2\) is a scaling invariant). Thus, Theorem 2.2.2(i) (right-angled triangles) on \((M, ds^2)\) reads as before, but with every argument \(x\) replaced by \(\tau x\):

\[
\cosh \tau c = \cosh \tau a \cosh \tau b,
\]

etc. If the curvature is non-constant, the identities can be replaced by inequalities. We give four examples.

2.5.8 Theorem. For any right-angled geodesic triangle \(a, b, c\) in \(M\) with right angle \(\gamma\) we have

\[
\cosh \kappa a \cosh \kappa b \geq \cosh \kappa c,
\]

\[
\cosh \omega c \geq \cosh \omega a \cosh \omega b.
\]

**Proof.** We introduce polar coordinates centered at the vertex of \(\gamma\). By the above remark, \(a\) and \(b\) are the sides of a right-angled triangle with respect to all three metrics. The corresponding hypothenuses \(c\), \(c\), and \(c\) are distinct curves, in general. Using the fact that geodesics are shortest connecting curves, we obtain from (2.5.5)

\[
\ell_c(c) \geq \ell(c) \geq \ell_a(c) \geq \ell_o(c),
\]

where in our notation \(\ell(c) = c\). The theorem follows now from (2.5.7).

2.5.9 Theorem. For any right-angled geodesic pentagon in \(M\) with consecutive sides \(a, b, \alpha, c, \beta\), we have

\[
\sinh \kappa a \sinh \kappa b \geq \cosh \kappa c,
\]

\[
\cosh \omega c \geq \sinh \omega a \sinh \omega b.
\]

**Proof.** We use polar coordinates with the common vertex \(p_0\) of sides \(a\) and \(b\) as center, and introduce the comparison metrics (2.5.4) for \(\tau = \kappa\) and \(\tau = \omega\). To prove the first inequality we draw in \((M, ds^2)\) the perpendiculars \(\beta'\) and \(\alpha'\) at the endpoints \(a\) and \(b\). In \((M, ds^2)\) the curves \(\beta'\) and \(\alpha'\) are generally not geodesics but they are orthogonal to \(a\) and \(b\) with respect to both metrics. This follows from (2.5.2).

We check that Fig. 2.5.1 is drawn correctly in the sense that \(\beta'\) and \(\alpha'\) do not intersect the open strip between \(\beta\) and \(\alpha\). For this we let \(p' \in \beta'\) (respectively, \(p' \in \alpha'\)) and consider the geodesic arc (with respect to either metric) \(w\) from \(p_0\) to \(p'\).
Dropping in $(M, ds^2)$ the perpendicular $u$ from $p'$ to the geodesic extension of $a$, we obtain a right-angled triangle $u, v, w$, where $v$ is on the extension of $a$ and $w$ is the hypotenuse. In $(M, ds_x^2)$, $w$ is the hypotenuse of a right-angled triangle $u', a, w$, where $u'$ lies on $\beta'$. Theorem 2.5.8 yields

$$\cosh kv \cosh \kappa \ell(u) \geq \cosh kw = \cosh \kappa a \cosh \kappa \ell_x(u').$$

Since in negative curvature the perpendiculars are shortest connecting curves, we have by (2.5.5)

$$\ell_x(u') \geq \ell(u') \geq \ell(u).$$

With this and the preceding inequality, $v \geq a$ follows. Since $(M, ds^2)$ has negative curvature, the perpendiculars $u$ and $\beta$ either coincide or else are disjoint. This proves that $p'$ is not contained in the open strip between $\beta$ and $\alpha$.

We conclude that $\beta'$ and $\alpha'$ have positive distance in $(M, ds_x^2)$. Namely, any connecting curve $c'$ from $\beta'$ to $\alpha'$ intersects $\beta$ and $\alpha$ so that, by (2.5.5),

$$\ell_x(c') \geq \ell(c') \geq \ell(c) = c.$$

In $(M, ds_x^2)$, $\beta'$ and $\alpha'$ therefore have a common perpendicular. We let $c'$ be this perpendicular. It follows that

$$\sinh \kappa a \sinh \kappa b = \cosh \kappa \ell_x(c') \geq \cosh \kappa c.$$

The second inequality of the theorem is proved in a similar way, with $\beta'$ and $\alpha'$ in $(M, ds_\omega^2)$, where now $\beta'$ and $\alpha'$ do not intersect the exterior of the strip between $\beta$ and $\alpha$. It possible that $\beta'$ and $\alpha'$ intersect or have a common endpoint at infinity. In this case we have $\sinh \omega a \sinh \omega b \leq 1$, and the second inequality of the theorem is trivially true. $\diamondsuit$
2.5.10 Corollary. Under the hypothesis of Theorem 2.5.9, we have
\[
\cosh \kappa c \geq \coth \kappa \alpha \coth \kappa \beta, \\
\cosh \omega c \leq \coth \omega \alpha \coth \omega \beta.
\]

Proof. Twice applying the first inequality of Theorem 2.5.9 (with cyclic permutation) yields
\[
\cosh^2 \kappa \beta \leq \sinh^2 \kappa \alpha (\cosh^2 \kappa b - 1) \\
\leq \sinh^2 \kappa \alpha (\sinh^2 \kappa \beta \sinh^2 \kappa c - 1) \\
= \sinh^2 \kappa \alpha (\sinh^2 \kappa \beta \cosh^2 \kappa c - \cosh^2 \kappa \beta).
\]
This gives the first inequality. The second inequality is proved in the same way. 

Our final example will be applied in Section 4.2 to obtain a sharp lower bound for the length of a closed geodesic with transversal self-intersections.

2.5.11 Theorem. For any right-angled convex geodesic hexagon in \( M = (M, ds^2) \) with consecutive sides \( a, \gamma, b, \alpha, c, \beta \), the following hold.
\[
\sinh \kappa a \sinh \kappa b \cosh \kappa \gamma - \cosh \kappa a \cosh \kappa b \geq \cosh \kappa c, \\
\cosh \omega c \geq \sinh \omega a \sinh \omega b \cosh \omega \gamma - \cosh \omega a \cosh \omega b.
\]

Proof. We begin with the second inequality. By the scaling invariance of curvature \( \times \) length\(^2\), we may assume that \( \omega = 1 \). The idea is to use a comparison hexagon in the hyperbolic plane. For this we draw the consecutive orthogonal geodesic arcs \( a, \gamma, b \) in \( \mathbb{H} \) and extend this configuration into a hexagon. If this is impossible, then we are done because then
\[
1 \geq \sinh a \sinh b \cosh \gamma - \cosh a \cosh b.
\]
(Increase \( \gamma \) until the hexagon comes into existence, then use Theorem 2.4.1 and observe that the function \( t \mapsto \sinh a \sinh b \cosh t - \cosh a \cosh b \) is

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{figure2.5.2}
\end{array}
\]

Figure 2.5.2
monotone increasing.)

Assume therefore that the completion, say \( a, \gamma, b, \alpha', c', \beta' \), exists. Divide both hexagons into right-angled pentagons as shown in Fig. 2.5.2 (the existence of \( r \) is proved as in the case of constant curvature with the Arzelà-Ascoli theorem). By the second inequality in Theorem 2.5.9, applied to the pentagon on the left-hand side of the first hexagon, we have \( r \geq r' \).

Since \( \sinh r' \sinh b_1' = \cosh a \geq \sinh r \sinh b_1 \), we find that \( b_1' \geq b_1 \) and \( b_2 \geq b_2' \). Hence,

\[
\cosh c \geq \sinh r \sinh b_2 \geq \sinh r' \sinh b_2' = \cosh c',
\]

where \( \cosh c' = \sinh a \sinh b \cosh \gamma - \cosh a \cosh b \).

The first inequality of the theorem will be proved in a similar way, except that the inequalities obtained at intermediate steps will at the same time be used to prove that the comparison hexagon exists. For this we assume that \( \kappa = 1 \), that is, we assume the curvature bounds \(-1 \leq K < 0\).

Draw orthogonal sides \( a, \gamma \), in \( H \). This configuration can be completed into a right-angled pentagon \( a, \gamma, b_1'', r'', \beta''_1 \). (This has been shown in the proof of Theorem 2.5.9, see Fig. 2.5.1, where \( a \) and \( b \) play the roles of \( a \) and \( \gamma \) of Fig. 2.5.2). By Theorem 2.5.9 we have \( r \leq r'' \), \( b_1'' \leq b_1 \), and \( b_2 \leq b_2'' \), where \( b_1'' + b_2'' = b \). For the same reason, there exists a pentagon with sides \( r, b_2 \) in \( H \). Hence, we complete the bigger sides \( r'' \) and \( b_2'' \) to find the right-angled pentagon \( r'', b_2'', \alpha'', c'', \beta''_2 \), where, by Theorem 2.5.9 and by the monotonicity of the involved trigonometric functions, \( c'' \geq c \). Pasting the two pentagons together along \( r'' \), we get the desired comparison hexagon. \( \diamond \)

For better reference we restate the last part as a corollary.

**2.5.12 Corollary.** Let \( a, \gamma, b, \alpha, c, \beta \) be a right-angled convex geodesic hexagon in \( M \), where \( M \) is a complete surface of curvature \( K \) satisfying \(-1 \leq K < 0\). Then there exists a right-angled convex geodesic hexagon \( a, \gamma, b, \alpha'', c'', \beta'' \) in \( H \) and this hexagon satisfies \( c'' \geq c \). \( \diamond \)

We do not know whether the inequalities above, which involve the lower curvature bound, remain valid if the curvature is allowed to assume positive values. In the proofs we needed the non-positive curvature assumption in order to work with polar coordinates. In some cases however, stronger methods can be applied which are not restricted to non-positive curvature. As an illustration, we apply Toponogov's theorem to prove the following lemma. (The lemma will be needed in Section 4.3 to extend the collar theorem to variable curvature.)
2.5.13 Lemma. Let $G$ be a (simply connected) right-angled geodesic pentagon in a surface of arbitrary curvature $K \geq -1$. Then any pair of adjacent sides $a_1, a_2$ satisfies the relation \( \sinh a_1 \sinh a_2 > 1 \).

Proof. We use the notation of Fig. 2.5.3. A new feature is that we no longer have the uniqueness theorems for connecting geodesics arcs and perpendiculars. However, we still have the existence of geodesic arcs in $G$ of minimal length in the various homotopy classes. This follows, as always, from the Arzelà-Ascoli theorem, now applied to the compact metric space $G$. In the arguments which follow, "minimal" is meant with respect to $G$. Also, all arcs considered will be arcs in $G$.

Replace each side $a_i$ of $G$ by a shortest geodesic arc $a'_i$ in $G$ having the same endpoints. Since the arcs have minimal length, they do not intersect each other, except for the common vertices, and we obtain a geodesic pentagon $G'$ in $G$. We triangulate $G'$ as shown in Fig. 2.5.3(a) with diagonals which again have minimal length in their homotopy classes. By Toponogov's comparison theorem (Cheeger-Ebin [1]), there exists a geodesic pentagon $G''$ in $H$ with sides $a''_i$ of length $\ell(a''_i) = \ell(a'_i)$, $i = 1, \ldots, 5$, such that all interior angles of $G''$ are less than or equal to the corresponding angles of $G'$. In particular, all interior angles of $G''$ are less than or equal to $\pi/2$.

In $H$ we drop the perpendiculars $\alpha_1$ from $A''_1$ to $a''_1$, $\alpha_2$ from $A''_2$ to $a''_2$, $\alpha_3$ from $A''_3$ to $a''_2$ and $\alpha_4$ from $A''_4$ to $a''_3$ (Fig. 2.5.3(b)). We obtain a right-angled geodesic pentagon $\tilde{G}$ in $H$ with sides $\tilde{a}_i$ of length $\ell(\tilde{a}_i) \leq \ell(a''_i) \leq \ell(a'_i)$. By virtue of the pentagon formula (Theorem 2.3.4(i)) this proves the lemma. \[\Box\]
2.6 Appendix: The Hyperboloid Model Revisited

In this section we describe a more algebraic aspect of the hyperboloid model of the hyperbolic plane. The new feature is the vector product associated with the bilinear form \( h \). This will allow the unification of the hyperbolic plane and its isometry group into a quaternion algebra which one may call the quaternion model of the hyperbolic plane. As an application we prove the triple trace theorem (Theorem 2.6.16 and Corollary 2.6.17) and an improved version of the general sine and cosine laws of Thurston [1] (Theorem 2.6.20).

**The Quaternion Model**

As in Section 2.1 we consider the bilinear form \( h : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \) given by

\[
h(X, Y) = x_1y_1 + x_2y_2 - x_3y_3.
\]

\( h \) is a symmetric bilinear form of signature \((2, 1)\). To simplify the notation we shall write

\[
h(X, Y) = XY.
\]

\( \mathbb{R}^3 \) together with this product will be denoted by \( \mathcal{H}_0 \).

In Section 2.1 the hyperboloid model was introduced as the surface

\[
H = \{ X \in \mathcal{H}_0 \mid h(X, X) = -1, \, x_3 > 0 \}.
\]

The geodesics are the sets \( H \cap \Gamma \), where \( \Gamma \) is a plane through the origin of \( \mathcal{H}_0 \). Using the action of \( \Omega \) (cf. Lemma 2.1.4), we easily check that if \( C \in \mathcal{H}_0 - \{ 0 \} \) is a vector \( h \)-orthogonal to \( \Gamma \), then \( h(C, C) > 0 \). Conversely, for every non-zero vector \( C \) with \( h(C, C) > 0 \), the \( h \)-orthogonal plane \( \Gamma \) of \( C \) intersects \( H \) in a geodesic. We may thus say that \( C \) represents a geodesic whenever \( h(C, C) > 0 \), and \( C \) represents a point whenever \( h(C, C) < 0 \). This leads to the following

**2.6.1 Definition.** A vector \( X \in \mathcal{H}_0 - \{ 0 \} \) is point-like if \( h(X, X) < 0 \), geodesic-like if \( h(X, X) > 0 \), and infinity-like if \( h(X, X) = 0 \).

Our point of view is to see \( \mathcal{H}_0 \) as an example of hyperbolic geometry with the basic geometric quantities: points and lines represented by point-like and geodesic-like vectors. For this we call vectors equivalent if and only if they differ by a non-zero factor. The equivalence classes of the point-like, geodesic-like and infinity-like vectors are respectively the points, the geodesics (or lines)
and the points at infinity. As usual we shall identify equivalence classes with their representatives. The set of all points, lines and points at infinity is denoted by \( \mathcal{H}' \).

### 2.6.2 Definition.
Two vectors \( X, Y \in \mathcal{H}_0 - \{0\} \), along with the equivalence classes represented by these vectors, are called *incident* iff \( XY = 0 \).

To compare this with the hyperboloid model \( H \), we use the following notation. If \( X \) is point-like we denote by \( X^* \) the intersection point of \( H \) with the straight line spanned by \( X \). If \( X \) is geodesic-like we denote by \( X^* \) the geodesic in \( H \) which is cut out by the \( h \)-orthogonal plane of \( X \). In this way we have a natural one-to-one correspondence between the points and lines of \( \mathcal{H}' \) and the points and geodesics of \( H \). Finally, if \( X \) is infinity-like we denote by \( X^* \) the straight line spanned by \( X \) and call \( X^* \) a *point at infinity* of \( H \).

We now interpret the above incidence relation. If \( X \in \mathcal{H}_0 \) is point-like, we may, by virtue of Lemma 2.1.4 (the twice transitivity of \( \Omega \)), assume that \( X = (0, 0, 1) \) and then easily check that \( X \) is incident with \( Y \) if and only if \( Y \) is geodesic-like with the corresponding geodesic \( Y^* \) containing the point \( X^* \).

If \( X \) is infinity-like, then \( X \) can be incident with \( Y \) in exactly two cases: (i) \( Y^* = X^* \), and (ii) \( Y \) is geodesic-like and \( X^* \) is an endpoint at infinity of the geodesic \( Y^* \). (Again use Lemma 2.1.4.)

If both \( X \) and \( Y \) are geodesic-like, then \( X \) and \( Y \) are incident if and only if the geodesics \( X^* \) and \( Y^* \) are orthogonal.

With our definition of incidence we can restate and complete Theorems 1.1.4 - 1.1.6 as follows.

### 2.6.3 Theorem.
If \( \alpha \) and \( \beta \) are distinct elements of \( \mathcal{H}' \), then there exists a unique \( \gamma \in \mathcal{H}' \) which is incident with \( \alpha \) and \( \beta \).

**Proof.** Check the various cases, or wait until the end of the proof of Theorem 2.6.7. \( \diamond \)

The theorem becomes more tangible if we introduce as a new tool the vector product associated with \( h \).

Let \( I = (1, 0, 0), J = (0, 1, 0) \) and \( K = (0, 0, 1) \). Then \( \{I, J, K\} \) is an orthonormal basis of \( \mathcal{H}_0 \) in the sense that \( IJ = IK = JK = 0 \) and

\[
II = JJ = -KK = 1.
\]

### 2.6.4 Definition.
The vector product associated with \( h \) is the unique antisymmetric bilinear mapping \( \wedge : \mathcal{H}_0 \times \mathcal{H}_0 \to \mathcal{H}_0 \) satisfying
\[ I \wedge J = K, \quad I \wedge K = J, \quad J \wedge K = -I. \]

In the following lemma, \( \det(A, B, C) \) is the determinant of the matrix formed by the coordinate vectors of \( A, B, C \) with respect to the basis \( \{ I, J, K \} \).

2.6.5 Lemma. The vector product has the following properties:

(i) \( (A \wedge B)C = A(B \wedge C) = -\det(A, B, C) \),
(ii) \( A \wedge (B \wedge C) = (AB)C - (AC)B \),
\[ (A \wedge B) \wedge C = (BC)A - (AC)B, \]
(iii) \( (A \wedge B)(C \wedge D) = (AD)(BC) - (AC)(BD) \),
(iv) \( (A \wedge B) \) is \( h \)-orthogonal to \( A \) and \( B \).

Proof. By the linearity in all arguments it suffices to check (i) and the first identity in (ii) on the above orthonormal basis. The second identity in (ii) is a consequence of the first. (iii) is a consequence of (i) and the first identity in (ii). The final statement, (iv), is a consequence of (i). \( \diamond \)

From the lemma follows that up to a factor \( \pm 1 \) the vector product \( \wedge \) is independent of our particular choice of the orthonormal basis \( \{ I, J, K \} \). In fact, let \( A, B, C \) be \( h \)-orthonormal vectors. Then from the incidence relations (or by using \( \Omega \)) we see that two of the vectors are geodesic-like and one is point-like. We may therefore assume that \( AA = BB = -CC = 1 \). By (iii) we have \( (A \wedge B)(A \wedge B) = -1 \). Therefore, by (iv), \( A \wedge B = eC \) with \( e = \pm 1 \). Now (i) and (iv) imply \( A \wedge C = eB \) and \( B \wedge C = -eA \). Hence, the vector product based on \( \{ A, B, C \} \) differs from \( \wedge \) only by the factor \( e \).

As in the Euclidean case we have the following lemma.

2.6.6 Lemma. \( A \wedge B = 0 \) iff \( A \) and \( B \) are linearly dependent.

Proof. For each \( B = b_1 I + b_2 J + b_3 K \), the linear mapping \( A \mapsto A \wedge B \) is represented by the following matrix (operating on column vectors) with respect to the basis \( \{ I, J, K \} \)

\[
\begin{pmatrix}
0 & -b_3 & b_2 \\
-b_3 & 0 & -b_1 \\
b_2 & b_1 & 0
\end{pmatrix}.
\]

If \( B \neq 0 \), this matrix has rank 2, so the kernel of this mapping consists of the multiples of \( B \). \( \diamond \)

Here is a first application of the vector product.
2.6.7 Theorem. If $A, B \in \mathcal{H}_0$ represent distinct elements of $\mathcal{H}'$, then $A \wedge B$ represents the unique element which is incident with $A$ and $B$.

Proof. By Lemma 2.6.6 we have $A \wedge B \neq 0$. By Lemma 2.6.5(iv) $A \wedge B$ is incident with $A$ and $B$. If $C$ is another vector incident with $A$ and $B$, then by Lemma 2.6.5(ii), $(A \wedge B) \wedge C = 0$. By Lemma 2.6.6, $C$ is a multiple of $A \wedge B$, i.e. $C$ and $A \wedge B$ represent the same element of $\mathcal{H}'$. This proves the theorem. At the same time this provides a proof of Theorem 2.6.3. \hfill \diamond

Let us next look at the isometries of $\mathcal{H}_0$, or, more precisely at the endomorphisms of $\mathcal{H}_0$ which preserve $h$. It is not difficult to check that the subgroup $\Omega \subset \text{GL}(3, \mathbb{R})$ of Section 2.1 acts by isometries and preserves the vector product. However, we want to redevelop the isometries in a more algebraic way which uses quaternions. We begin by adding an additional dimension to $\mathcal{H}_0$ by taking the direct sum of vector spaces:

$$\mathcal{H} = \{ \mathcal{A} = a + A \mid a \in \mathbb{R}, A \in \mathcal{H}_0 \}.$$ 

$\mathcal{H}$ is a vector space with basis $\{ 1, I, J, K \}$. The vector product is extended to $\mathcal{H}$ as follows.

2.6.8 Definition. For $A, B \in \mathcal{H}_0$ we define

$$A \ast B = AB + A \wedge B,$$

and for $A = a + A, B = b + B \in \mathcal{H}$ we define

$$\mathcal{A} \ast \mathcal{B} = ab + aB + bA + A \ast B.$$ 

This is the distributive extension of all previous products and we check that $\mathcal{H}$ together with $+$ and $\ast$ is an algebra. More precisely, $\mathcal{H}$ is a quaternion algebra of type $(1, 1)$ (Vignéras [3]). $(\mathcal{H}, \ast)$ is the quaternion model of hyperbolic geometry.

The hyperbolic plane in this model is the set of all equivalence classes of point-like vectors of $\mathcal{H}_0 \subset \mathcal{H}$. The geodesics are the equivalence classes of the geodesic-like vectors. Since we identify equivalence classes by their representatives, we may rephrase this by saying that $\mathcal{H}$ contains the hyperbolic plane and that $\mathcal{H}$ also contains the geodesics. We next describe how the isometries sit in $\mathcal{H}$ and how the various relationships may be expressed in terms of the algebra.

The following are basic concepts of quaternion algebras (for a general introduction to quaternion algebras and applications to Riemann surfaces we refer to Vignéras [3]).
2.6.9 Definition. For every $\mathcal{A} = a + A \in \mathcal{H}$ we define the quaternion conjugate $\bar{\mathcal{A}} := a - A$, the norm $n(\mathcal{A}) := \mathcal{A} \ast \bar{\mathcal{A}}$, and the trace $\text{tr}(\mathcal{A}) := a$.

Observe that the trace allows us to rewrite the product $h$ on $\mathcal{H}_0$ in terms of the quaternion algebra:

$$h(A, B) = AB = \text{tr}(A \ast B), \quad A, B \in \mathcal{H}_0.$$  

2.6.10 Lemma. $\text{tr} : \mathcal{H} \rightarrow \mathbb{R}$ is a vector space homomorphism. Quaternion conjugation is a vector space homomorphism which satisfies the rule

$$\bar{\mathcal{A}} \ast \bar{\mathcal{B}} = \bar{\mathcal{B}} \ast \bar{\mathcal{A}},$$

and $n : \mathcal{H} \rightarrow \mathbb{R}$ is a multiplicative homomorphism with respect to $\ast$.

Proof. Clearly, trace and conjugation are vector space homomorphisms, and the rule is easily checked on the basis vectors $1, I, J$ and $K$. With this rule we compute $n(\mathcal{A} \ast \mathcal{B}) = \mathcal{A} \ast \mathcal{B} \ast \bar{\mathcal{A}} \ast \bar{\mathcal{B}} = \mathcal{A} \ast \mathcal{B} \ast \bar{\mathcal{B}} \ast \bar{\mathcal{A}} = \mathcal{A} \ast n(\mathcal{B})\bar{\mathcal{A}} = n(\mathcal{A})n(\mathcal{B})$. 

We introduce the subsets $\mathcal{H}^1$ and $\mathcal{H}^{-1}$.

2.6.11 Definition.

$$\mathcal{H}^1 = \{ a + A \in \mathcal{H} | a^2 - AA = 1 \}, \quad \mathcal{H}^{-1} = \{ a + A \in \mathcal{H} | a^2 - AA = -1 \}.$$  

Since $n$ is a $\ast$-homomorphism, $\mathcal{H}^1$ and $\mathcal{H}^1 \cup \mathcal{H}^{-1}$ are groups with respect to $\ast$. For $\mathcal{A} = a + A \in \mathcal{H}^1 \cup \mathcal{H}^{-1}$ the inverse $\mathcal{A}^{-1}$ is given by

$$\mathcal{A}^{-1} = a - A \text{ if } \mathcal{A} \in \mathcal{H}^1, \quad \mathcal{A}^{-1} = -a + A \text{ if } \mathcal{A} \in \mathcal{H}^{-1}.$$  

For $\mathcal{A} \in \mathcal{H}^1 \cup \mathcal{H}^{-1}$ and $\mathcal{B} \in \mathcal{H}$ we compute that

$$\text{tr}(\mathcal{A}^{-1} \ast \mathcal{B} \ast \mathcal{A}) = \text{tr}(\mathcal{B}).$$

Since $\mathcal{H}_0$ is the set of quaternions with trace zero, it follows that $\mathcal{H}^1$ and $\mathcal{H}^1 \cup \mathcal{H}^{-1}$ act on $\mathcal{H}_0$ by conjugation (conjugation in the sense of groups). Moreover, the following is also true.

2.6.12 Lemma. For each $\mathcal{A} \in \mathcal{H}^1 \cup \mathcal{H}^{-1}$ the vector space endomorphism $X \mapsto \mathcal{A}^{-1} \ast X \ast \mathcal{A}, X \in \mathcal{H}_0$ is an isometry with respect to $h$. This isometry commutes with the vector product.

Proof. We use the fact that for $X, Y \in \mathcal{H}_0$ the bilinear form $h$ can be written as $h(X, Y) = \text{tr}(X \ast Y)$ (see above). Hence $h(\mathcal{A}^{-1} \ast X \ast \mathcal{A}, \mathcal{A}^{-1} \ast Y \ast \mathcal{A}) = \text{tr}(\mathcal{A}^{-1} \ast X \ast \mathcal{A} \ast \mathcal{A}^{-1} \ast Y \ast \mathcal{A}) = \text{tr}(\mathcal{A}^{-1} \ast Y \ast \mathcal{A}) = h(X, Y)$. This proves
the first claim. The equation $X \wedge Y = X \ast Y - h(X, Y)$ proves the second claim.

Note in particular that for $A \in \mathcal{H}^{-1}$ conjugation with $A$ is commutative and not, as on might expect, anticommutative with the vector product. For the action of $A \in \mathcal{H}^1 \cup \mathcal{H}^{-1}$ on $\mathcal{H}_0$ we use the notation

$$A^{-1} \ast X \ast A = A[X].$$

In axiomatic geometry the isometries are studied by looking at the half-turns (point symmetries) and the reflections (axial symmetries). This is easy to undertake in $\mathcal{H}$. In the following, $\delta$ is either 1 or $-1$.

2.6.13 Proposition. Let $A \in \mathcal{H}_0$ with $AA = -\delta$. Set $A = A$ and interpret $A$ as an element of $\mathcal{H}^\delta$ acting on $\mathcal{H}_0$ by conjugation. Then

(1) $A[A] = A$,

(2) $A[X] = -X$ for all $X \in \mathcal{H}_0$ which are incident with $A$.

Proof. (i) is clear. For (ii), with Lemma 2.6.5 we compute that

$$(iii) \quad A[X] = -2\delta(AX)A - X$$

and recall that, by definition, $X$ is incident with $A$ if and only if $AX = 0$.

If we interpret $A$ as acting on $\mathcal{H}'$ and denote, as earlier, by $A^*, X^*$, etc. the elements in $\mathcal{H}'$ represented by $A$, $X$, etc., then (iii) shows that $A \neq id$. Moreover, $A$ is a half-turn about $A^*$ if $A$ is point-like (it fixes each geodesic $X^*$ through $A^*$), and $A$ is a symmetry with axis $A^*$ if $A$ is geodesic-like (it fixes the points $X^*$ on $A^*$ and the geodesics $Y^*$ orthogonal to $A^*$).

The two-fold transitive action of $\mathcal{H}^1$ on $\mathcal{H}'$ can now be proved in the same way as in axiomatic geometry:

If $X^*$ and $Y^*$ are points, we can chose the representatives $X$ and $Y$ in $\mathcal{H}_0$ such that $XX = YY$ and such that $XY < 0$. This implies that $(X + Y)$ is again point-like and we can further normalize the representatives such that, in addition $(X + Y)(X + Y) = -1$. The proposition below will show that under these normalizations $(X + Y)^*$ is the mid-point of $X^*$ and $Y^*$ and, if $(X + Y)$ is interpreted as an element of $\mathcal{H}^1$, then $(X + Y)$ is the half-turn about $(X + Y)^*$ which interchanges $X^*$ and $Y^*$.

Similarly, if $X^*$ and $Y^*$ are geodesics, we can choose the representatives $X$ and $Y$ such that $XX = YY$ and $(X + Y)(X + Y) = 1$. The following proposition will show that $(X + Y)^*$ is either an angle bisector or the median of $X^*$ and $Y^*$, and that $(X + Y)$, if seen as element of $\mathcal{H}^{-1}$, is the symmetry with axis $(X + Y)^*$ and interchanges $X^*$ and $Y^*$. 
2.6.14 Proposition. Let $X, Y \in \mathcal{H}_0$ be linearly independent such that $XX = YY$ and $(X + Y)(X + Y) = -\delta$, where $\delta = \pm 1$. Set $\mathcal{A} = (X + Y)$. Then $\mathcal{A}$, when seen as an isometry, satisfies the following relations.

$$\mathcal{A}[X] = Y, \mathcal{A}[Y] = X \quad \text{and} \quad \mathcal{A}[X \wedge Y] = -X \wedge Y.$$  

Proof. This is obtained by a straightforward computation. The statement about $X \wedge Y$ can also be concluded from Proposition 2.6.13 because $X \wedge Y$ is incident with $X$ and $Y$ (cf. Theorem 2.6.7).

For every $\mathcal{A} \in \mathcal{H} \cup \mathcal{H}^{-1}$ the action of $\mathcal{A}$ on $\mathcal{H}$ coincides with the action of $-\mathcal{A}$. By looking at the fixed points (respectively, the fixed geodesics) we see that the kernel of the action is $\{ \pm 1 \}$ and that $\mathcal{H}^1 / \{ \pm 1 \}$ can be identified with the group $\Omega$ of Section 2.1.

In order to see the relation between $\mathcal{H}^1$ and the isometries in the upper half-plane model we associate with each basis vector of $\mathcal{H}$ a $(2 \times 2)$ matrix as follows.

\[
(2.6.15) \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

This mapping extends to an isomorphism between the quaternion algebra $\mathcal{H}$ and $\text{GL}(2, \mathbb{R})$, and the restriction to $\mathcal{H}^1$ is an isomorphism between $\mathcal{H}^1$ and $\text{SL}(2, \mathbb{R})$.

Resumé. The quaternion algebra $\mathcal{H}$ contains the points, the lines and the points at infinity in the subset $\mathcal{H}_0$; the incidence of elements $X, Y \in \mathcal{H}_0$ is given by the relation $\text{tr}(X \ast Y) = 0$; and the isometries are contained in $\mathcal{H}^1 \cup \mathcal{H}^{-1}$ which acts twice transitively by conjugation.

A Trace Relation

We postpone the introduction of distances and angles to the end of the section and now prove, as a first application, a theorem about traces in finitely generated subgroups of $\text{SL}(2, \mathbb{R})$. We start with a more general version in $\mathcal{H}^1$.

Consider $n$ distinct elements $X_1, \ldots, X_n \in \mathcal{H}^1$, (e.g. generators of a discrete subgroup). We can write words in $X_1, \ldots, X_n$ as follows. Let $m \in \mathbb{N}$, and consider a finite sequence

$$\omega = v_1, n_1, v_2, n_2, \ldots, v_m, n_m,$$

where every $v_i$ is an index, $v_i \in \{ 1, \ldots, n \}$, and every $n_i$ is an exponent,
\[ n_i \in \mathbb{Z}, i = 1, \ldots, m. \text{ Then write} \]
\[ W = W_\omega(X_1, \ldots, X_n) = X_{v_1}^{n_1} \ast X_{v_2}^{n_2} \ast \cdots \ast X_{v_m}^{n_m}. \]

We want to express this word in a different form using the traces from the following list
\[ \begin{align*}
\text{tr } X_i, & \quad i = 1, \ldots, n, \\
\text{tr}(X_i \ast X_j), & \quad 1 \leq i < j \leq n, \\
\text{tr}(X_i \ast X_j \ast X_k), & \quad 1 \leq i < j < k \leq n.
\end{align*} \]

For simplicity we rewrite this list as a sequence \( \tau_1, \ldots, \tau_N \), where \( \tau_i = \tau(X_1, \ldots, X_n) \), \( i = 1, \ldots, N \). (Later on, the group elements \( X_1, \ldots, X_n \) will be varied.) The following theorem has its roots in Fricke-Klein [1], p. 366.

2.6.16 Theorem. (Triple trace theorem I). For every sequence \( \omega = \nu_1, n_1, \nu_2, n_2, \ldots, \nu_m, n_m \) as defined above, there exists a sequence of real polynomial functions
\[ f_i = f_i(x_1, \ldots, x_N), \quad i = 0, \ldots, n, \]
\[ f_{ij} = f_{ij}(x_1, \ldots, x_N), \quad 1 \leq i < j \leq n \]
\(((x_1, \ldots, x_N) \in \mathbb{R}^N) \) which depend only on \( \omega \) such that any word \( W = W_\omega(X_1, \ldots, X_n) \) with \( X_1, \ldots, X_n \in \mathcal{H}^1 \) can be written in the form
\[ W = f_0(\tau_1, \ldots, \tau_N) + \sum_i f_i(\tau_1, \ldots, \tau_N)X_i + \sum_{i<j} f_{ij}(\tau_1, \ldots, \tau_N)X_i \ast X_j. \]

Proof. We first consider some elementary relations. If \( X_i = x_i + X_i \) with \( x_i = \text{tr}(X_i) \) and \( X_i \in \mathcal{H}_0 \), then \( X_i^{-1} = x_i - X_i = 2x_i - X_i \). This yields the relation
\[ X_i^{-1} = 2\text{tr}X_i - X_i, \]
and since \( X_i \in \mathcal{H}^1 \), we have
\[ X_iX_i = \text{tr}^2(X_i) - 1. \]
If \( X_j = x_j + X_j \), then, by the definition of the \( \ast \) -product,
\[ X_iX_j = \text{tr}(X_i \ast X_j) = \text{tr}(X_i) \text{tr}(X_j). \]
We also compute that
\[ X_i \wedge X_j = X_i \ast X_j - \text{tr}(X_i)X_j - \text{tr}(X_j)X_i - \text{tr}(X_i \ast X_j) + 2\text{tr}(X_i) \text{tr}(X_j). \]
Finally, if \( X_k = x_k + X_k \), then
\[ \text{tr}(X_i \ast X_j \ast X_k) = x_iX_jX_k + x_kX_iX_j + x_iX_jX_k + x_jX_iX_k + (X_i \wedge X_j)X_k. \]
The theorem is now proved by a straightforward induction: if \( W = X_i^{\pm 1} \), the
claim follows from (1). Assume therefore that the claim holds for $W$ and let $W' = W \ast X_k^{\pm 1}$ for some $k$. By (1) we may assume that $W' = W \ast X_k$. Then
\[ W' = f_0 X_k + \sum_i f_i X_i \ast X_k + \sum_{i<j} f_{ij} X_i \ast X_j \ast X_k. \]

By (2), (3) and (4), we can rewrite each term $X_i \ast X_k$ in the form given in the theorem. The same is verified for the terms $X_i \ast X_k \ast X_k$ if we apply (5) to $(X_i \wedge X_j)X_k$ and use that $(X_i \wedge X_j) \wedge X_k = (X_j X_k)X_i - (X_i X_k)X_j$ (Lemma 2.6.5(ii)). Hence, $W'$ can be rewritten in the desired form. \hfill \Box

With the isomorphism $\mathcal{H}^1 \to \text{SL}(2, \mathbb{R})$ given by (2.6.15) we immediately obtain the following.

**2.6.17 Corollary.** (Triple trace theorem II). Let $v_1, n_1, v_2, n_2, \ldots, v_m, n_m$, be a sequence as given above. Then there exists a polynomial function $f = f(x_1, \ldots, x_N)$ such that for any sequence $A_1, \ldots, A_n \in \text{SL}(2, \mathbb{R})$ the word
\[ W = A_{v_1}^{n_1} A_{v_2}^{n_2} \cdots A_{v_m}^{n_m} \]
has trace $\text{tr} W = f(\tau_1, \ldots, \tau_N)$, where $\tau_1, \ldots, \tau_N$ is the sequence of all traces $\text{tr} A_i, \text{tr} A_i A_j, \text{tr} A_i A_j A_k$, $1 \leq i < j < k \leq n$.

**Proof.** Use the correspondence given by (2.6.15) and observe that the trace of each $A \in \mathcal{H}^1$ is half the trace of its corresponding matrix in $\text{SL}(2, \mathbb{R})$. \hfill \Box

The General Sine and Cosine Formula

The final step in this essay is to introduce length and angular measure and then to give a new proof of the general sine and cosine formulae of Thurston [1], p. 218.

Our point of view is that most frequently, a trigonometric formula has to be applied to a configuration given by a drawn figure. We have therefore tried to find a formula together with an adjustment rule so that a reader who is used to reading figures may conveniently adjust the general formula to the given configuration.

It turns out that there are quite tedious sign conventions. In order to make them more applicable, we have decided to introduce oriented points and oriented arcs, but work with non-oriented measures. The orientations will then be used in the adjustment rule in order to rewrite the formula for the given case. The so adjusted formula then applies to the non-oriented lengths and angles.
The general sine and cosine law will be given in Theorem 2.6.20. Prior to it there will be a rule telling how to set arrows in the figure, and a list (Fig. 2.6.1) telling how to interpret the terms occurring in the general rule. The definitions which now follow are only needed for the proof.

For $U \in \mathcal{H}_0$ we denote again by $U^*$ the element in the hyperboloid $H$ represented by $U$ (cf. the paragraph after Definition 2.6.2). $\Omega$ is the twice transitive group generated by the isometries $L_\sigma$ and $M_\rho$ as defined in (2.1.2).

To orient the geodesics we first consider the case $U = I$. Here $I^*$ is a geodesic in $H$ which passes through the point $p_0 = (0, 0, 1)$ in an $h$-orthogonal plane of $I$. We parametrize $I^*$ with unit speed such that the tangent vector at $p_0$ becomes $-J$. This introduces an orientation on $I^*$ and we let $I^*$ denote the geodesic $I^*$ together with this orientation. If $U \in \mathcal{H}_0$ is an arbitrary geodesic-like vector, normalized so that $UU = 1$, then there exists an isometry $\phi \in \Omega$ which sends $I$ to $U$ and $I^*$ to $U^*$, and we denote by $U^*$ the geodesic $U^*$ together with the orientation induced from $I^*$ by $\phi$. We abbreviate this by writing

$$U^* = \phi(I^*).$$

Note that with our convention the tangent vector of $I^*$ at $p_0$ is $I^*$. Since every isometry of $\Omega$ which fixes $I$ also fixes the orientation of $I^*$, this definition is independent of the particular choice of $\phi$, and the following compatibility with $\Omega$ holds:

$$\Psi(U^*) = (\Psi(U))^* \text{ for all } \Psi \in \Omega.$$ 

Finally, if $U$ is geodesic-like satisfying $UU = a^2$, $a > 0$, then we define $U^* = (U/a)^*$. Note that $(-U)^*$ has opposite orientation.

To define the orientation of a point, we let $U \in \mathcal{H}_0$ be a point-like vector and denote by $U^*$ the point $U^* \in H$ together with an orientation which is defined to be positive if $U \mathcal{K} < 0$ (i.e. if $U$ has a positive third component) and negative otherwise. Again $(-U)^*$ has the opposite orientation of $U^*$.

Points at infinity will not be considered.

In the figures which follow, we adopt the following conventions. The orientation of a geodesic will be marked by an arrow (indicating the sense of the parametrization), and the orientation of a point will be marked by an oriented curve which goes around the point in the counterclockwise sense if the orientation is positive and in the clockwise sense otherwise. This convention commutes with the action of $\Omega$: if $V = \Psi(U)$ with $\Psi \in \Omega$, then marking $V$ in the figure is the same as first marking $U^*$ and then letting $\Psi$ operate on the figure. Finally, we adopt the convention that a counterclockwise rotation of $\pi/2$ is needed to rotate $I^*$ into $J^*$. (This is our definition of counterclockwise.)
In the next step we introduce a non-oriented measure and a parity for each of the following configurations (cf. Fig. 2.6.1 further down): (1) an angle flanked by two oriented geodesic arcs, (2) a geodesic arc flanked by two oriented perpendicular geodesics, (3) a geodesic arc flanked by two oriented points, (4) a geodesic arc flanked by an oriented point and an oriented perpendicular geodesic.

If the geodesic arcs in (1) intersect in opposite directions the curve marking the orientation of the vertex, then we define the parity \( \delta \) of the configuration to be 1. If the intersections are in the same direction we set \( \delta = -1 \). For the configurations (2) - (4), the parity is defined in a similar way (cf. Fig. 2.6.1). Note that in case (3) the parity is 1 if both endpoints have the same orientation. We also remark that the parity is preserved under the action of \( \Omega \).

We next define the non-oriented measures. To simplify the notation we use the following abbreviation for \( X \in \mathcal{H}_0 \):

\[ |X| = (|XX|)^{1/2}. \]

**2.6.18 Definition and Remarks.** (1) First let \( U^* \) and \( V^* \) with \( U, V \in \mathcal{H}_0 \) be the oriented geodesics intersecting at \( p \in H \) and carrying the geodesic arcs of the first configuration. There exists a unique \( \phi \in \Omega \) and a unique \( \sigma \in [-\pi, \pi] \) such that \( \phi(p) = p_0, \phi(U^*) = I^*, \phi(-\delta V^*) = L_\sigma(I^*) \), where \( \delta \) denotes the parity and \( L_\sigma \) is the rotation as in (2.1.2) with fixed point \( p_0 \) and angle \( \sigma \). We define \( a = |\sigma| \) to be the measure of the angle flanked by \( U^* \) and \( V^* \). This is just the ordinary angular measure and is independent of the orientations of the arcs. Using \( \phi \) we easily check that \( UV = -\delta \cos a \, |U||V| \) and \( |U \wedge V| = \sin a \, |U||V| \). For this configuration we define

\[ C(a) := \cos a, \quad S(a) := \sin a. \]

(2) Now let \( U^* \) and \( V^* \) be oriented geodesics in \( H \) having a common perpendicular, say \( \gamma \). There exists a unique \( \phi \in \Omega \) and a unique \( a \geq 0 \) such that \( \phi \) sends the intersection point of \( U^* \) and \( \gamma \) to \( p_0 \) and the intersection point of \( V^* \) and \( \gamma \) to \( M_a(p_0) \), where \( M_a \) is the isometry from (2.1.2) with axis \( f^* \) and displacement length \( a \). We define \( a \) to be the distance from \( U^* \) to \( V^* \). Again using \( \phi \), we verify the two equations \( UV = -\delta \cosh a \, |U||V| \) and \( |U \wedge V| = \sinh a \, |U||V| \). Here we define

\[ C(a) := \cosh a, \quad S(a) := \sinh a. \]

(3) If \( U^* \) and \( V^* \) are the oriented endpoints of a geodesic arc, we have a unique \( \phi \in \Omega \) and a unique \( a \geq 0 \) such that \( \phi \) sends \( U^* \) to \( p_0 \) and \( V^* \) to \( M_a(p_0) \), (disregarding the orientations). We define \( a \) to be the distance of \( U^* \) and \( V^* \), and check that \( UV = -\delta \cosh a \, |U||V|, |U \wedge V| = \sinh a \, |U||V| \). Here we define
C(a) := \cosh a, \quad S(a) := \sinh a.

(4) The remaining case is that \( U^* \) is a point and \( V^* \) a geodesic (or vice-versa). We let \( \gamma \) be the (non-oriented) geodesic through \( U^* \) orthogonal to \( V^* \). There exist uniquely determined \( \phi \in \Omega \) and \( a \geq 0 \), satisfying \( \phi(U^*) = p_0 \) and \( \phi(V^* \cap \gamma) = M_\alpha(p_0) \). (Again we disregard the orientation of curves.) We define \( a \) to be the distance from the point to the geodesic. In contrast to the above cases (1) - (3) we now have \( UV = -\delta \sinh a \|U\|\|V\| \) and \( |U \wedge V| = \cosh a \|U\|\|V\| \), and we define

\[
C(a) := \sinh a, \quad S(a) := \cosh a.
\]

Using the above isometries \( \phi \in \Omega \), we check that in cases (2) - (4) the arrow which marks the orientation of \( (U \wedge V)^* \) points from \( U \) to \( V \) if the parity is \(-1\), and from \( V \) to \( U \) if the parity is \(+1\). This is also true in case (1) if the arrow of the marking curve is drawn on the part which is inside the angular region of angle \( a \) flanked by the two geodesic arcs. We abbreviate this fact by saying that \((-\delta U \wedge V)^* \) points from \( U^* \) to \( V^* \). The above arguments yield:

2.6.19 Lemma. In each of the configurations (1) - (4) the following hold.

\[
UV = -\delta C(a) \|U\|\|V\|, \quad |U \wedge V| = S(a) \|U\|\|V\|,
\]

and \((-\delta U \wedge V)^* \) points from \( U^* \) to \( V^* \).

\[
\begin{align*}
(1) & \quad C(a) = \cos a, \quad S(a) = \sin a, \\
(2) & \quad C(a) = \cosh a, \quad S(a) = \sinh a, \\
(3) & \quad C(a) = \cosh a, \quad S(a) = \sinh a, \\
(4) & \quad C(a) = \sinh a, \quad S(a) = \cosh a.
\end{align*}
\]

\[\delta = 1 \quad \delta = -1\]

Figure 2.6.1
Rule for setting the arrows. Let a generalized triangle $a, \gamma, b, \alpha, c, \beta$ in $H$ be given. Mark each element of angle type by a small arc connecting the adjacent sides and contained in the sector of angle $\leq \pi$ as shown in Fig. 2.6.2. Then orient $a, \gamma$, etc. by setting arrows in such a way that $a$ points from $\beta$ to $\gamma$, $\gamma$ points from $a$ to $b$, and so on (cf. Fig. 2.6.2). The configurations $\beta a \gamma, a \gamma b, \gamma b \alpha$, etc. now have well defined parities which we denote by $\delta_a, \delta_\gamma, \delta_b$, etc. Finally, we define $\varepsilon_\gamma = 1$ if $\gamma$ is a geodesic arc, and $\varepsilon_\gamma = -1$ if $\gamma$ is an angle.

![Figure 2.6.2](image)

2.6.20 Theorem. With the above definitions each generalized triangle $a, \gamma, b, \alpha, c, \beta$ satisfies the cosine law

$$C(c) = \varepsilon_\gamma \delta_a \delta_b \delta_c \left[ \delta_\gamma S(a)S(b)C(\gamma) - C(a)C(b) \right]$$

and the sine law

$$S(a) : S(\alpha) = S(b) : S(\beta) = S(c) : S(\gamma).$$

Proof. Choose $A, B, C \in \mathcal{H}_0$ with $|A| = |B| = |C| = 1$ such that with respect to the given orientations $A^* = \alpha, B^* = \beta, C^* = \gamma$ (recall that $|X| = |XX|^{1/2}$). Lemma 2.6.5(iii) implies that

$$(AB)(CC) = (B \wedge C)(C \wedge A) + (BC)(CA).$$

From Lemma 2.6.19 we have

$$AB = -\delta_c C(c), \quad BC = -\delta_a C(a), \quad CA = -\delta_b C(b),$$

$$|B \wedge C| = S(a), \quad \langle -\delta_b B \wedge C \rangle^* = a^*,$$

$$|C \wedge A| = S(b), \quad \langle -\delta_b C \wedge A \rangle^* = b^*,$$

where $a^*$ and $b^*$ are the oriented geodesics carrying $a$ and $b$ and having the orientations of $a$ and $b$. By the first relation of Lemma 2.6.19 we have therefore
\[
\delta_a \delta_y (B \wedge C)(C \wedge A) = -\delta_y C(\gamma) |B \wedge C| |C \wedge A| = -\delta_y C(\gamma) S(a) S(b),
\]
and the cosine law follows. To prove the sine law we note from Lemma 2.6.19 and Lemma 2.6.5 that
\[
S(\gamma)|B \wedge C| |C \wedge A| = |(B \wedge C) \wedge (C \wedge A)| = |\text{det}(A, B, C)|.
\]
Hence \( S(c) : S(\gamma) = |A \wedge B| |B \wedge C| |C \wedge A| : |\text{det}(A, B, C)|, \) where the right-hand side is invariant under cyclic permutation.

As an example we compute the cosine formula for the generalized triangles in Fig. 2.6.2. In the first triangle the signs are \( \epsilon_\gamma = 1, \delta_a = \delta_b = 1, \delta_c = \delta_\gamma = -1. \) The formula becomes
\[
cosh c = \sinh a \sin b \sinh \gamma + \cosh a \cos b.
\]
In the second triangle the signs are \( \epsilon_\gamma = -1, \delta_b = \delta_c = \delta_\gamma = 1, \delta_a = -1. \) The formula becomes
\[
\cos c = \cosh a \cosh b \cos \gamma - \sinh a \sinh b.
\]
One may now again compute the formulae for the configurations in the earlier sections. We recall that in the present section the angles and sides have non-oriented measures. When oriented measures are used, as for example in (2.3.2), then the signs in the formula must be adjusted accordingly.
Geometry and Spectra of Compact Riemann Surfaces
Buser, P.
2010, XIV, 456 p. 145 illus., Softcover
ISBN: 978-0-8176-4991-3
A product of Birkhäuser Basel