Nonlinearly Perturbed Stochastic Processes and Systems

Dmitrii S. Silvestrov
Stockholm University, Stockholm, Sweden

Abstract: This paper is a survey of results presented in the recent book Gyllenberg and Silvestrov [GS08]. This book is devoted to studies of quasi-stationary phenomena for nonlinearly perturbed stochastic processes and systems. New methods of asymptotic analysis for nonlinearly perturbed stochastic processes based on asymptotic expansions for perturbed renewal equation and recurrence algorithms for construction of asymptotic expansions for Markov type processes with absorption are presented. Asymptotic expansions are given in mixed ergodic (for processes) and large deviation theorems (for absorption times) for nonlinearly perturbed regenerative processes, semi-Markov processes, and Markov chains. Applications to analysis of quasi-stationary phenomena in nonlinearly perturbed queueing systems, population dynamics and epidemic models, and for risk processes are presented. The book also contains an extended bibliography of works in the area.

Keywords and phrases: Nonlinear perturbation, Quasi-stationary phenomenon, Pseudo-stationary phenomenon, Stochastic process, Stochastic system, Renewal equation, Asymptotic expansion, Ergodic theorem, Limit theorem, Large deviation, Regenerative process, Regenerative stopping time, Semi-Markov process, Markov chain, Absorption time, Queueing system, Population dynamics, Epidemic model, Lifetime, Risk process, Ruin probability, Cramér-Lundberg approximation, Diffusion approximation

2.1 Introduction

The book mentioned above presents new methods of asymptotic analysis of nonlinearly perturbed stochastic processes and systems with random lifetimes.

Usually the behaviour of a stochastic system can be described in terms of some Markov type stochastic process \( \eta^{(\varepsilon)}(t) \) and its lifetime defined to be the time \( \mu^{(\varepsilon)} \) at which the process \( \eta^{(\varepsilon)}(t) \) hits a special absorption subset of the phase space of this process for the first time. A typical situation is when the process \( \eta^{(\varepsilon)}(t) \) and the absorption time \( \mu^{(\varepsilon)} \) depend on a small parameter \( \varepsilon \geq 0 \) in the sense that some of their local “transition” characteristics depend on the parameter \( \varepsilon \). The parameter \( \varepsilon \) is involved in the model in such a way that the corresponding local characteristics are continuous at the point \( \varepsilon = 0 \), if regarded as functions of \( \varepsilon \). These continuity conditions permit to consider the process \( \eta^{(\varepsilon)}(t) \), for \( \varepsilon > 0 \), as a perturbed version of the process \( \eta^{(0)}(t) \).

The object of interest is the joint distribution of the process \( \eta^{(\varepsilon)}(t) \) subject to a condition of non-absorption of the process up to a moment \( t \), i.e., the probabilities \( P\{\eta^{(\varepsilon)}(t) \in A, \mu^{(\varepsilon)} > t\} \). In models with perturbations, it is natural to study the asymptotic behaviour of these probabilities when the time \( t \to \infty \) and the perturbation parameter \( \varepsilon \to 0 \) simultaneously. The corresponding asymptotic results describe so-called quasi-stationary and pseudo-stationary phenomena for processes \( \eta^{(\varepsilon)}(t) \). These phenomena differ by the asymptotic behaviour of the absorption times \( \mu^{(\varepsilon)} \). These random variables are stochastically bounded or unbounded as \( \varepsilon \to 0 \), respectively, in the quasi-stationary and pseudo-stationary cases.

The principal novelty of results presented in book [GS08] is that the models with nonlinear perturbations are studied. Local transition characteristics that were mentioned above are usually some scalar or vector moment functionals \( p^{(\varepsilon)} \) of local transition probabilities for the corresponding processes. By a nonlinear perturbation we mean that these characteristics are nonlinear functions of the perturbation parameter \( \varepsilon \) and that the assumptions made imply that the characteristics can be expanded in an asymptotic power series with respect to \( \varepsilon \) up to and including some order \( k \), i.e.,

\[
p^{(\varepsilon)} = p^{(0)} + p^{[1]}\varepsilon + \cdots + p^{[k]}\varepsilon^k + o(\varepsilon^k).
\]

The case \( k = 1 \) corresponds to models with usual linear perturbations while the cases \( k > 1 \) correspond to models with nonlinear perturbations.

The classes of processes for which this program is realised include nonlinearly perturbed regenerative processes, semi-Markov processes, and continuous time Markov chains with absorption. The approach is based on advanced techniques, developed in the book, of nonlinearly perturbed renewal equations. Applications to the analysis of quasi-stationary phenomena in models of nonlinearly perturbed stochastic systems considered in the book pertain to models of highly reliable queueing systems, M/G queueing systems with quick service, stochastic systems of birth–death type, including epidemic and population dynamics models, metapopulation dynamic models, and perturbed risk processes.

The book [GS08] contains an extended introduction, where the main problems, methods, and algorithms that constitute the content of the book are presented in informal form. In Chaps. 1 and 2, results which deal with a generalisation of the classical renewal theorem to a model of the perturbed renewal equation are presented. These results are interesting by their own and, as we think, can find various applications beyond the areas mentioned in the book. In Chaps. 3–5 quasi- and pseudo-stationary asymptotics is studied for nonlinearly perturbed regenerative processes, semi-Markov processes, and continuous time Markov chains with absorption. Chapters 6 and 7 are devoted to applications of the theoretical results to studies of quasi-stationary phenomena for various nonlinearly perturbed models of stochastic systems. In Chap. 6, quasi-stationary phenomena are studied for highly reliable queueing systems, M/G
queueing systems with quick service, stochastic systems of birth–death type, including epidemic and population dynamics models, and metapopulation dynamic models; Chap. 7 deals with perturbed risk processes. Finally, Chap. 8 contains three supplements. The first one gives some basic operation formulas for scalar and matrix asymptotic expansions. In the second supplement, some new prospective directions for future research in the area are discussed. In the last supplement, bibliographical remarks to the bibliography that includes more than 1000 references are given.

2.2 Nonlinearly Perturbed Renewal Equation

Let us consider the family of renewal equations,

\[ x^{(\varepsilon)}(t) = q^{(\varepsilon)}(t) + \int_{0}^{t} x^{(\varepsilon)}(t - s)F^{(\varepsilon)}(ds), \quad t \geq 0, \quad (2.1) \]

where, for every \( \varepsilon \geq 0 \), we have the following: (a) \( q^{(\varepsilon)}(t) \) is a real-valued function on \([0, \infty)\) that is Borel measurable and locally bounded, i.e., bounded on every finite interval, and (b) \( F^{(\varepsilon)}(s) \) is a distribution function on \([0, \infty)\) which is not concentrated at 0 but can be improper, i.e., \( F^{(\varepsilon)}(\infty) \leq 1 \).

As well known, there exists the unique Borel measurable and bounded on every finite interval solution \( x^{(\varepsilon)}(t) \) of (2.1).

In the model of perturbed renewal, the forcing function \( q^{(\varepsilon)}(t) \) and distribution \( F^{(\varepsilon)}(s) \) depend on some perturbation parameter \( \varepsilon \geq 0 \) and converge in some sense to \( q^{(0)}(t) \) and \( F^{(0)}(s) \) as \( \varepsilon \to 0 \).

The fundamental fact of the renewal theory connected with this equation is the renewal theorem given in its final form by Feller [Fel66]. This theorem describes the asymptotic behavior of solution in the form of asymptotic relation \( x^{(0)}(t) \to x^{(0)}(\infty) \) as \( t \to \infty \) for non-perturbed renewal equation.

The renewal theorem is a very powerful tool for proving ergodic theorems for regenerative stochastic processes. This class of processes is very broad. It includes Markov processes with discrete phase space. Moreover, Markov processes with a general phase space can be included, under some minor conditions, in a model of regenerative processes with the help of the procedure of artificial regeneration.

Applying the renewal theorem to ergodic theorems for regenerative type processes is based on the well known fact that the distribution of a regenerative process at a moment \( t \) satisfies a renewal equation. This makes it possible to apply the renewal theorem and to describe the asymptotic behaviour of the distribution of the regenerative process as \( t \to \infty \).

Theorems that generalise the classical renewal theorem to a model of the perturbed renewal equation was proved in papers Silvestrov [Sil76, Sil78, Sil79]. These results are presented in Chap. 1 of De Gruyter Expositions in Mathematics [GS08].

As usual the symbol \( F^{(\varepsilon)}(\cdot) \Rightarrow F^{(0)}(\cdot) \) as \( \varepsilon \to 0 \) means weak convergence of the distribution functions that is, the pointwise convergence in each point of continuity of the limiting distribution function.
Further, the following notations are used,

\[ f^{(e)} = 1 - F^{(e)}(\infty), \quad m_n^{(e)} = \int_0^\infty s^n F^{(e)}(ds), \quad n \geq 1. \]

We assume that the functions \( q^{(e)}(t) \) and the distributions \( F^{(e)}(s) \) satisfy the following continuity conditions at the point \( \varepsilon = 0 \), if regarded as functions of \( \varepsilon \):

- **D\(_1\)**: \( F^{(e)}(\cdot) \Rightarrow F^{(0)}(\cdot) \) as \( \varepsilon \to 0 \), where \( F^{(0)}(s) \) is a proper non-arithmetic distribution function;
- **M\(_1\)**: \( m_1^{(e)} \to m_1^{(0)} < \infty \) as \( \varepsilon \to 0 \);
- **F\(_1\)**: (a) \( \lim_{u \to 0} \lim_{0 \leq \varepsilon < u} \sup_{v \leq u} |q^{(e)}(t + v) - q^{(0)}(t)| = 0 \) almost everywhere with respect to the Lebesgue measure on \([0, \infty)\);
  (b) \( \lim_{0 \leq \varepsilon < \infty} \sup_{0 \leq t \leq T} |q^{(e)}(t)| < \infty \) for every \( T \geq 0 \);
  (c) \( \lim_{T \to \infty} \lim_{0 \leq \varepsilon < h} \sum_{r \geq T/h} \sup_{h \leq t \leq (r+1)h} |q^{(e)}(t)| = 0 \) for some \( h > 0 \).

It is easy to show that, under conditions **D\(_1\)**, \( f^{(e)} \to f^{(0)} = 0 \) as \( \varepsilon \to 0 \).

Let also assume the following condition that balances the rate at which time \( t^{(e)} \) approaches infinity, and the convergence rate of the defect \( f^{(e)} \) to zero as \( \varepsilon \to 0 \):

- **B**: \( 0 \leq t^{(e)} \to \infty \) and \( f^{(e)} \to 0 \) as \( \varepsilon \to 0 \) in such a way that \( f^{(e)} t^{(e)} \to \lambda \), where \( 0 \leq \lambda \leq \infty \).

The starting point for the research studies presented in book [GS08] is the following theorem (Silvestrov [Sil76, Sil78, Sil79]).

**Theorem 1.** Let conditions **D\(_1\)**, **M\(_1\)**, **F\(_1\)**, and **B** hold. Then,

\[ x^{(e)}(t^{(e)}) \to e^{-\lambda/m_1^{(0)}} \int_0^{m_1^{(0)}} q^{(0)}(s)ds \quad \text{as} \quad \varepsilon \to 0. \] (2.2)

**Remark 1.** It is worth to note that this theorem reduces to the classical renewal theorem in the case of non-perturbed renewal equation, i.e., where the forcing functions \( q^{(e)}(t) \equiv q^{(0)}(t) \) and distribution functions \( F^{(e)}(s) \equiv F^{(0)}(s) \) do not depend on \( \varepsilon \). In particular, condition **D\(_1\)** reduces to the assumption that \( F^{(0)}(s) \) is a proper non-arithmetic distribution function; **M\(_1\)** to the assumption that the expectation \( m_1^{(0)} \) is finite; and **F\(_1\)** to the assumption that the function \( q^{(0)}(t) \) is directly Riemann integrable on \([0, \infty)\).

In this case, the defect \( f^{(e)} \equiv 0 \) and the balancing condition **B** holds for any \( t^{(e)} \to \infty \) as \( \varepsilon \to 0 \) with the parameter \( \lambda = 0 \).

Note that condition **D\(_1\)** does not require and does not provide that the pre-limit \((\varepsilon > 0)\) distribution functions \( F^{(e)}(s) \) are non-arithmetic.

Also, condition **F\(_1\)** does not provide that the pre-limit \((\varepsilon > 0)\) forcing functions \( q^{(e)}(t) \) are directly Riemann integrable on \([0, \infty)\). However, this condition does imply that the limit forcing functions \( q^{(0)}(t) \) has this property.

In the general case, the balancing condition **B** restrict the rate of growth for time \( t^{(e)} \). This restriction becomes unnecessary if an additional Cramér type condition is imposed on the distributions \( F^{(e)}(s) \).
In this case, one can also weaken condition \( D_1 \) and accept also the possibility for the limit distribution be improper:

\[ D_2: (a) \quad F^{(\varepsilon)}(\cdot) \Rightarrow F^{(0)}(\cdot) \quad \text{as} \quad \varepsilon \to 0, \quad \text{where} \quad F^{(0)}(t) \quad \text{is a non-arithmetic distribution function (possibly improper);} \]

\[ (b) \quad f^{(\varepsilon)}(t) \to f^{(0)}(t) \in [0, 1] \quad \text{as} \quad \varepsilon \to 0. \]

The Cramér type condition mentioned above takes the following form:

\[ C_1: \text{There exists} \quad \delta > 0 \quad \text{such that:} \]

\[ (a) \quad \lim_{0 \leq \varepsilon \to 0} \int_0^\infty e^{\delta s} F^{(\varepsilon)}(ds) < \infty; \]

\[ (b) \quad \int_0^\infty e^{\delta s} F^{(0)}(ds) > 1. \]

Let us introduce the moment generation function,

\[ \phi^{(\varepsilon)}(\rho) = \int_0^\infty e^{\rho s} F^{(\varepsilon)}(ds), \quad \rho \geq 0. \]

Consider the following characteristic equation,

\[ \phi^{(\varepsilon)}(\rho) = 1. \tag{2.3} \]

Under conditions \( D_2 \) and \( C_1 \), there exists \( \varepsilon_1 > 0 \) such that \( \phi^{(\varepsilon)}(\delta) \in (1, \infty) \), and, therefore, (2.3) has a unique non-negative root \( \rho^{(\varepsilon)} \) and \( \rho^{(\varepsilon)} \leq \delta \), for every \( \varepsilon \leq \varepsilon_1 \). Also, \( \rho^{(\varepsilon)} \to \rho^{(0)} \) as \( \varepsilon \to 0 \).

Note also that (a) \( \rho^{(0)} = 0 \) if and only if \( f^{(0)} = 0 \) and (b) \( \rho^{(0)} > 0 \) if and only if \( f^{(0)} > 0 \).

In this case, condition \( F_1 \) takes the following modified form:

\[ F_2: (a) \quad \lim_{u \to 0} \sup_{0 \leq \varepsilon \leq u} \sup_{|v| \leq \varepsilon} |q^{(\varepsilon)}(t + v) - q^{(0)}(t)| = 0 \quad \text{almost everywhere with respect to the Lebesgue measure on} \quad [0, \infty); \]

\[ (b) \quad \lim_{0 \leq \varepsilon \to 0} \sup_{0 \leq t \leq T} |q^{(\varepsilon)}(t)| < \infty \quad \text{for every} \quad T \geq 0; \]

\[ (c) \quad \lim_{T \to \infty} \lim_{0 \leq \varepsilon \to 0} h \sum_{r \geq T/h} \sup_{r h \leq t \leq (r+1)h} e^{\gamma t} |q^{(\varepsilon)}(t)| = 0 \quad \text{for some} \quad h > 0 \quad \text{and} \quad \gamma > \rho^{(0)}. \]

Let us denote,

\[ \tilde{x}^{(\varepsilon)}(\infty) = \frac{\int_0^\infty e^{\rho^{(\varepsilon)} s} q^{(\varepsilon)}(s) m(ds)}{\int_0^\infty s e^{\rho^{(\varepsilon)} s} F^{(\varepsilon)}(ds)} , \]

where \( m(ds) \) is the Lebesgue measure on a real line.

Conditions \( D_2, C_1, \) and \( F_2 \) imply, due to relation \( \rho^{(\varepsilon)} \to \rho^{(0)} \) as \( \varepsilon \to 0 \), that there exists \( 0 < \varepsilon_2 \leq \varepsilon_1 \) such that \( \rho^{(\varepsilon)} < \gamma \) and \( \int_0^\infty e^{\rho^{(\varepsilon)} s} q^{(\varepsilon)}(s) m(ds) < \infty \) for \( \varepsilon \leq \varepsilon_2 \). Thus, the functional \( \tilde{x}^{(\varepsilon)}(\infty) \) is well defined for \( \varepsilon \leq \varepsilon_2 \).

The following theorem was also proved in Silvestrov [Sil76, Sil78, Sil79].

**Theorem 2.** Let conditions \( D_2, C_1, \) and \( F_2 \) hold. Then,

\[ \frac{\tilde{x}^{(\varepsilon)}(t^{(\varepsilon)})}{e^{-\rho^{(\varepsilon)} t^{(\varepsilon)}}} \to \tilde{x}^{(0)}(\infty) \quad \text{as} \quad \varepsilon \to 0. \tag{2.4} \]

The asymptotic relation (2.4) given in Theorem 2 should be compared with the asymptotic relation (2.2) given in Theorem 1, in the case where \( \rho^{(0)} = 0 \).
Indeed, relation (2.2) can be re-written in the form given in (2.4), with coefficients \( \rho^{(\varepsilon)} = f^{(\varepsilon)}/m_1^{(\varepsilon)} \). The Cramér type condition \( C_1 \) makes it possible to use in (2.4) an alternative coefficients \( \rho^{(\varepsilon)} \) defined as the solution of the characteristic equation (2.3). The latter coefficients provide better fitting of the corresponding exponential approximation for solution of renewal equation. That is why the asymptotic relation (2.4) does not restrict the rate of growth for time \( t^{(\varepsilon)} \) while the asymptotic relation (2.2) does impose such restriction.

Remark 2. It is worth to note that this theorem reduces to the variant of renewal theorem for improper renewal equation in the case of non-perturbed renewal equation, also given in Feller [Fel66]. Condition \( D_2 \) reduces to the assumption that \( F^{(0)}(s) \) is a non-arithmetic distribution function with defect \( f^{(0)} \in [0,1) \); \( C_1 \) to the assumption that the exponential moment \( \phi^{(0)}(\delta) \in (1,\infty) \); and \( F_2 \) to the assumption that the function \( e^{tq^{(0)}(t)} \) is directly Riemann integrable on \([0,\infty)\) for some \( \gamma > \rho^{(0)} \).

The results formulated in Theorems 1 and 2 created the base for further research studies in the area. For example, Shurenkov [Shu80a,Shu80b] generalised the results of these theorems to the case of perturbed matrix renewal equation using possibility of imbedding the matrix model to the scalar model considered in Theorems 1 and 2.

A new improvement was achieved in the paper Silvestrov [Sil95] and then in the papers Gyllenberg and Silvestrov [GS99a,GS00a]. Under natural additional perturbation conditions, which assume that the defect \( f^{(\varepsilon)} \) and the corresponding moments of the distribution \( F^{(\varepsilon)}(s) \) can be expanded in power series with respect to \( \varepsilon \) up to and including an order \( k \), explicit expansions for the corresponding characteristic roots were given, and the corresponding exponential expansions were obtained for solutions of nonlinearly perturbed renewal equations. In [Sil95], the case with asymptotically proper distributions \( F^{(\varepsilon)}(s) \) was considered, while, in Gyllenberg and Silvestrov [GS99a,GS00a], the case with asymptotically improper distributions \( F^{(\varepsilon)}(s) \) was investigated. These results are presented in Chap. 2 of De Gruyter Expositions in Mathematics [GS08].

Let us introduce the mixed power-exponential moment generating functions,

\[
\phi^{(\varepsilon)}(\rho,n) = \int_0^\infty s^n e^{\rho s} F^{(\varepsilon)}(ds), \quad \rho \geq 0, \quad n = 0, 1, \ldots
\]

Note that by the definition \( \phi^{(\varepsilon)}(\rho,0) = \phi^{(\varepsilon)}(\rho) \). Under conditions \( D_2 \) and \( C_1 \), for any \( 0 < \delta' < \delta \), there exists \( 0 < \varepsilon_3 < \varepsilon_2 \) such that \( \phi^{(\varepsilon)}(\delta',n) < \infty \) for \( n = 0, 1, \ldots \) and \( \varepsilon \leq \varepsilon_3 \). Also, \( \phi^{(\varepsilon)}(\rho,n) \to \phi^{(0)}(\rho,n) \) as \( \varepsilon \to 0 \) for \( n = 0, 1, \ldots \) and \( \rho \leq \delta' \). Let \( \delta' \) is chosen such that \( \phi^{(0)}(\delta') \in (1,\infty) \). In this case, the characteristic root \( \rho^{(0)} < \delta' \) and also there exists \( 0 < \varepsilon_4 < \varepsilon_3 \), such that the characteristic roots \( \rho^{(\varepsilon)} < \delta' \) for \( \varepsilon \leq \varepsilon_4 \).

The basic role plays the following nonlinear perturbation condition:

\[
P_1: \phi^{(\varepsilon)}(\rho^{(0)},n) = \phi^{(0)}(\rho^{(0)},n) + b_{1,n}\varepsilon + \cdots + b_{k-n,n}\varepsilon^{k-n} + o(\varepsilon^{k-n}) \quad \text{for} \quad n = 0, \ldots, k,
\]

where \( |b_{i,n}| < \infty, i = 1, \ldots, k - n, \quad n = 0, \ldots, k \).

It is convenient to define \( b_{0,n} = \phi^{(0)}(\rho^{(0)},n), n = 0, 1, \ldots. \) From the definition of \( \rho^{(0)} \) it is clear that \( b_{0,0} = \phi^{(0)}(\rho^{(0)},0) = 1 \).

It should be noted that, in the case \( f^{(0)} = 0 \), where characteristic root \( \rho^{(0)} = 0 \), the perturbation condition \( P_1 \) involves usual power moments of distributions \( F^{(\varepsilon)}(s) \). While in the case \( f^{(0)} > 0 \), where characteristic root \( \rho^{(0)} > 0 \), the perturbation condition involves mixed power-exponential moments of distributions \( F^{(\varepsilon)}(s) \).
Let us also formulate the following condition that balances the rate at which time \( t(\varepsilon) \) approaches infinity and the convergence rate of perturbation in different asymptotic zones, for \( 1 \leq r \leq k \):

\[
\mathcal{B}(r): 0 \leq t(\varepsilon) \to \infty \text{ in such a way that } \varepsilon^r t(\varepsilon) \to \lambda_r, \text{ where } 0 \leq \lambda_r < \infty.
\]

The following theorem is given in Silvestrov [Sil95] and Gyllenberg and Silvestrov [GS99a,GS00a,GS08].

**Theorem 3.** Let conditions \( D_2, C_1, \) and \( P_1^{(k)} \) hold. Then,

(i) The root \( \rho(\varepsilon) \) of (2.3) has the asymptotic expansion

\[
\rho(\varepsilon) = \rho(0) + a_1 \varepsilon + \cdots + a_k \varepsilon^k + o(\varepsilon^k),
\]

where the coefficients \( a_n \) are given by the recurrence formulas \( a_1 = -b_{1,0}/b_{0,1} \) and, in general, for \( n = 1, \ldots, k \),

\[
a_n = -b_{0,1}^{-1}(b_{n,0} + \sum_{q=1}^{n-1} b_{n-q,1} a_q + \sum_{2 \leq m \leq n \leq q=m} \sum_{n_1, \ldots, n_{q-1} \in D_{m,q}} \prod_{p=1}^{q-1} a_p^{n_p}/n_p!),
\]

where \( D_{m,q} \), for every \( 2 \leq m \leq q < \infty \), is the set of all nonnegative, integer solutions of the system

\[
n_1 + \cdots + n_{q-1} = m, \quad n_1 + \cdots + (q-1)n_{q-1} = q.
\]

(ii) If \( b_{i,0} = 0, i = 1, \ldots, n \), for some \( 1 \leq n \leq k \), then \( a_1, \ldots, a_n = 0 \). If \( b_{i,0} = 0, i = 1, \ldots, n-1 \) but \( b_{n,0} < 0 \), for some \( 1 \leq n \leq k \), then \( a_1, \ldots, a_{n-1} = 0 \) but \( a_n > 0 \).

(iii) If, additionally, conditions \( \mathcal{B}(r) \), for some \( 1 \leq r \leq k \), and \( F_2 \) hold, then the following asymptotic relation holds:

\[
\frac{x(\varepsilon) t(\varepsilon)}{\exp\{-\rho(0) + a_1 \varepsilon + \cdots + a_{r-1} \varepsilon^{r-1} \}} \to e^{-\lambda_r \varepsilon^{r}} x(0) \text{ as } \varepsilon \to 0.
\]

The asymptotic relation (2.8) given in Theorem 3 should be compared with the asymptotic relation (2.4) given in Theorem 2.

The asymptotic relation (2.4) looks nicely but has actually a serious drawback. Indeed, the exponential normalisation with the coefficient \( \rho(\varepsilon) \) is not so effective because of this coefficient is given us only as the root of the nonlinear equation (2.3), for every \( \varepsilon \geq 0 \).

Relation (2.8) essentially improves the asymptotic relation (2.4) replacing this simple convergence relation by the corresponding asymptotic expansion. The exponential normalisation with the coefficient \( \rho(0) + a_1 \varepsilon + \cdots + a_{r-1} \varepsilon^{r-1} \) involves the root \( \rho(0) \). To find it one should solve only one nonlinear equation (2.3), for the case \( \varepsilon = 0 \). As far as the coefficients \( a_1, \ldots, a_r \) are concerned, they are given in the explicit algebraic recurrence form.
Moreover, the root \( \rho^{(0)} = 0 \) in the most interesting case, where \( f^{(0)} = 0 \), i.e., the limit renewal equation is proper. Here, the non-linear step connected with finding of the root of (2.3) can be omitted.

If there exist \( 0 < \varepsilon' \leq \varepsilon_5 \) such that the conditions listed in Remark 2 holds for the distribution function \( F^{(\varepsilon)}(s) \) and the forcing function \( q^{(\varepsilon)}(t) \) for every \( \varepsilon \leq \varepsilon' \), then according to Theorem 2, the following asymptotic relation holds for every \( \varepsilon \leq \varepsilon' \),

\[
\frac{x^{(\varepsilon)}(t)}{e^{-\rho^{(\varepsilon)}t}} \rightarrow \tilde{x}^{(\varepsilon)}(\infty) \quad \text{as} \quad t \rightarrow \infty. \tag{2.9}
\]

Let us now define mixed power-exponential moment functionals for the forcing functions,

\[
\omega^{(\varepsilon)}(\rho, n) = \int_0^{\infty} s^n e^{\rho s} q^{(\varepsilon)}(s) m(ds), \quad \rho \geq 0, \ n = 0, 1, \ldots
\]

Under conditions \( C_1 \) and \( F_2 \), for any \( 0 < \gamma' < \gamma \), there exists \( 0 < \varepsilon_6 < \varepsilon_5 \) such that \( \bar{\omega}^{(\varepsilon)}(\gamma', n) = \int_0^{\infty} s^n e^{\gamma' s} |q^{(\varepsilon)}(s)| m(ds) < \infty \) for \( n = 0, 1, \ldots \) and \( \varepsilon \leq \varepsilon_6 \). Also, \( \omega^{(\varepsilon)}(\rho, n) \rightarrow \omega^{(0)}(\rho, n) \quad \text{as} \quad \varepsilon \rightarrow 0 \) for \( n = 0, 1, \ldots \) and \( \rho \leq \gamma' \). Let \( \gamma' \) is chosen such that \( \rho^{(0)} < \gamma' \). In this case, there exists \( 0 < \varepsilon_7 < \varepsilon_6 \) such that the characteristic roots \( \rho^{(\varepsilon)} < \gamma' \) for \( \varepsilon \leq \varepsilon_7 \).

Note that the renewal limit \( \tilde{x}^{(\varepsilon)}(\infty) \) is well defined for \( \varepsilon \leq \varepsilon_7 \) even without the non-arithmetic assumption made above in order to provide asymptotic relation (2.9) and, moreover,

\[
\tilde{x}^{(\varepsilon)}(\infty) = \frac{\omega^{(\varepsilon)}(\rho^{(\varepsilon)}, 0)}{\phi^{(\varepsilon)}(\rho^{(\varepsilon)}, 1)}. \tag{2.10}
\]

Let us now formulate a perturbation condition for mixed power-exponential moment functionals for the forcing functions:

\( \mathbf{P}_2^{(k)}: \omega^{(\varepsilon)}(\rho^{(0)}, n) = \omega^{(0)}(\rho^{(0)}, n) + c_{1,n} \varepsilon + \cdots + c_{k-n,n} \varepsilon^{k-n} + o(\varepsilon^{k-n}) \) for \( n = 0, \ldots, k \), where \( |c_{i,n}| < \infty, i = 1, \ldots, k - n, n = 0, \ldots, k \).

It is convenient to set \( c_{0,n} = \omega^{(0)}(\rho^{(0)}, n), n = 0, 1, \ldots \).

The following theorem supplements Theorem 3.

**Theorem 4.** Let conditions \( \mathbf{D}_2, \mathbf{C}_1, \mathbf{F}_2, \mathbf{P}_1^{(k+1)} \), and \( \mathbf{P}_2^{(k)} \) hold. Then the functional \( \tilde{x}^{(\varepsilon)}(\infty) \) has the following asymptotic expansions:

\[
\tilde{x}^{(\varepsilon)}(\infty) = \frac{\omega^{(0)}(\rho^{(0)}, 0) + f'_1 \varepsilon + \cdots + f'_k \varepsilon^k + o(\varepsilon^k)}{\phi^{(0)}(\rho^{(0)}, 1) + f''_1 \varepsilon + \cdots + f''_k \varepsilon^k + o(\varepsilon^k)}
\begin{align*}
&= \frac{\omega^{(0)}(\rho^{(0)}, 0)}{\phi^{(0)}(\rho^{(0)}, 1)} + \frac{f'_1 \varepsilon + \cdots + f'_k \varepsilon^k}{f''_1 \varepsilon + \cdots + f''_k \varepsilon^k + o(\varepsilon^k)}, \tag{2.11}
\end{align*}
\]

where the coefficients \( f'_n, f''_n \) are given by the formulas \( f'_0 = \omega^{(0)}(\rho^{(0)}, 0) = c_{0,0} \), \( f'_1 = c_{1,0} + c_{0,1} a_1 \), \( f''_1 = \phi^{(0)}(\rho^{(0)}, 1) = b_{0,1} \), \( f''_1 = b_{1,1} + b_{0,2} a_1 \), and in general for \( n = 0, \ldots, k \),

\[
f'_n = c_{n,0} + \sum_{q=1}^{n} c_{n-q,1} a_q + \sum_{2 \leq m \leq n \ q=m} c_{n-q,m} \sum_{n_1, \ldots, n_{q-1} \in D_{m,q}} \prod_{p=1}^{q-1} a_p^{n_p}/n_p!, \tag{2.12}
\]
\[ f_n'' = b_{n,1} + \sum_{q=1}^{n} b_{n-q,2}a_q \]
\[ + \sum_{2 \leq m \leq n} \sum_{q=m}^{n} b_{n-q,m+1} \cdot \sum_{n_1,\ldots,n_{q-1} \in D_{m,q}} \prod_{p=1}^{q-1} a_{n_p}^p/n_p! , \quad (2.13) \]

and the coefficients \( f_n \) are given by the recurrence formulas \( f_0 = \tilde{x}^{(0)}(\infty) = f_0'/f_0'' \) and in general for \( n = 0, \ldots, k \),
\[ f_n = \frac{f_n' - \sum_{q=0}^{n-1} f_n''f_q}{f_0''}. \quad (2.14) \]

It should be noted that the perturbation condition \( P_{k+1}(k+1) \) stronger than \( P_k(k) \) is required in Theorem 4. This is because of the former condition is needed to get the corresponding expansion for \( \phi^{(k)}(\rho^{(k)},1) \) in an asymptotic power series with respect to \( \varepsilon \) up to and including the order \( k \).

Chapter 2 of De Gruyter Expositions in Mathematics [GS08] also contains asymptotic results based on more general perturbation conditions.

It worth to mention that discrete time analogues of some of the results presented above are given in papers by Englund and Silvestrov [ES97], Englund [Eng00, Eng01], and Silvestrov [Sil00b]. Also, exponential asymptotic expansions for renewal equation with non-polynomial perturbations are studied in papers Englund and Silvestrov [ES97, Eng01], and Ni, Silvestrov and Malyarenko [NSM08].

2.3 Nonlinearly Perturbed Regenerative Processes

Method of asymptotic analysis of nonlinearly perturbed renewal equation can be directly used in studies of quasi- and pseudo-stationary asymptotics for nonlinearly perturbed regenerative processes. The corresponding results are presented in Chap. 3 of De Gruyter Expositions in Mathematics [GS08]. This chapter is partly based on the results of the papers Gyllenberg and Silvestrov [GS99a, GS00b].

Let \( \xi^{(\varepsilon)}(t), t \geq 0 \) be a regenerative process with a measurable phase space \( X \) and regeneration times \( \tau_n^{(\varepsilon)} \), \( n = 1, 2, \ldots, \) and \( \mu^{(\varepsilon)} \) be a regenerative stopping time that regenerates jointly with the process \( \xi^{(\varepsilon)}(t) \), at times \( \tau_n^{(\varepsilon)} \).

Both the regenerative process \( \xi^{(\varepsilon)}(t) \) and the regenerative stopping time \( \mu^{(\varepsilon)} \) are assumed to depend on a small perturbation parameter \( \varepsilon \geq 0 \). The processes \( \xi^{(\varepsilon)}(t) \), for \( \varepsilon > 0 \) are considered as a perturbation of the process \( \xi^{(0)}(t) \), and therefore we assume some weak continuity conditions for certain characteristic quantities of these processes regarded as functions of \( \varepsilon \) at point \( \varepsilon = 0 \).

As far as the regenerative stopping times are concerned, we consider two cases. The first one is a pseudo-stationary case, where the random variables \( \mu^{(\varepsilon)} \) are stochastically
unbounded, i.e., $\mu^{(e)}$ tend to $\infty$ in probability as $\varepsilon \to 0$. The second one is the quasi-stationary case, where the random variables $\mu^{(e)}$ are stochastically bounded as $\varepsilon \to 0$.

The object of studies is the probabilities $P^{(e)}(t, A) = P\{\xi^{(e)}(t) \in A, \mu^{(e)} > t\}$. These probabilities satisfy the following renewal equation,

$$P^{(e)}(t, A) = q^{(e)}(t, A) + \int_0^\infty P^{(e)}(t - s, A)F^{(e)}(ds), \ t \geq 0,$$  \hspace{1cm} (2.15)

where the forcing function $q^{(e)}(t, A) = P\{\xi^{(e)}(t) \in A, \tau_1^{(e)} \land \mu^{(e)} > t\}$ and distribution function $F^{(e)}(s) = P\{\tau_1^{(e)} \leq s, \mu^{(e)} \geq \tau_1^{(e)}\}$.

Note that the distribution $F^{(e)}(s)$ has the defect $f^{(e)} = P\{\mu^{(e)} < \tau_1^{(e)}\}$.

In this case, the mixed power-exponential moment generating function $\phi^{(e)}(\rho, n) = E(\tau_1^{(e)})^n e^{\rho \tau_1^{(e)}} \chi(\mu^{(e)} \geq \tau_1^{(e)})$ and the characteristic equation (2.3) takes the form $\phi^{(e)}(\rho, 0) = 1$.

The corresponding perturbation condition assumes that function $\phi^{(e)}(\rho, n)$ (taken in point $\rho^{(0)}$ which is the root of the limit characteristic equation) can be expanded in a power series with respect to $\varepsilon$ up to and including the order $k - n$ for every $n = 0, \ldots, k$.

The relationship between the rate with which $\varepsilon$ tends to zero and the time $t$ tends to infinity has a delicate influence upon the results. The balance between the rate of perturbation and the rate of growth of time is characterized by the following asymptotic relation $e^{rt^{(e)}} \to \lambda_r < \infty$ as $\varepsilon \to 0$ that is assumed to hold for some $1 \leq r \leq k$.

The direct application of Theorem 3 to the renewal equation (2.15) yields, under the corresponding conditions, the following exponential asymptotic expansion,

$$\frac{P\{\xi^{(e)}(t^{(e)}) \in A, \mu^{(e)} > t^{(e)}\}}{\exp\{-(\rho_0 + a_1 \varepsilon + \cdots + a_{r-1} \varepsilon^{r-1})t^{(e)}\}} \to \tilde{\pi}^{(0)}(A)e^{-\lambda_ra_r} \text{ as } \varepsilon \to 0,$$  \hspace{1cm} (2.16)

where

$$\tilde{\pi}^{(e)}(A) = \frac{\int_0^\infty e^{\rho^{(e)}t}q^{(e)}(s, A)m(ds)}{\int_0^\infty se^{\rho^{(e)}s}F^{(e)}(ds)}.$$

Also, Theorem 4, applied to the renewal equation (2.15), yields, under the corresponding conditions, the asymptotic expansions for the renewal limits $\tilde{\pi}^{(e)}(A)$ and then the following asymptotic expansion for the quasi-stationary distributions $\pi^{(e)}(A) = \tilde{\pi}^{(e)}(A)/\tilde{\pi}^{(e)}(X)$,

$$\pi^{(e)}(A) = \pi^{(0)}(A) + g_1(A)\varepsilon + \cdots + g_k(A)\varepsilon^k + o(\varepsilon^k).$$  \hspace{1cm} (2.17)

Both asymptotic expansions (2.16) and (2.17) are provided by the explicit algorithms for calculating the coefficients in these expansions as rational functions of the coefficients in the expansions involved in the initial perturbation conditions.

The case $\rho_0 = 0$ corresponds to a model with stochastically unbounded random variables $\mu^{(e)}$, while the case $\rho_0 > 0$ corresponds to a model with stochastically bounded random variables $\mu^{(e)}$. The asymptotic relation (2.16) describes in these cases, respectively, pseudo-stationary and quasi-stationary phenomena for perturbed regenerative processes.
To clarify the meaning of the asymptotic relation (2.16) let us consider the pseudo-stationary case, where $\rho_0 = 0$. Note that in this case $\tilde{\pi}^{(0)}(X) = 1$ and, therefore, $\tilde{\pi}^{(0)}(A) = \pi^{(0)}(A)$.

If $k = 1$, then the only case $r = 1$ is possible for the above balancing condition for the rate of perturbation and the rate of growth of time. In this case, the asymptotic relation (2.16) is equivalent to the asymptotic relation $P\{\xi^{(e)}(t^{(e)}) \in A, \mu^{(e)} > t^{(e)}\} \rightarrow \pi^{(0)}(A)e^{-\lambda_1\varepsilon t}$ as $\varepsilon \rightarrow 0$. It shows that the position of the regenerative process $\xi(t^{(e)})$ and the normalised regenerative stopping time $\varepsilon \mu^{(e)}$ are asymptotically independent and have, in the limit, a stationary distribution and an exponential distribution, respectively. This can be interpreted as a mixed ergodic theorem (for the regenerative processes) and a limit theorem (for regenerative stopping times).

If $k = 2$, then two cases, $r = 1$ and $r = 2$, are possible for the balancing condition. The case $r = 1$ was already commented and interpreted above. In this case, relation (2.16) can be given in the equivalent alternative form, $P\{\xi^{(e)}(t^{(e)}) \in A, \mu^{(e)} > t^{(e)}\} \rightarrow \pi^{(0)}(A)\exp(-a_1\varepsilon t)$ as $\varepsilon \rightarrow 0$, for non-zero sets such that $\pi^{(0)}(A) \neq 0$. It shows that probability $P\{\xi^{(e)}(t^{(e)}) \in A, \mu^{(e)} > t^{(e)}\}$ can be approximated by the exponential type mixed tail probability $\pi^{(0)}(A)\exp(-a_1\varepsilon t)$, with the zero asymptotic relative error, in every asymptotic time zone which is determined by the relation $\varepsilon t \rightarrow \lambda_1$ as $\varepsilon \rightarrow 0$, where $0 \leq \lambda_1 < \infty$.

In the case $r = 2$, the asymptotic relation (2.16) reduces to the asymptotic relation $P\{\xi^{(e)}(t^{(e)}) \in A, \mu^{(e)} > t^{(e)}\} \rightarrow e^{-a_2\lambda_2}$ as $\varepsilon \rightarrow 0$. It shows that probability $P\{\xi^{(e)}(t^{(e)}) \in A, \mu^{(e)} > t^{(e)}\}$ can be approximated by the exponential type mixed tail probability $\pi^{(0)}(A)\exp(-a_1\varepsilon t)$ as $\varepsilon \rightarrow 0$, with the asymptotic relative error $1 - e^{-a_2\lambda_2}$, in every asymptotic time zone which is determined by the relation $\varepsilon^2 t \rightarrow \lambda_2$ as $\varepsilon \rightarrow 0$, where $0 \leq \lambda_2 < \infty$.

If $\lambda_2 = 0$, then $\varepsilon t \rightarrow O(\varepsilon^{-1})$ and the asymptotic relative error is 0. Note that this case also covers the situation where $\varepsilon t$ is bounded, which corresponds to the asymptotic relation (2.16) with $k = 1$. This is already an extension of this asymptotic result since it is possible that $\varepsilon t \rightarrow \infty$.

If $\lambda_2 > 0$, then $\varepsilon t = O(\varepsilon^{-1})$, and the asymptotic relative error is $1 - e^{-\lambda_2a_2}$. It differs from 0. Therefore, $O(\varepsilon^{-1})$ is an asymptotic bound for the large deviation zone with the asymptotic relative error 0.

To get the approximation with zero asymptotic relative error in the asymptotic time zone which are determined by the relation $\varepsilon^2 t \rightarrow \lambda_2$ one should approximate the mixed tail probabilities $P\{\xi^{(e)}(t^{(e)}) \in A, \mu^{(e)} > t^{(e)}\}$ by the exponential type mixed tail probabilities $\pi^{(0)}(A)e^{-(a_1\varepsilon + a_2\varepsilon^2)t^{(e)}} \sim \pi^{(0)}(A)e^{-a_1\varepsilon t^{(e)} - a_2\lambda_2}$, i.e., to introduce the corresponding corrections for the parameters in the exponents.

The comments above let one interpret relation (2.16) in the case $r = 2$ as a new type of mixed ergodic and large deviation theorem for the nonlinearly perturbed process $\eta^{(e)}(t)$ and the lifetime $\mu^{(e)}_0$.

A similar interpretation can be made for the asymptotic relation (2.16) if $k > 2$ and in the quasi-stationary case, where $\rho^{(0)} > 0$.

Finally, the above asymptotic results is expanded to the model of nonlinearly perturbed regenerative processes with transition period.
2.4 Nonlinearly Perturbed Semi-Markov Processes

The asymptotic results obtained in Chaps. 1–3 play the key role in further studies. In Chaps. 4 and 5 of De Gruyter Expositions in Mathematics [GS08], they are applied to analysis of pseudo- and quasi-stationary phenomena for perturbed semi-Markov processes with a finite set of states. The results presented in this chapter are partly based on papers Gyllenberg and Silvestrov [GS99a, GS00a] and Silvestrov [Sil07a, Sil07b].

A semi-Markov process \( \eta^{(\varepsilon)}(t), t \geq 0 \), with a phase space \( X = \{0, \ldots, N\} \) and transition probabilities \( Q_{ij}^{(\varepsilon)}(u) \) is considered. The first hitting time \( \mu^{(\varepsilon)}_0 \) to the state 0 plays the role of an absorption time. Asymptotic behaviour of probabilities \( P_{ij}^{(\varepsilon)}(t) = P_{i} \{ \eta^{(\varepsilon)}(t) = j, \mu^{(\varepsilon)}_0 > t \} \) is an object of studies.

This can be done by using the facts that a semi-Markov process can be considered as a regenerative process with regeneration times which are subsequent return moments to any fixed state \( j \neq 0 \) and the first hitting time to the absorption state 0 is a regenerative stopping time. The asymptotic results mentioned above are obtained by applying the corresponding results for regenerative processes given in Chap. 3.

Not only the generic case, where the limiting semi-Markov process has one communication class of recurrent-without absorption states, is considered in details, but also the case, where the limiting semi-Markov process has one communication class of recurrent-without absorption states and, additionally, the class of non-recurrent-without absorption states. The latter model covers a significant part of applications.

In this case, the distribution function \( \varrho G_{jj}^{(\varepsilon)}(t) \) of the return-without absorption time in a state \( j \neq 0 \) generates the renewal equation. The corresponding characteristic equation takes the form \( \int_0^\infty e^{\rho s} \varrho G_{jj}^{(\varepsilon)}(ds) = 1 \). It is shown that the characteristic root \( \rho^{(\varepsilon)} \) of this equation does not depend on the choice of a recurrent-without absorption state \( j \neq 0 \).

It is natural to formulate the perturbation conditions in terms of transition probabilities \( Q_{ij}^{(\varepsilon)}(u) \). In particular, nonlinear perturbation conditions are imposed on these transition probabilities, which assume that mixed power-exponential moment generation functions \( P_{ij}^{(\varepsilon)}(\varrho, n) = \int_0^\infty s^n e^{\varrho s} Q_{ij}^{(\varepsilon)}(ds), i \neq 0, j \in X \) (taken in point \( \rho^{(0)} \) which is the root of the corresponding characteristic equation for the limit case \( \varepsilon = 0 \)) can be expanded in a power series with respect to \( \varepsilon \) up to and including the order \( k - n \) for every \( n = 1, \ldots, k \).

Conditions and expansions formulated for regenerative processes are specified in terms of expansions for the moments of regeneration times. As was pointed above the return times play the role of regeneration moments for semi-Markov processes. Therefore, the corresponding asymptotic expansions for absorption probabilities and the moments of return and hitting times for perturbed semi-Markov processes must be derived from the nonlinear perturbation conditions imposed on transition probabilities \( Q_{ij}^{(\varepsilon)}(u) \). Then, the corresponding asymptotic results for regenerative processes can be applied.

Thus, as the first step, asymptotic expansions for hitting probabilities, power and mixed power-exponential moments of hitting times are constructed using a procedure that is based on recursive systems of linear equations for hitting probabilities and
moments of hitting times. These moments satisfy recurrence systems of linear equations with the same perturbed coefficient matrix and the free terms connected by special recurrence systems of relations. In these relations, the free terms for the moments of a given order are given as polynomial functions of moments of lower orders. This permits to build an effective recurrence algorithm for constructing the corresponding asymptotic expansions. Each sub-step in this recurrence algorithm is of a matrix but linear type, where the solution of the system of linear equations with nonlinearly perturbed coefficients and free terms should be expanded in asymptotic series. These expansions are also provided with a detailed analysis of their pivotal properties. These results have their own values and possible applications beyond the problems studied in the book.

As soon as asymptotic expansions for moments of return-without-absorption times are constructed, the second nonlinear but scalar step of construction asymptotic expansions expansions for the characteristic root $\rho^{(\varepsilon)}$ can be realised.

The separation of two steps described above, the first one matrix and recurrence but linear and the second one nonlinear but scalar, significantly simplify the whole algorithm.

The asymptotic expansions for the quasi-stationary distributions require two more steps, which are needed for constructing asymptotic expansions at the point $\rho^{(\varepsilon)}$ for the corresponding moment generation functions, giving expressions for quasi-stationary probabilities in the quotient form, and for transforming the corresponding asymptotic quotient expressions to the form of power asymptotic expansions.

As a result, one get an effective algorithm for a construction of the asymptotic expansions given in relations (2.16) and (2.17). It seems, that the method used for obtaining the expansions mentioned above has its own value and great potential for future studies.

As a particular but important example, the model of nonlinearly perturbed continuous time Markov chains with absorption is also considered. In this case, it is more natural to formulate the perturbation conditions in terms of generators of the perturbed Markov chains. Here, an additional step in the algorithms is needed, since the initial perturbation conditions for generators must be expressed in terms of the moments for the corresponding semi-Markov transition probabilities. Then, the basic algorithms obtained for nonlinearly perturbed semi-Markov processes can be applied.

Chapters 1–5 present a theory that can be applied in studies of pseudo- and quasi-stationary phenomena in nonlinearly perturbed stochastic systems.

2.5 Nonlinearly Perturbed Stochastic Systems

Chapter 6 of book [GS08] deals with applications of the results obtained in Chaps. 1–5 to an analysis of pseudo- and quasi-stationary phenomena in nonlinearly perturbed stochastic systems. This chapter is partly based on the results of the papers Gyllenberg and Silvestrov [GS94, GS99a, GS00a].

Examples of stochastic systems under consideration are queueing systems, epidemic, and population dynamics models with finite lifetimes. In queueing systems, the lifetime is usually the time at which some kind of a fatal failure occurs in the system. In epidemic
models, the time of extinction of the epidemic in the population plays the role of the lifetime, while in population dynamics models, the lifetime is usually the extinction time for the corresponding population.

Several classical models being the subject of long term research studies were selected. These models serve nowadays mainly as platforms for demonstration of new methods and innovation results. Our goal also is to show what kind of new types results related to quasi-stationary asymptotics can be obtained for such models with nonlinearly perturbed parameters.

As the first example, a M/M queueing system with highly reliable main servers is considered. This queueing system is our first choice because of its function can be described by some nonlinearly perturbed continuous time Markov chains with absorption. Here, all conditions take a very explicit and clear form.

Also a M/G queueing system with quick service and a bounded queue buffer is considered. In this case, the perturbed stochastic processes, which describe the dynamics of the queue in the system, belong to the class of so-called stochastic processes with semi-Markov modulation. These processes admit a construction of imbedded semi-Markov processes and are more general than semi-Markov processes. This example was chosen because it shows in which way the main results obtained in the book can be applied to stochastic processes more general than semi-Markov processes, in particular, to stochastic processes with semi-Markov modulation.

The next example is based on classical semi-Markov and Markov birth-and-death type processes. Some classical models of queueing systems, epidemic or population dynamic models can be described with the use of such processes. We show in which way nonlinear perturbation conditions should be used and what form will take advanced quasi- and pseudo-stationary asymptotics developed in Chaps. 1–5.

Finally, an example of nonlinearly perturbed metapopulation model is considered. This example is interesting since it brings, for the first time, the discussion on advanced quasi- and pseudo-stationary asymptotics in this actual area of research in mathematical biology.

2.6 Nonlinearly Perturbed Risk Processes

The classical risk processes are still the object of intensive research studies as show, for example, references given in the bibliography of De Gruyter Expositions in Mathematics [GS08]. Of course, the purpose of these studies is not any more to derive formulas relevant for field applications. These studies intend to illustrate new methods and types of results that can later be expanded to more complex models. The same approach was used by us when choosing this model. The aim was to show that the innovative methods of analysis for nonlinearly perturbed processes developed in the book can yield new results for this classical models.

Chapter 7 of De Gruyter Expositions in Mathematics [GS08] contains results that extend the classical Cramér–Lundberg and diffusion approximations for the ruin probabilities to a model of nonlinearly perturbed risk processes. Both approximations are presented in a unified way using the techniques of perturbed renewal equations
developed in Chaps. 1 and 2. This chapter is partly based on the results of the papers Gyllenberg and Silvestrov [GS99a, GS99b, GS00b] and Silvestrov [Sil00a, Sil07b].

The main new element in the results presented in Chap. 7 is a high order exponential asymptotic expansion in these approximations for nonlinearly perturbed risk processes. Correction terms are obtained for the Cramér-Lundberg and diffusion type approximations, which provide the right asymptotic behaviour of relative errors in the perturbed model. We study the dependence of these correction terms on the relations between the rate of perturbation and the rate of growth of the initial capital.

Also, various variants of the diffusion type approximation, including the asymptotics for increments and derivatives of the ruin probabilities are given.

Finally, we give asymptotic expansions in the Cramér–Lundberg and diffusion type approximations for distribution of the capital surplus prior and at ruin for nonlinearly perturbed risk processes.

It seems to us that results presented in Chaps. 6 and 7 illustrate well a potential of asymptotic methods developed in the book.

The works of Englund [Eng99a, Eng99b, Eng01] and Ni, Silvestrov, and Malyarenko [NSM08] may also be mentioned. They also deal with applications of methods based on perturbed renewal to asymptotic analysis of nonlinearly perturbed queuing systems and nonlinearly perturbed risk processes, but for models with non-polynomial nonlinear perturbations.

2.7 Conclusion

The last Chap. 8 of De Gruyter Expositions in Mathematics [GS08] contains three supplements.

The first supplement presents some basic arithmetic operation formulas for scalar and matrix asymptotic expansions.

In the second supplement, some new directions in the research concerned pseudo- and quasi-stationary phenomena for perturbed stochastic processes and systems that relate to the theory developed in this book are discussed and commented on. There is a hope that this discussion will be especially useful for young researchers and stimulate their interest to research studies in these areas. The corresponding extended comments can also be found in Silvestrov [Sil08].

The third supplement in Chap. 8 contains the brief bibliographical remarks.

The extended and carefully gathered bibliography has more than 1000 references to works in related areas, dealing with ergodic and quasi-ergodic theorems, stability theorems, limit and large deviation theorems for lifetime-type functionals and asymptotic aggregation theorems for regenerative, Markov, and semi-Markov type processes, as well as applications of such theorems to queueing systems, models of population dynamics, epidemic models, and other stochastic systems.

Here, we would like to point some originating and survey papers and books related to perturbation problems for stochastic processes. These are Vere-Jones [Ver62], Hanen [Han63], Kingman [Kin63], Kato [Kat66], Korolyuk and Turbin [KT76], [KT78], Wentzell and Freidlin [WF79], Silvestrov [Sil80], Seneta [Sen81], Solov’ev [Sol83], Asmussen [Asm87], [Asm00], Kalashnikov and Rachev [KR88], Stewart and Sun Ji
Quasi-stationary phenomena and related problems are a subject of intensive studies during several decades. However, the development of theory of quasi-stationary phenomena is still far from its completion. The part of the theory related to conditions of existence of quasi-stationary distributions is comparatively well developed while computational aspects of the theory are underdeveloped. The content of the book [18] is concentrated in this area. The book presents new effective methods for asymptotic analysis of pseudo- and quasi-stationary phenomena for nonlinearly perturbed stochastic processes and systems. Moreover, the results presented in the book unite, for the first time, research studies of pseudo- and quasi-stationary phenomena in the frame of one theory. Methods of asymptotic analysis for nonlinearly perturbed stochastic processes and systems developed in the book have their own values and possible applications beyond the problems studied in the book.

The results presented in the book will be interesting to specialists, who work in such areas of the theory of stochastic processes as ergodic, limit, and large deviation theorems, analytical and computational methods for Markov chains, regenerative, Markov, semi-Markov, risk and other classes of stochastic processes, renewal theory, and their queueing, reliability, population dynamics, and other applications. There is a hope that the book will also attract attention of those researchers, who are interested in new analytical methods of analysis for nonlinearly perturbed stochastic processes and systems, especially those who like serious analytical work.

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