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Degradation Processes: An Overview

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Abstract: In this chapter we survey research on different types of degradation processes. Over time, a device is subject to degradation, generally described as an increasing stochastic process. The device has a threshold Y, and it fails once the degradation level exceeds the threshold. Life distribution properties and maintenance policies for such devices are discussed.

Keywords and phrases: Degradation processes, life distributions, increasing failure rate, increasing failure rate average, maintenance and replacement policies

2.1 Introduction

Degradation processes received a great attention over the last decades, in this chapter we review some of the research done on these processes. At the beginning of the work done in reliability, engineers described the uncertainties about the failure times using the survival function; knowing the shape of such a function they can determine and study the properties of the failure rate and based on that they can determine the best possible maintenance policies. To estimate the survival function accurately (from statistical point of view), one has to observe the failure times of many items and these failure random variables are assumed to be independent. In practice, it is not always possible to observe many failures, and even if such failure times are possible to obtain, they are not independent as they all might be effected by a common environment. The other approach is to assess the failure of a device based on the characteristics of the process that caused its failure, normally a degradation process. Such an approach is common in assessing the amount of crack, the amount of erosion, and creep, and amount of contamination. Since many devices fail because of degradation, the degradation process is some type of stochastic process. In this chapter we examine different candidates for such processes. Such candidates include Lévy processes, functionals of Wiener processes, as well as pure jump Markov processes.

In Section 2.2, we give different classes of degradation processes. In Section 2.3, we define classes of life distributions and discuss dependence of random variables. In

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Section 2.4, we examine the behavior of the failure rate of devices subject to deterioration and give conditions that insure that the distribution of the failure time belongs to the defined classes of life distributions. In Section 2.5, maintenance and replacement policies of devices subject to degradation are discussed. Detailed proofs of the results given in this chapter are found in the references indicated at the end of this chapter.

2.2 Lévy and Pure Jump Processes

Throughout we let R to be the set of real numbers, $R_+ = [0, \infty)$, N be the set of integers, and $N_+ = \{0, 1, \ldots\}$. In this section we will define Lévy process, as well as pure jump Markov processes. We will discuss different examples of Lévy processes and give their corresponding Lévy measures. The relationship between Lévy processes and pure jump Markov processes is also examined.

2.2.1 Lévy processes

A stochastic process $X = \{X_t, t \geq 0\}$ is said to be a Lévy process if the following holds:

(i) The distribution of $X_{t+s} - X_t$, is independent of t.
(ii) The process is additive, i.e., for every $t, s \geq 0$, $P\{X_{t+s} - X_t \in A| \ X_u, u \leq t\} = P\{X_{t+s} - X_t \in A\}$.
(iii) X is stochastically continuous, i.e., for every $t \geq 0$ and $\varepsilon > 0$: $\lim_{s \to t} P\{|X_t - X_s| > \varepsilon\} = 0$.

That is to say a Lévy process is a stochastically continuous process with stationary and independent increments.

If $\Phi(z)$ is the characteristic function of a Lévy process, then $\ln \Phi(z)$ is of the form

$$t\left\{iza - \frac{z^2 b}{2} + \int_R [\exp(izx) - 1 - izx I_{\{|x|<1\}}]\nu(dx)\right\},$$

where $a \in R$, $b \in R_+$, and ν is a measure on R satisfying $\nu(\{0\}) = 0$, $\int_R (1 \wedge x^2)\nu(dx) < \infty$.

Examples of such processes include the following:

(1) ***The Brownian motion***. A Lévy process is said to be a Brownian motion with drift μ, and volatility rate σ^2, if $\mu = a$, $b = \sigma^2$, and $\nu(R) = 0$.

Another way to define Brownian motion is as follows: for every $t, s \in R_+$, $X_{t+s} - X_t$ has a normal distribution with mean $s\mu$, and a variance $s\sigma^2$.

It follows that the sample paths of the Brownian motion are non-differentiable, continuous and has finite variation on every open interval, almost everywhere. Furthermore, such a process is a strong Markov process. For every $x \in R_+$, we define the time of first passage through the threshold x by

$$T_x = \inf\{t \geq 0 : X_t \geq x\}.$$

For any $\alpha > 0$, we define $\gamma = (\mu^2 + 2\alpha\sigma^2)^{1/2}$. Abdel-Hameed and Nakhi [6] show that, for $y < x$, the Laplace transform of T_x is given by the equation

$$E_y(\exp(-\alpha T_x)) = \exp\{-(\gamma - \mu)(x - y)/\sigma^2\}.$$

Inverting the right-hand side of the above equation with respect to α, we get the probability density function of T_x as follows:

$$f_y(x, t|\mu, \sigma^2) = \frac{(x - y)}{\sqrt{\sigma^2 t^3}} \exp\left\{\frac{-(x - y - \mu t)^2}{2t\sigma^2}\right\}.$$

Although the proof of this formula has been obtained by other authors before, however, our proof is completely different.

For any $t \geq 0$, we let

$$M_t = \sup\{X_s, s \leq t\}.$$

Then,

$$P\{M_t < x\} = P\{T_x > t\}.$$

(2) ***Increasing Lévy Processes***. A Lévy process is said to be increasing if its sample paths are increasing.

Every such process must satisfy the following: $b = 0$, $\nu(-\infty, 0) = 0$, $\int (x \wedge 1)\nu(dx) < \infty$, and $d = a - \int_0^1 x\nu(dx) \geq 0$.

The measure ν characterizes the size and frequency of the jumps. If the measure is infinite, then the process has infinitely many jumps of very small sizes in any small interval. The constant d defined above is called the drift term of the process. If $d = 0$, then the process is a pure jump process that changes from one state to another state only through jump.

We now mention some examples of the Lévy process.

The compound Poisson process.

If the process is a compound Poisson process with jump rate λ, and the distribution of the jump sizes is denoted by F, then $\nu(dx) = \lambda F(dx)$, in this case the Lévy measure is finite. Actually, the finiteness of the Lévy measure characterizes the Poisson process.

Stable processes.

Stable processes are increasing Lévy processes, with Lévy measure given by

$$\nu(dy) = \alpha y^{-(1+\beta)},$$

where the constant α is non-negative, β is between 0 to 1, and d=0.

The gamma process.

The gamma process is an increasing Lévy process with Lévy measure

$$\nu(dx) = \delta x^{-1} \exp(-\eta x) dx,$$

where δ and η are non-negative real numbers and d=0.

2.2.2 Pure jump Markov processes

A Markov process (X) is called a pure jump process if for each $t \geq 0$

$$X_t = X_0 + \sum_{s \leq t} (X_s - X_{s-}).$$

Cinlar and Jacod [10] show that if X is such a process, then there exists a Poisson random measure N on $R_+ \times R_+$ whose mean measure at the point $(s, z) \in R_+ \times R_+$ is $dsdz/z^2$ and a deterministic function c defined on $R_+ \times (0, \infty)$ that is increasing in the first argument such that

$$\sum_{s \leq t} f(X_{s-}, X_s) = \int_{[0,t] \times R_+} N(ds, dz) f(X_{s-}, X_{s-} + c(X_{s-}, z)) \tag{2.1}$$

almost surely for each function f defined on $R_+ \times R_+$, for which $f(x, x) = 0$ for all x in R_+. In particular, it follows that

$$X_t = X_0 + \int_{[0,t] \times R_+} N(ds, dz) c(X_{s-}, z).$$

The above formula has the following interpretation:

$t \to X_t(w)$ jumps at s if the Poisson random measure $N(w, .)$ has an atom at (s, z) and the jump is from the left-hand limit $X_{s-}(w)$ to the right-hand limit: $X_s = X_{s-} + c(X_{s-}, z)$.

We note that an increasing Lévy process is a special case of the pure jump Markov processes with $c(x, z) = z$.

2.3 Life Distributions and Dependence Between Random Variables

In this section we discuss classes of life distributions and different notions of dependence between random variables.

2.3.1 Classes of life distributions

Let X be a non-negative random variable describing the lifetime of a given device. Let F be the distribution function of X and let $\bar{F} = 1 - F$ be the survival function, and cumulative hazard function $R = -\ln \bar{F}$. Then $\bar{F}$ is to have, or to be

(i) increasing failure rate (IFR) if R is a convex function. The IFR property is equivalent to saying that the failure rate is increasing, whenever it exists.
(ii) increasing failure rate average (IFRA) if $(1/t)R(t)$ is increasing function in t.
(iii) new better than used if for every non-negative t, s $\bar{F}(t + s) \leq \bar{F}(t)\bar{F}(s)$.

Assuming that $\bar{F}(0) = 1$, the following implications do hold:

$$\text{IFR} \Longrightarrow \text{IRA} \Longrightarrow \text{NBU}.$$

There are dual life distribution classes parallel to the above-mentioned classes and are obtained by reversing the direction of inequality or monotonicity in the above definitions. These classes are the decreasing failure rate (DFR), decreasing failure rate average (DFRA), and new worse than used (NWU) classes. Knowing the behavior of the failure rate of any device enables us to determine the appropriate maintenance and replacement policy for such device.

Excellent treatment of the classes of life distributions can be found in Barlow and Proschan [9].

2.3.2 Dependence between random variables

Components of systems exhibit some degree of dependence between their performances, indicated by their lifetimes. These dependence could be positive, as in the failure times of components subject to the same environment. Negative dependence arises in competing risk applications, where items are competing for a fixed amount of resources. The statistical literature is full with references to different measures of dependence. The simplest is the correlation and partial correlation coefficients. The notion of association between random variables has many applications in reliability and statistics. While there are many other measures of dependence between random variables, we will only discuss a few of them that will be needed in the following sections. One of the strongest notion of dependence is given in the following

Definition. A function $f : R^2 \longrightarrow R_+$ is said to be totally positive of order r (TP_r) if $det(f(x_i, y_j)) \geq 0$ for each choice $x_i \leq x_2 \leq ... \leq x_k$ and $y_i \leq x_2 \leq \cdots \leq y_k$, $1 \leq k \leq r, r \geq 1$.

If f is the joint probability density function of two random variables, then f is TP_r implies that the two random variables, loosely speaking, exhibit a very strong positive dependence. More detailed investigation of this matter can be found in Barlow and Proschan [9].

Definition. A function $f : R \longrightarrow R_+$ is said to be a Pólya frequency function of order r (PF_r) if $f(x - y)$ is TP_r in x, y.

2.4 The Degradation Process and Properties of the Resulting Life Distribution

2.4.1 Non-stationary gamma degradation process

Abdel-Hameed [1] discusses the case where the degradation process (X) is a non-stationary gamma process, with transition density function given by, for $x, t \geq 0$

$$p(t, x) = \exp(-\lambda x) \ (\lambda x)^{\Lambda(t)} / x\Gamma(\Lambda(t)),$$

where Γ is the gamma function, and $\Lambda(t)$ is a non-negative increasing function in its argument. Assume that the device has a resistance level random variable (denoted by Y) with right-tail probability $\bar{G}$ and the process X and the random variable Y

are independent of each other. The device fails when the degradation level crosses its resistance level. Let ρ be the failure time of the device, then

$$\rho = \inf\{t : X_t \geq Y\}$$

which is the time of first crossing of the process X to the random boundary Y. It follows that for $t \geq 0$,

$$\begin{aligned}
\bar{F}(t) &= P\{\rho > t\} \\
&= P\{X_t \leq Y\} \\
&= E(\bar{G}(X_t)) \\
&= \int_0^\infty P\{X_t \leq y) G(dy) \\
&= \int_0^\infty \gamma(\Lambda(t), \lambda y) G(dy) / \gamma(\Lambda(t)\ , \infty)
\end{aligned}$$

where $\gamma(t, y)$ is the incomplete gamma function defined as $\int_0^y \exp{(-x)} x^{t-1} dx$. He shows that (under appropriate conditions on $\Lambda(t)$) life distribution properties of the resistance level Y are inherited as corresponding properties of the failure time ρ. In particular, if the resistance level has increasing failure rate and $\Lambda(t)$ is a convex function, then the failure-time distribution has increasing failure rate as well. As a by-product, if the resistance level is constant and if $\Lambda(t)$ is convex, then the failure-time distribution has increasing failure rate.

2.4.2 Increasing Lévy and pure jump degradation processes

Abdel-Hameed [3] extends the results above for the gamma degradation process, assuming that the degradation process is a non-homogeneous Lévy process. Let μ be the Radon–Nikodym derivatives of the Lévy measure ν. The main results are as follows:

(a) If G has increasing failure rate, $\Lambda(t)$ is a convex function, and μ is PF_2, then F has increasing failure rate.
(b) If G has increasing failure rate average, the function $\Lambda(t)/t$ is increasing in its argument, then F has increasing failure rate average.
(c) If G is NBU, the function $\Lambda(t)$ is super-additive ($\Lambda(t+s) \leq \Lambda(t) + \Lambda(s)$), then F is NBU.

Dual results for the DFR, DFRA, and NWU classes are shown to hold.

Abdel-Hameed [4] extends the above results to include the case where the degradation process is an increasing pure jump process. He finds conditions on the resistance distribution function and the function c in (1) that insure that the distribution of the failure time ρ is IFR, IFRA, NBU, DFR, etc.

2.4.3 Brownian motion like degradation processes

Durham and Padget [11], Park and Padget [12] study the case where the degradation process is a functional of Brownian motion. In the second paper the authors proposed to approximate the distribution function of the degradation process as follows:

$$P\{X_t \leq x\} \approx \Phi(\frac{x-\mu t}{\sigma\sqrt{t}})$$

for every $x \geq 0$, where Φ is the standard normal distribution function.

2.5 Maintenance Policies of Devices Subject to Degradation

Assume that a device is subject to degradations and the degradation process is of the gamma type. Abdel-Hameed [2] study the case where the device is replaced at failure (corrective maintenance) or at when the deterioration level exceeds a predetermine level (preventive maintenance). The cost of corrective maintenance is fixed, while the cost of preventive maintenance depends on the deterioration level at the time when the maintenance is performed. He obtains an explicit formula for the long-run average cost per a unit of time. He also discuss a discrete version of the model, where the damage is observed at discrete points in time, and not continuously. Noortwijk [13] applies this result to maintenance of a cylinder on a swing bridge.

Abdel-Hameed [5] treats the optimal inspection policy of a device when the degradation process is an increasing pure jump Markov process, the degradation is monitored periodically. There is a penalty cost of observing the degradation level, and a cost of preventive maintenance as well as the cost of corrective maintenance. In this model, he considers two decision variables, the inspection interval and the preventive maintenance level. A failure is detected only by inspection. He finds the optimal maintenance policy that minimizes the long-run average cost per unit of time.

Abdel-Hameed and Nakhi [7] treat the maintenance policy for devices subject to degradation, when the degradation process is an increasing semi-Markov process. Specifically, let the degradation process (Z) be an increasing semi-Markov process with embedded Markov renewal process $(X, T) = (X_n, T_n; n \in N)$, where $X_n = Z(T_n)$. Let $Q = \{Q(x, A, t), x, t \in R_+, , A \subset R_+ \}$ be the semi-Markov kernel associated with (X, T), that is, $Q(x, A, t) = \Pr\{X_{n+1} \in A, T_{n+1} - T_n \leq t | X_n = x\}$, and Markov renewal kernel $R = \{R(x, A, t), x, t \in R_+, , A \subset R_+ \}$, where $R(x, A, t) = \sum_{n=0}^{\infty} Q^{(n)}(x, A, t)$. The system has a resistance level (denoted by random variable Y), and the device fails once the degradation level crosses the resistance level. The resistance level and the degradation process are assumed to be independent. We denote the failure time by ρ. Let $\hat{Z}$ be the degradation process, obtained by killing the process Z at the failure time, that is, $\hat{Z} = (Z_t, t < \rho)$. Define $\hat{Q}$ and $\hat{R}$ as the corresponding semi-Markov kernel and Markov renewal kernel, respectively. The system can be replaced before or at failure and is maintained continuously. The maintenance and non-failure costs are state dependent. They determine the optimal maintenance policy, using the total discounted as well as the long-run average cost per unit of time. Let $g : R_+ \to R$ be the function describing the maintenance rate. In the case, where the state space is countable, we define for degradation levels i, j in the state space

$$\begin{aligned} q(i,j) &= P\{X_{n+1} = j | X_n = i\}, \\ m(i) &= E_i(T), \\ \hat{q}(i,j) &= q(i,j)\frac{\bar{G}(j)}{\bar{G}(i)}; \\ \hat{h}(i) &= P_i\{T_1 = \rho\}. \end{aligned}$$

Assume that the costs of a preventative (corrective) maintenance are c_1 and c_2, ($c_2 > c_1$), respectively, and define the matrix $Q = (q(i,j))$. The optimal replacement policy that minimizes the long-run average cost per a unit time can be summarized in the following algorithm. For more detailed explanations, the reader is referred to the reference above.

Algorithm. Assume that the degradation level at time zero is equal to i, normally taken equal to zero.

Step 1. let $j = i$.

Step 2. Compute the matrix $\hat{R}$, using the well-known relationship $\hat{R} = [I - \hat{Q}]^{-1}$, where I is the identity matrix of proper dimensions.

Step 3. For i, j let $\hat{r}(i,j) = \hat{R}(i,j) - \hat{R}(i,j-1)$.

Step 4. Compute

$$b_j(k) = c_2 \left[\frac{m(k)\hat{h}(j)}{m(j)} - \hat{h}(k) \right] + m(k)(g(j) - g(k)),$$

for $k = i, \ldots, j$.

Step 5. Compute

$$F(j) = \sum_{i}^{j} \hat{r}(i,j) b_j(k).$$

Step 6. If $F(j) \geq c_1$, then j is the optimum replacement level, otherwise $j = j+1$ and go to step 2.

Abdel-Hameed [8] considers the optimal maintenance policy for a system subject to degradation. The degradation level is only observed at successive inspection times. It follows that the degradation levels at inspections and the times of successive inspections form a Markov renewal process. Failure is detected only by inspection; at this point in time the system goes through a corrective maintenance. The system is also maintained when the degradation exceeds a predetermined level (preventive maintenance). He determines the optimal maintenance policy using both the total discounted and the long-run average cost criteria.

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