Preface

"Where shall I begin, please your Majesty?" he asked. "Begin at the beginning," the King said, gravely, "and go on till you come to the end: then stop."

— Lewis Carroll, *Alice in Wonderland*

What control strategy will transfer a space vehicle from one circular orbit to another in minimum time or in such a manner as to minimize fuel consumption? How should a batch distillation column be operated to maximize the yield, subject to specified constraints on purity of the product? Practical questions such as these underlie the field of Optimal Control. In the language of mathematics, Optimal Control concerns the properties of control functions that, when inserted into a differential equation, give solutions which minimize a "cost" or measure of performance.\(^1\) In engineering applications the control function is a control strategy. The differential equation describes the dynamic response of the mechanism to be controlled, which, of course, depends on the control strategy employed.

Systematic study of optimal control problems dates from the late 1950s, when two important advances were made. One was the the Maximum Principle, a set of necessary conditions for a control function to be optimal. The other was Dynamic Programming, a procedure that reduces the search for an optimal control function to finding the solution to a partial differential equation (the Hamilton–Jacobi Equation).

In the following decade, it became apparent that progress was being impeded by a lack of suitable analytic tools for investigating local properties of functions which are nonsmooth; i.e., not differentiable in the traditional sense.

Nonsmooth functions were encountered at first attempts to put Dynamic Programming on a rigorous footing, specifically attempts to relate value functions and solutions to the Hamilton–Jacobi Equation. It was found that, for many optimal control problems of interest, the only "solutions" to

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\(^1\)This is a simplification: the field also concerns optimization problems with dynamic constraints which might be functional differential equations, difference equations, partial differential equations, or take other forms.
the Hamilton–Jacobi Equation have discontinuous derivatives. How should we interpret these solutions? New ideas were required to answer this question since the Hamilton–Jacobi Equation of Optimal Control is a nonlinear partial differential equation for which traditional interpretations of generalized solutions, based on the distributions they define, are inadequate.

Nonsmooth functions surfaced once again when efforts were made to extend the applicability of necessary conditions such as the Maximum Principle. A notable feature of the Maximum Principle (and one that distinguishes it from necessary conditions derivable using classical techniques), is that it can take account of pathwise constraints on values of the control functions. For some practical problems, the constraints on values of the control depend on the vector state variable. In flight mechanics, for example, the maximum and minimum thrust of a jet engine (a control variable) will depend on the altitude (a component of the state vector). The Maximum Principle in its original form is not in general valid for problems involving state dependent control constraints. One way to derive necessary conditions for these problems, and others not covered by the Maximum Principle, is to reformulate them as generalized problems in the Calculus of Variations, the cost integrands for which include penalty terms to take account of the constraints. The reformulation comes at a price, however. To ensure equivalence with the original problems it is necessary to employ penalty terms with discontinuous derivatives. So the route to necessary conditions via generalized problems in the Calculus of Variations can be followed only if we know how to adapt traditional necessary conditions to allow for nonsmooth cost integrands.

Two important breakthroughs occurred in the 1970s. One was the end product of a long quest for effective, local descriptions of “non-smooth” functions, based on generalizations of the concept of the “subdifferential” of a convex function, to larger function classes. F. H. Clarke's theory of generalized gradients, by achieving this goal, launched the field of Nonsmooth Analysis and provided a bridge to necessary conditions of optimality for nonsmooth variational problems (and in particular optimal control problems reformulated as generalized problems in the Calculus of Variations). The other breakthrough, a somewhat later development, was the concept of Viscosity Solutions, due to M. G. Crandall and P.-L. Lions, that provides a framework for proving existence and uniqueness of generalized solutions to Hamilton–Jacobi equations arising in Optimal Control.

Nonsmooth Analysis and Viscosity Methods, introduced to clear a bottleneck in Optimal Control, have had a significant impact on Nonlinear Analysis as a whole. Nonsmooth Analysis provides an important new perspective: useful properties of functions, even differentiable functions, can be proved by examining related nondifferentiable functions, in the same
way that trigonometric identities relating to real numbers can sometimes simply be derived by a temporary excursion into the field of complex numbers. Viscosity Methods, on the other hand, provide a fruitful approach to studying generalized solutions to broad classes of nonlinear partial differential equations which extend beyond Hamilton–Jacobi equations of Optimal Control and their approximation for computational purposes.

The Calculus of Variations (in its modern guise as Optimal Control) continues to uphold a long tradition, as a stimulus to research in other fields of mathematics.

Clarke’s influential book, *Optimization and Nonsmooth Analysis*, [38] of 1982 covered many important advances of the preceding decade in Nonsmooth Analysis and its applications to the derivation of necessary conditions in Optimal Control. Since then, Optimal Control has remained an active area of research. In fact, it has been given fresh impetus by a number of developments. One is widespread interest in nonlinear controller design methods, based on the solution of an optimal control problem at each controller update time and referred to as Model Predictive Control [117],[103]. Another is the role of Optimal Control and Differential Games in generalizations of H–infinity controller design methods to nonlinear systems [90]. There has been a proliferation of new nonsmooth necessary conditions, distinguished by the hypotheses under which they are valid and by their ability to eliminate from consideration certain putative minimizers that are not excluded by rival sets of necessary conditions. However recent work has helped to clarify the relationships between them. This has centered on the Extended Euler–Lagrange Condition, a generalization of Euler’s Equation of the classical Calculus of Variations. The Extended Euler–Lagrange Condition subsumes the Hamilton Inclusion, a key early nonsmooth necessary condition, and is valid under greatly reduced hypotheses. A number of other necessary conditions, such as the Nonsmooth Maximum Principle, follow as simple corollaries. Necessary conditions for problems involving pathwise state constraints have been significantly improved, and degenerate features of earlier necessary conditions for such problems have been eliminated. Sharper versions of former nonsmooth transversality conditions have been introduced into the theory. Techniques for deriving improved necessary conditions for free time problems are now also available. Extensions of Tonelli’s work on the structure of minimizers have been carried out and used to derive new necessary conditions for variational problems in cases when traditional necessary conditions do not even make sense. In fact, there have been significant advances in virtually all areas of nonsmooth Optimal Control since the early 1980s.

Over the last two decades, viscosity techniques have had a growing following. Applications of Viscosity Methods are now routine in Stochastic Control, Mathematical Finance, Differential Games and other fields be-
sides Optimal Control.

Dynamic Programming is well served by a number of up-to-date expository texts, including Fleming and Soner's book *Controlled Markov Processes and Viscosity Solutions* [66], Barles's lucid introductory text, *Solutions de Viscosité des Équations de Hamilton–Jacobi*, and Bardi and Capuzzo Dolcetta's comprehensive monograph, *Optimal Control and Viscosity Solutions of Hamilton–Jacobo-Bellman Equations* [14]. This cannot be said of applications of Nonsmooth Analysis to the derivation of necessary conditions in Optimal Control. Expository texts such as Loewen's *Optimal Control Via Nonsmooth Analysis* [94] and Clarke's *Methods of Dynamic and Nonsmooth Optimization* [38] give the flavor of contemporary thinking in these areas of Optimal Control, as do the relevant sections of Clarke et al.'s recent monograph, *Nonsmooth Analysis and Control Theory* [54]. But details of recent advances, dispersed as they are over a wide literature and written in a wide variety of styles, are by no means easy to follow.

The main purpose of this book is to bring together as a single publication many major developments in Optimal Control based on Nonsmooth Analysis of recent years, and thereby render them accessible to a broader audience. Necessary conditions receive special attention. But other topics are covered as well. Material on the important, and unjustifiably neglected, topic of minimizer regularity provides a showcase for the application of nonsmooth necessary conditions to derive qualitative information about solutions to variational problems. The chapter on Dynamic Programming stands a little apart from other sections of the book, as it is complementary to recent mainstream research in the area based on Viscosity Methods (and which in any case is already the subject matter of substantial expository texts). Instead we concentrate on aspects of Dynamic Programming well matched to the analytic techniques of this book, notably the characterization (in terms of the Hamilton–Jacobi Equation) of extended-valued value functions associated with problems having endpoint and state constraints, inverse verification theorems, sensitivity relationships and links with the Maximum Principle.

A subsidiary purpose is to meet the needs of readers with little prior exposure to modern Optimal Control who seek quick answers to the questions: what are the main results, what were the deficiencies of the "classical" theory and to what extent have they been overcome? Chapter 1 provides, for their benefit, a lengthy overview, in which analytical details are suppressed and the emphasis is placed instead on communicating the underlying ideas.

To render this book self-contained, preparatory chapters are included on Nonsmooth Analysis, measurable multifunctions, and differential inclusions. Much of this material is implicit in the recent books of R. T.
Rockafellar and J. B. Wets [125] and Clarke et al. [53], and the somewhat older book by J.-P. Aubin and H. Frankowska [12]. It is expected, however, that readers, whose main interest is in Optimization rather than in broader application areas of Nonsmooth Analysis, which require additional techniques, will find these chapters helpful, because of the strong focus on topics relevant to Optimization.

Optimal Control is now a large field, and our choice of material for inclusion in this book is necessarily selective. The techniques used here to derive necessary conditions of optimality are within a tradition of research pioneered and developed by Clarke, Ioffe, Loewen, Mordukhovich, Rockafellar, Vinter, and others, based on perturbation, elimination of constraints and passage to the limit. The necessary conditions are "state of the art," as far as this tradition is concerned. Alternative approaches are not addressed, such as that of H. Sussmann [132], a synthesis of traditional ideas for approximating reachable sets and of extensions to the Warga's theory of derivate containers, which permit a relaxation of hypotheses under which the Maximum Principle is valid in some respects and leads to different kinds of necessary conditions for problems in which the dynamic constraint takes the form of a differential inclusion. The topic of higher order necessary conditions, addressed for example in [155] and [102], is not entered into, nor are computational aspects, examined for example in [113], discussed.

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The webpage http://www.ps.ic.ac.uk/~rbv/oc.html will record errors, ambiguities, etc., in the book, as they come to light. Readers' contributions, via the e-mail address r.vinter@ic.ac.uk, are welcome.

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