2

Thurston Norm

In this chapter, we prove the following theorem.

**Theorem 2.1** (W. Thurston [Thu85]). *Suppose that $M$ is a compact atoroidal orientable 3-manifold such that rank $H_2(M, \partial M; \mathbb{Z}) \geq 2$. Then $M$ contains an embedded superincompressible surface that is not a fiber in a fibration of $M$ over $\mathbb{S}^1$ and that represents a nontrivial element of $H_2(M, \partial M; \mathbb{Z})$.*

The proof of this theorem will be finished in Section 2.3. Our proof is essentially the same as Thurston’s.

2.1. Norms defined over $\mathbb{Z}$

Consider the space $\mathbb{R}^n$ with the lattice $\mathbb{Z}^n$ embedded in the standard way. Suppose that $p: \mathbb{Z}^n \rightarrow \mathbb{Z}$ is a nonnegative function that satisfies the following axioms:

1. $p$ is linear along rays, i.e., $p(m \cdot z) = |m|p(z)$ for each $z \in \mathbb{Z}^n$, $m \in \mathbb{Z}$.

2. $p$ is convex over $\mathbb{Z}$, i.e., for each $k, m \in \mathbb{Z}$, and $z, w \in \mathbb{Z}^n$, we have

$$p(k \cdot z + m \cdot w) \leq |k|p(z) + |m|p(w).$$

3. $p$ is nondegenerate, i.e., $p(z) = 0$ iff $z = 0$.

A function $p$ satisfying these properties is called a *norm defined over $\mathbb{Z}$*. 

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Theorem 2.2. Each norm defined over $\mathbb{Z}$ extends to a (usual) norm defined on the whole space $\mathbb{R}^n$.

Proof. Suppose that $\alpha \in \mathbb{Q}^n - \{0\}$. Then there exists $m \in \mathbb{Z}$ such that $m \cdot \alpha \in \mathbb{Z}^n$, and we let $p(\alpha) = p(m \cdot \alpha)/m$. This extension of the function $p$ to $\mathbb{Q}^n$ is well defined since $p$ is linear on rays in $\mathbb{Z}^n$. Clearly, the function $p : \mathbb{Q}^n \to \mathbb{Q}$ is linear on rays and $\mathbb{Q}$-convex in the sense that $p(k\alpha + m\beta) \leq |k|p(\alpha) + |m|p(\beta)$ for all $k, m \in \mathbb{Q}$ and $\alpha, \beta \in \mathbb{Q}^n$.

Lemma 2.3. The function $p : \mathbb{Q}^n \to \mathbb{Q}$ extends continuously to a seminorm (again denoted by $p$) on $\mathbb{R}^n$.

Proof. The reader will verify that for each $x \in \mathbb{Q}^n$ the restriction of $p$ to the unit ball centered at $x$ is bounded by a constant $C$. For $y \in \mathbb{Q}^n$ choose $Y = Y(y) \in \mathbb{Q}^n$ such that $|Y - x| \leq 1$, $y = (1 - t)x + tY$, $t \in \mathbb{Q}$, and $t \to 0$ as $y \to x$. Therefore, by convexity,

$$\lim_{y \to x} p(y) \leq \lim_{t \to 0} ((1 - t)p(x) + tC) = p(x).$$

On the other hand, if $y \in \mathbb{Q}^n$ converges to $x$ and $x$ is the midpoint of the segment $[yz]$, then $z$ converges to $x$ and $p(x) \leq (p(y) + p(z))/2$. Hence

$$\lim_{y \to x} p(y) \geq p(x).$$

Lemma 2.4. $p : \mathbb{R}^n \to \mathbb{R}$ is a norm.

Proof. Suppose that $p(x) = 0$ and $x \neq 0$. The sublevel set $C = \{p(y) < 1/2\}$ is an open convex neighborhood of the line $\mathbb{R} \cdot x$. Thus $C$ contains at least one integral point $z \in \mathbb{Z}^n - \{0\}$. Contradiction.

This finishes the proof of Theorem 2.2.

We retain the name norm defined over $\mathbb{Z}$ for the extension of $p$ to $\mathbb{R}^n$. Let

$$B^{(p)}(r) = \{v \in \mathbb{R}^n : p(v) \leq r\}$$

denote the ball of radius $r$ centered at zero, with respect to the norm $p$.

Theorem 2.5. Any norm $p : \mathbb{R}^n \to \mathbb{R}$ defined over $\mathbb{Z}$ has the following property. Suppose that $z$ is an element of $\mathbb{Z}^n$. Then there exists a linear function $l$ such that

- $l(z/p(z)) = 1$;
- $l : \mathbb{Z}^n \to \mathbb{Z}$, i.e., the function $l$ is defined over $\mathbb{Z}$;
- the operator norm of $l$ with respect to $p$ is equal to 1, i.e., $|l(\alpha)| \leq 1$ for all $\alpha \in B^{(p)}(1)$.  

2.2. Variation of fiber-bundle structure

Proof. We consider the sequence of integral balls of radius \( r \) in \( \mathbb{R}^n \): 
\[
B^{(p)}(r) \cap \mathbb{Z}^n,
\]
where \( r \in \mathbb{Z}_+ \cdot p(z) \). Then
\[
\bigcup_{r \in \mathbb{Z}_+} \frac{1}{r} \cdot (B^{(p)}(r) \cap \mathbb{Z}^n) = B^{(p)}(1) \cap \mathbb{Q}^n.
\]

Note that the integer vector \( \frac{r}{p(z)} \cdot z \) belongs to the boundary of \( B^{(p)}(r) \) for each \( r \) as above. The convex hulls \( C(r) \) of \( B^{(p)}(r) \cap \mathbb{Z}^n \) are finite polyhedra whose faces are defined over \( \mathbb{Z} \) in the sense that each top-dimensional face is contained in the zero level set of a linear function with integer coefficients. Thus for each point \( \frac{r}{p(z)} \cdot z \) there is linear function \( l_r \) with integer coefficients such that \( l_r(\frac{r}{p(z)} \cdot z) = r \) and the level set \( \{ l_r(w) = r \} \) is disjoint from the interior of \( C(r) \). Hence \( l_r(z/p(z)) = 1 \), and the level set \( \{ l_r(v) = 1 \} \) is disjoint from the interior of \( \frac{1}{r} C(r) \). However the convex sets \( \frac{1}{r} C(r) \) exhaust the unit ball \( B^{(p)}(1) \). Thus the operator norms (with respect to \( p \)) of the linear functions \( l_r \) are convergent to 1 as \( r \to \infty \). This means that the sequence of linear functions \( l_r \) with integer coefficients is subconvergent as \( r \to \infty \) to a linear function \( l \) whose norm is equal to 1. Clearly \( l \) has integer coefficients and \( l(z/p(z)) = 1 \). \(\square\)

The ratios \( z/p(z), z \in \mathbb{Z}^n \), are dense in the unit sphere \( \partial B^{(p)}(1) \). Thus the unit ball \( B^{(p)}(1) \) is the intersection of the sublevel sets \( \{ l \leq 1 \} \), where the linear functions \( l \) are as in Theorem 2.5. However the number of such linear functions \( l \) is finite. We conclude that \( B^{(p)}(1) \) is a polyhedron with finitely many faces that are given by linear equations with integer coefficients.

Corollary 2.6. Suppose that \( v \) is a vertex of the polyhedron \( B^{(p)}(1) \). Then there exists an element \( z \in \mathbb{Z}^n \) such that \( v = z/p(z) \).

2.2. Variation of fiber-bundle structure

Suppose that \( M \) is a compact 3-manifold (smoothly) fibered over \( S^1 \). Let \( \mathcal{F} \) denote the fibration and \( F_t \) denote the fiber of \( \mathcal{F} \) over \( t \in S^1 \). The fibers \( F_t \) are transversal to the boundary of \( M \), and we assume that \( M \) is given a Riemannian metric such that \( F_t \) are orthogonal to \( \partial M \). The tangent planes to the fibers \( F_t \) define a plane subbundle \( P \subset T(M) \). Let \( L \) denote the unit vector field on \( M \) that is orthogonal to \( P \). Along each boundary curve \( \partial F_t \) we choose the unit tangent field \( X \) on \( F_t \) that is orthogonal to \( \partial M \). Thus we get a section \( X \) of \( P|_{\partial M} \). The pair \( (P, X) \) determines an element of \( H_2(M, \partial M)^* \equiv H^1(M) \) as follows. Suppose that \( h : (\Sigma, \partial \Sigma) \to (M, \partial M) \) is a proper smooth map from a surface \( \Sigma \) that is a representative of a relative cycle \( \zeta \in Z_2(M, \partial M) \). Then \( h^*(P) \) is a 2-dimensional vector bundle over \( \Sigma \) with the prescribed section \( h^*(X) \) over \( \partial \Sigma \). There is a well-defined obstruction (the relative Euler number) \( e(h^*(P), X) \in \mathbb{Z} \) to the extension of \( h^*(X) \) to a nonzero section of the bundle \( h^*(P) \). Thus we define \( \tau \in H_2(M, \partial M)^* \) as \( \tau([\zeta]) = e(h^*(P), X) \). It is
easy to see that $\tau$ is well defined. Obviously for the relative class $[\xi] \in H_2(M, \partial M)$ that is represented by a fiber $F_i$ we get $\tau([\xi]) = \chi(F_i)$.

There is a closed nondegenerate integer 1-form $\theta$ on $M$ whose kernel is the tangent subbundle of $\mathcal{F}$. Namely, if $f : M \to S^1$ is the fibration, then $\theta$ is the pullback under $f$ of the angle form $dt$ from $S^1$. The converse to this is true as well.

**Theorem 2.7** (D. Tishler [Tis70]). Suppose that $\omega$ is a closed nondegenerate 1-form on $M$ that has integer periods (i.e., it determines an element of $H^1(M, \mathbb{Z})$). Assume that the restriction of $\omega$ to $\partial M$ is nondegenerate. Then there exists a fibration $\mathcal{G}$ of $M$ over $S^1$ such that fibers of $\mathcal{G}$ are tangent to the kernel distribution of $\omega$.

**Sketch of the proof.** Choose a base point $p \in M$ and consider the indefinite integral

$$f(q) = \int_p^q \omega \in \mathbb{R}/\mathbb{Z}.$$

The function $f(q)$ is well defined and smooth, and local calculation shows that $f$ has maximal rank at each point. Thus $f$ is a fibration. \hfill \Box

Now suppose that $\omega$ is a closed rational nondegenerate 1-form on $M$ that is sufficiently close to $\theta$, namely we assume that the kernel distribution of $\omega$ is transversal to the vector field $L$. Since $\omega$ is rational, the kernel distribution $Q$ of $\omega$ is tangent to a fibration $G$ of $M$ by surfaces $G_t, t \in S^1$. The fibers of $\mathcal{G}$ determine a relative class $[\xi] \in H_2(M, \partial M)$.

**Lemma 2.8.** $\tau([\xi]) = \chi(G_t)$.

**Proof.** Since both $\mathcal{F}$ and $\mathcal{G}$ are transversal to the vector field $L$, we conclude that for each fiber $G = G_t$ of $\mathcal{G}$:

$$P|_G \cong T(M)|_G / \text{Span}(L) \cong Q|_G.$$

Thus the bundles $P|_G$ and $Q|_G$ are isomorphic and, moreover, the isomorphism $\lambda$ between them carries the section $X$ (of $P|_G$) to the tangent vector field $\lambda(X)$ on $G$ that is normal to $\partial G$. Therefore the obstruction $\chi(G)$ to the extension of $\lambda(X)$ to a nonzero field on $G$ is the same as the relative Euler number $\tau([\xi]) = e(P|_G, X)$.

\hfill \Box

### 2.3. Application to incompressible surfaces

We consider a compact irreducible atoroidal orientable manifold $M$ with (possibly empty) incompressible boundary of zero Euler characteristic. Suppose that $\xi \in H_2(M, \partial M; \mathbb{Z})$ is a relative homology class. Define **Thurston's norm** $x(\xi)$ as

$$x(\xi) = \|\xi\| := \min\{\|\chi(S)\| : (S, \partial S) \subset (M, \partial M) \text{ is an embedded surface representing the class } \xi\}.$$
2.3. Application to incompressible surfaces

Since the manifold $M$ is atoroidal, $x(\xi) \neq 0$ for each $\xi \neq 0$. There is a generalization of Thurston's norm that is defined by taking the minimum over immersed surfaces representing the given homology class. It turns out that two norms coincide; see [Gab83], [Per93].

**Lemma 2.9.** Every element $\xi \in H_2(M, \partial M; \mathbb{Z})$ is represented by an embedded oriented surface $S$. If $\xi/n \in H_2(M, \partial M; \mathbb{Z})$ for some $n > 0$, then any surface $S$ representing $\xi$ is the union of $n$ disjoint subsurfaces $S_j$ each representing $\xi/n$.

**Remark 2.10.** The surfaces $S_j$ could be disconnected.

**Proof.** Recall that (by duality) each element $\zeta \in H_2(M, \partial M; \mathbb{Z})$ determines an element $\zeta^* \in H^1(\pi_1(M), \mathbb{Z}) \cong H^1(\pi_1(M), \mathbb{Z})$; the latter is a homomorphism $\zeta^* : \pi_1(M) \to \mathbb{Z}$. The homomorphism $\zeta^*$ is induced by a PL map $f : M \to S^1$. If $c$ is a regular value of $f$, then the surface $F = f^{-1}(c)$ is embedded, has canonical orientation, and represents the class $\zeta$. This proves the first assertion of the lemma (cf. Lemma 1.23).

Suppose that $\zeta \in H_2(M, \partial M; \mathbb{Z})$ is represented by an embedded relative 2-cycle $F$. The image of the first homology group of $F$ in $H_1(M, \mathbb{Z})$ lies in the kernel of $\zeta^*$; hence the restriction of $f$ to $F$ is homotopic to a constant, and we can choose $f$ within its homotopy class to be constant on $F$. We can also assume that $f$ has regular value at $c = f(F)$. The inverse image $f^{-1}(c)$ can be larger then the surface $F$. However, since $f^{-1}(c)$ is homologous to $F$, we conclude that $F_0 = f^{-1}(c) - F$ is homologically trivial.

If two maps $f, g : M \to S^1$ correspond to the same homology class $\zeta$, then $f$ is homotopic to $g$ (since they induce the same homomorphism of $\pi_1(M)$).

Now let us use this to prove the second assertion of the lemma. Suppose that $\alpha = \xi/n \in H_2(M, \partial M; \mathbb{Z})$, $f_\alpha, f_{\bar{\xi}} : M \to S^1$ are the corresponding maps, and the subsurface $S \subset f_{\bar{\xi}}^{-1}(c)$ represents the homology class $\bar{\xi}$. Let $\lambda : S^1 \to S^1$ be the $n$-covering, the inverse image $\lambda^{-1}(c)$ is a set $\{b_1, \ldots, b_n\}$. Hence the function $\lambda \circ f_\alpha$ corresponds to the class $n\alpha = \xi$; this function is homotopic to $f_{\bar{\xi}}$. Thus $f_{\bar{\xi}}$ lifts to a function $\tilde{f}_{\bar{\xi}} : M \to S^1$ via the covering $\lambda$. Therefore the surface

$$S \cup S_0 = f_{\bar{\xi}}^{-1}(c) = \bigcup_{i=1}^n \tilde{f}_{\xi}^{-1}(b_i)$$

is a disjoint union of $n$ subsurfaces $S_1, \ldots, S_n$. Recall that the relative homology class $[S_0]$ is trivial. Since the function $\tilde{f}_{\bar{\xi}}$ is homotopic to $f_\alpha$, we conclude that each of these subsurfaces represents the homology class $\alpha$. \qed

From now on we shall assume that all components of surfaces representing nontrivial homology classes in $H_2(M, \partial M; \mathbb{Z})$ have negative Euler characteristic. (We can make this assumption since $\partial M$ is incompressible and $M$ is atoroidal and aspherical.)
Theorem 2.11. Suppose that $M$ is a 3-manifold as above. Then the function

$$x : H_2(M, \partial M; \mathbb{Z}) \rightarrow \mathbb{Z}$$

is a norm defined over $\mathbb{Z}$.

Proof. According to Lemma 2.9, we have

$$x(n\xi) \geq |n|x(\xi) \text{ for all } n \in \mathbb{Z}.$$ 

The opposite inequality is obvious. Thus $x(n\xi) = |n|x(\xi)$ for all $n \in \mathbb{Z}$, i.e., $x$ is linear on rays.

Lemma 2.12. If $\alpha, \beta \in H_2(M, \partial M; \mathbb{Z})$, then $x(\alpha + \beta) \leq x(\alpha) + x(\beta)$.

Proof. Represent the classes $\alpha, \beta$ by PL-embedded oriented surfaces $A, B$. We can assume that these surfaces intersect transversally: their intersection is a 1-dimensional submanifold $\Gamma$. By Theorem 1.23 we can assume that each surface $A, B$ is incompressible. Consider components of $A - \Gamma$ and $B - \Gamma$. If a component of, say, $A - \Gamma$ is a disk $D_A$ bounded by a loop $\gamma \subset A$, then $\gamma$ bounds a disk $D_B$ on $B$ and we can “trade” these disks: take $A := A - D_A \cup D_B, B := B - D_B \cup D_A$ (see Figure 2.1).

![Figure 2.1: Trading disks and pushing surfaces apart.](image)

By pushing the new surfaces apart near $\gamma$ we eliminate the intersection along $\gamma$. This procedure preserves the homology classes of $A$ and $B$ (since $M$ is aspherical and the sphere $D_A \cup D_B$ is contractible in $M$) and preserves the Euler characteristics of the surfaces. Thus we may assume that each component of $A - \Gamma$ and $B - \Gamma$ is different from the disk. The union of surfaces $A \cup B$ with the natural orientation of simplices on $A, B$ represents the class $\alpha + \beta$. Now we remove $\Gamma$ from both $A$ and $B$ and glue the components of the complement as follows. Suppose that $C_A \subset A - \Gamma$ is adjacent to a loop $\gamma \subset \Gamma$. There are exactly two components of $B - \Gamma \cap \text{Nbd}(\gamma)$ that are adjacent to $\gamma$. We glue $C_A$ to the one that induces an orientation on $\gamma$ opposite to the orientation induced by $C_A$ and do the same with the component $C_B \subset B - \Gamma$ (exactly the same way as we did with disks $D_A, D_B$).
above; see Figure 2.1). We repeat this procedure for all components of \( A \cup B - \Gamma \).

The resulting singular 2-cycle \( R(A \cup B) \) has nontransversal self-intersections along \( \Gamma \). Clearly \( R(A \cup B) \) has the same homology class as \( \alpha + \beta \). Thus we push the self-intersections apart, and the result is an embedded oriented surface \( A \oplus B \) that represents the homology class \( \alpha + \beta \).

Direct calculation shows that \( \chi(A) + \chi(B) = \chi(A \oplus B) \). Also, no component of \( A \oplus B \) is a sphere. This proves the lemma. \( \square \)

The above lemma and linearity of the function \( x \) on rays imply that \( x \) is convex. Finally, the norm \( x \) is nondegenerate since there are no elements of \( H_2(M, \partial M; \mathbb{Z}) - \{0\} \) that are represented by tori. \( \square \)

Theorem 2.2 implies that the norm \( x \) (defined over \( \mathbb{Z} \)) extends from \( H_2(M, \partial M; \mathbb{Z}) \) to \( H_2(M, \partial M; \mathbb{R}) \).

**Theorem 2.13.** Suppose that \( \dim(H_2(M, \partial M; \mathbb{R})) \geq 2 \). Let \( \zeta \in H_2(M, \partial M; \mathbb{Z}) \) be such that \( v = \zeta / x(\zeta) \) is a vertex of the polyhedron \( B^{(x)}(1) \). Then \( \zeta \) cannot be represented by a fiber in a fibration of \( M \) over \( S^1 \).

**Proof.** Suppose that \( \zeta \) is a class in \( H_2(M, \partial M; \mathbb{Z}) \) represented by a fibration of \( M \) over \( S^1 \). Recall that such a fibration determines a nonzero element \( \tau \in H_2(M, \partial M)^* \). We proved that there exists a positive number \( \epsilon \) such that the linear function \( \tau \) satisfies the following properties:

- The linear function \( \tau \) is integral, i.e., \( \tau : H_2(M, \partial M; \mathbb{Z}) \to \mathbb{Z} \).
- For all integer classes \( \xi \in H_2(M, \partial M; \mathbb{Z}) \) such that
  \[
  \|\xi / x(\xi) - v\| \leq \epsilon,
  \]
  we have \( x(\xi) = -\tau(\xi) \).

Rational classes are dense in the cone

\[
C_\varepsilon(v) = \{\alpha \in H_2(M, \partial M; \mathbb{R}) : \|\alpha / x(\alpha) - v\| \leq \epsilon\}.
\]

Thus we conclude that in this cone the integer linear function \( -\tau \) coincides with Thurston's norm \( x \). Therefore the unit vector \( v \) belongs to the interior of a top-dimensional face of the unit sphere \( \partial B^{(x)}(1) \), which locally (near \( v \)) is given by the equation \( \tau(\beta) = -1 \). This contradicts our assumptions. \( \square \)

Now we can prove the main theorem of this chapter (Theorem 2.1).

**Proof.** Take any vertex \( v \) of the polyhedron \( B^{(x)}(1) \). According to Corollary 2.6 there exists an integral class \( \zeta \in H_2(M, \partial M; \mathbb{Z}) \) such that \( v = \zeta / x(\zeta) \). On the other hand, by Theorem 2.13, the class \( \zeta \) is not represented by a fiber in any fibration of \( M \) over \( S^1 \). The class \( \zeta \) is nontrivial, and thus it can be represented by a superincompressible surface \( S \) according to Theorem 1.23. \( \square \)
Corollary 2.14. Suppose that $M$ is an orientable atoroidal 3-manifold that fibers over the circle with the fiber $\Sigma$. Assume that $M$ has at least two boundary components. Then $M$ contains an embedded superincompressible surface that is not a fiber in a fibration of $M$ over $S^1$ and that represents a nontrivial element of $H_2(M, \partial M; \mathbb{R})$.

Proof. It is clear that $\dim H_2(M, \partial M; \mathbb{R}) \geq 1$ since the relative homology class $[\Sigma]$ is nontrivial. By duality $H_2(M, \partial M; \mathbb{R}) \cong H_1(M, \mathbb{R})$. Let $T_1, T_2$ be distinct boundary tori of $M$. Suppose that $[\gamma] \in H_1(T_1, \mathbb{R}) - \{0\}$ is in the kernel of the homomorphism $H_1(T_1, \mathbb{R}) \to H_1(M, \mathbb{R})$. Then $\gamma = \partial \sigma$ where $[\sigma] \in H_2(M, T_1; \mathbb{R})$. Hence the image of $[\sigma]$ in $H_1(T_2, \mathbb{R})$ is zero. Therefore $[\sigma]$ and $[\Sigma]$ are linearly independent since $[\Sigma]$ has nontrivial image in $H_1(T_j, \mathbb{R})$ for each boundary torus $T_j$. Thus $\dim H_2(M, \partial M; \mathbb{R}) \geq 2$ in this case. The remaining case is that $H_1(T_1, \mathbb{R})$ injects into $H_1(M, \mathbb{R})$, and $\dim H_1(M, \mathbb{R}) \geq 2$. This again implies that $\dim H_2(M, \partial M; \mathbb{R}) \geq 2$. Thus the assertion of the corollary follows from Theorem 2.1.

Corollary 2.15. Suppose that $M$ is an orientable atoroidal 3-manifold that fibers over the circle with the fiber $\Sigma$. Assume that $M$ has at least one boundary component. Then $M$ admits a finite cover $M' \to M$ such that $M'$ contains an embedded superincompressible surface that is not a fiber in a fibration of $M$ over $S^1$ and that represents a nontrivial element of $H_2(M, \partial M; \mathbb{R})$.

Proof. According to the above corollary, it suffices to find a finite covering over $M$ that has at least two boundary components. Let $\Sigma' \to \Sigma$ be the characteristic covering corresponding to the kernel of the homomorphism $\pi_1(\Sigma) \to H_1(\Sigma, \mathbb{Z}_2)$. Then the surface $\Sigma'$ has at least two boundary components. The manifold $M$ is the mapping torus of a homeomorphism $\tau : \Sigma \to \Sigma$. The characteristic subgroup $\pi_1(\Sigma')$ is invariant under $\tau_*$. There is $n \geq 1$ such that $\tau^n$ lifts to a homeomorphism $\tau'$ of $\Sigma'$ that maps each boundary circle to itself. The mapping torus of $\tau'$ is the required finite covering $M'$ over $M$. 

\qed
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