Chapter 2
Group Theory and Symmetry

2.1 The Concept of Groups

A group is a collection or set of elements together with an inner binary operation, conventionally called multiplication, satisfying some special rules discussed below. It is not necessary to specify what the elements are in order to discuss the group that they constitute. Of particular interest in the context of this book are groups formed by permutations of sets of small numbers of objects, particularly the roots of algebraic equations; such groups are called permutation groups. Frequently the properties of such abstract permutation groups are most readily visualized by considering analogous or isomorphic groups formed by sets of symmetry operations on polyhedra or other readily visualized spatial objects. Groups formed by symmetry operations on three-dimensional objects are called symmetry point groups. Some features of both of these types of groups will be discussed in this chapter.

A set of elements combined with an binary operation (multiplication) to form a mathematical group must satisfy the following four conditions or rules:

1. The product of any two elements in the group and the square of each element must be an element of the group. A product of two group elements, \( AB \), is obtained by applying the binary operation (multiplication) to them and the square of a group element, \( A^2 \), is the product of an element with itself. This definition can be extended to higher powers of group elements. The multiplication of two group elements is said to be commutative if the order of multiplication is immaterial, i.e., if \( AB = BA \). In such a case \( A \) is said to commute with \( B \). The multiplication of two group elements is not necessarily commutative.

2. One element in the group must commute with all others and leave them unchanged. This element is conventionally called the identity

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element and often designated as $E$. This condition may be concisely stated as $EX =XE = X$.

(3) The associative law of multiplication must hold. This condition may be expressed concisely as $A(BC) = (AB)C$, i.e., the result must be the same if $C$ is multiplied by $B$ to give $BC$ followed by multiplication of $BC$ by $A$ to give $A(BC)$ or if $B$ is multiplied by $A$ to give $AB$ followed by multiplication of $AB$ by $C$ to give $(AB)C$.

(4) Every element must have an inverse, which is also an element in the group. The element $Z$ is the inverse of the element $A$ if $AZ = ZA = E$. The inverse of an element $A$ is frequently designated by $A^{-1}$. Note that multiplication of an element by its inverse is always commutative.

These defining characteristics of a group have been summarized concisely by defining a group as "...a mathematical system consisting of elements with inverses which can be combined by some operation without going outside the system."

The number of elements in a group is called the order of the group. This book will be concerned exclusively with finite groups, i.e., groups with finite numbers of elements. Within a given group it may be possible to select various smaller sets of elements, each set including the identity element $E$, which are themselves groups. Such smaller sets are called subgroups. A subgroup of a group $G$ is thus defined as a subset $H$ of the group $G$ which is itself a group under the same multiplication operation of $G$. The fact that $H$ is a subgroup of $G$ may be written $H \subset G$. The order of a subgroup must be an integral factor of the order of the group. Thus if $H$ is a subgroup of $G$ and $|H|$ and $|G|$ are the orders of $H$ and $G$, respectively, then the quotient $|G|/|H|$ must be an integer. This quotient is called the index of the subgroup $H$ in $G$.

Let $A$ and $X$ be two elements in a group. Then $X^{-1}AX = B$ will be equal to some element in the group. The element $B$ is called the similarity transform of $A$ by $X$ and $A$ and $B$ may be said to be conjugate. Conjugate elements have the following properties:

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(1) **Every element is conjugate with itself.** Thus for any particular element \( A \) there must be at least one element \( X \) such that \( A = X^{-1}AX \).

(2) **If \( A \) is conjugate with \( B \), then \( B \) is conjugate with \( A \).** Thus if \( A = X^{-1}BX \), then there must be some element, \( Y \), in the group such that \( B = Y^{-1}AY \).

(3) **If \( A \) is conjugate with \( B \) and \( C \), then \( B \) and \( C \) are conjugate with each other.**

A complete set of elements of a group which are conjugate to one another is called a *class* (or more specifically a *conjugacy class*) of the group. The number of elements in a conjugacy class is called its *order*; the orders of all conjugacy classes must be integral factors of the order of the group.

An important property of each element of a group is its *period*. In this context the period of an element is the minimum number of times that the element must be multiplied by itself before the identity element \( E \) is obtained, i.e., the smallest positive integer \( n \) such that \( A^n = E \). The period of the identity element \( E \) is, of course, 1. The period of an element is sometimes also called its *order* but this is confusing because the term "order" is also used to describe the number of elements in a group or conjugacy class (see above).

Certain elements \( g_1, g_2, \ldots, g_m \) of a given finite group \( G \) are called a set of *generators* if every element of \( G \) can be expressed as a finite product of their powers (including negative powers).\(^3\) A set of generators may be denoted by the symbol \( \{g_1, g_2, \ldots, g_m\} \). A set of relations satisfied by the generators of a group is called an *abstract definition* or *presentation* of the group if every relation satisfied by the generators is an algebraic consequence of these particular relations. A group with only one generator (i.e., \( m = 1 \)) is a *cyclic group*, \( \{g\} \cong C_n \), whose order \( n \) is the period of the single generator \( g \), i.e., \( g^n = E \), where \( E \) is the identity element. The cyclic group \( C_1 \) is the trivial group consisting solely of the identity element.

A group in which every element commutes with every other element is called a *commutative* group or an *Abelian* group after the famous Norwegian mathematician, Abel (1802–1829). In an Abelian group every element is in a

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conjugacy class by itself, i.e., all conjugacy classes are of order one. A normal subgroup $N$ of $G$, written $N \triangleleft G$, is a subgroup which consists only of entire conjugacy classes of $G$.\textsuperscript{4} A normal chain of a group $G$ is a sequence of normal subgroups $C_1 \triangleleft N_{a_1} \triangleleft N_{a_2} \triangleleft N_{a_3} \triangleleft \cdots \triangleleft N_{a_s} \triangleleft G$, in which $s$ is the number of normal subgroups (besides $C_1$ and $G$) in the normal chain. A simple group has no normal subgroups other than the identity group $C_1$. Simple groups are particularly important in the theory of finite groups.\textsuperscript{5,6} If a normal chain starts with the identity group $C_1$ and leads to $G$ and if all of the quotient groups $N_{a_1}/C_1 = C_{a_1}, N_{a_2}/N_{a_1} = C_{a_2}, \ldots, G/N_{a_s} = C_{a_{s+1}}$ are cyclic, then $G$ is a composite or soluble group.

2.2 Symmetry Groups

Many of the properties of groups are most readily visualized by studying the groups consisting of symmetry operations of polyhedra or other concrete three-dimensional objects. In this context a symmetry operation is a movement of an object such that, after completion of the movement, every point of the body coincides with an equivalent point or the same point of the object in its original orientation. The position and orientation of an object before and after carrying out a symmetry operation are indistinguishable. Thus a symmetry operation takes an object into an equivalent configuration.

The symmetry operations for objects in ordinary three-dimensional space can be classified into four fundamental types each of which is defined by a symmetry element around which the symmetry operation takes place. The four fundamental types of symmetry operations and their corresponding symmetry elements are listed in Table 2–1.

The identity operation, designated as $E$, leaves the object unchanged. Although this operation may seem trivial, it is mathematically necessary in order

to convey the mathematical properties of a group on the set of all of the
symmetry operations applicable to a given object. The reflection operation,
designated as $\sigma$, involves reflection of the object through a plane, known as a
reflection plane. For example, in a reflection through the $xy$-plane
(conveniently designated as $\sigma_{xy}$) the coordinates of a point $(x, y, z)$ change to
$(x, y, -z)$—a reflection operation thus can result in the change of only a single
coordinate. A rotation operation, designated as $C_n$, consists of a $360^\circ/n$ rotation
around a line, known as a rotation axis. For example, a $C_2$ rotation around the
$z$-axis changes the coordinates of a point $(x, y, z)$ to $(-x, -y, z)$—a rotation
operation thus can result in a change of only two coordinates. An improper
rotation, designated as $S_n$, consists of a $360^\circ/n$ rotation around a line followed
by a reflection in a plane perpendicular to the rotation axis. An $S_2$ operation is
called an inversion and is designated by $i$; the intersection of the $C_2$-axis and the
perpendicular reflection plane is called an inversion center. Inversion through
the origin changes the coordinates of a point $(x, y, z)$ to $(-x, -y, -z)$—thus an
$S_n$ operation must change the signs of all three coordinates. An $S_1$ improper
rotation in which the $C_1$ proper rotation component is equivalent to the identity
$E$ corresponds to a reflection operation $\sigma$. Thus the reflection operation $\sigma$ is a
special type of improper rotation, namely $S_1$.

**Table 2-1: The Four Fundamental Types of Symmetry Operations**

<table>
<thead>
<tr>
<th>Symmetry Operation</th>
<th>Designation</th>
<th>Corresponding Symmetry Element</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identity (no change)</td>
<td>$E$</td>
<td>The entire object</td>
<td>3</td>
</tr>
<tr>
<td>Reflection</td>
<td>$\sigma$</td>
<td>Reflection plane</td>
<td>2</td>
</tr>
<tr>
<td>Rotation</td>
<td>$C_n$</td>
<td>Rotation axis</td>
<td>1</td>
</tr>
<tr>
<td>Improper rotation</td>
<td>$S_n$</td>
<td>Improper rotation axis*</td>
<td>0</td>
</tr>
</tbody>
</table>

*Point of intersection of a proper rotation axis and a perpendicular reflection plane
Consider the set of symmetry operations in ordinary three-dimensional space describing the symmetry of an actual object. Such a set of symmetry operations satisfies the properties of a group in the mathematical sense and is therefore called a symmetry point group. In most cases such a symmetry point group contains a finite number of operations and is therefore a finite group.

There is a systematic way to classify objects in three-dimensional space by their symmetry point groups based on the following sequence of questions:

1. **Is the object linear** (i.e., only "one-dimensional")?

   Linear objects are the only objects having infinite rather than finite symmetry point groups since the linear axis corresponds to an infinite order rotation axis, namely $C_\infty$. If there is a reflection plane perpendicular to the infinite order rotation axis (dividing the object into two equivalent halves), then the symmetry point group is $D_\infty$, if not the symmetry point group is $C_\infty$.

2. **Does the object have multiple "higher-order" rotation axes** (i.e., $C_{>2}$ axes)?

   Multiple axes in this question refer to noncollinear $C_{>2}$ axes. Such nonlinear objects have the symmetries of well-known regular polyhedra and are generally readily recognizable by containing the regular polyhedra in some manner. Such groups are sometimes called the polyhedral point groups. Thus the tetrahedral groups $T$, $T_h$, and $T_d$ with 12, 24, and 24 operations, respectively, have four noncollinear $C_3$ axes and are distinguished by the presence or absence of horizontal ($\sigma_h$) or diagonal ($\sigma_d$) reflection planes. The octahedral groups $O$ and $O_h$ with 24 and 48 operations, respectively, have not only four noncollinear $C_3$ axes but also three noncollinear $C_4$ axes and are distinguished by the presence or absence of reflection planes. Figure 2–1 shows the multiple rotation axes of the regular octahedron and the cube, both of which have the octahedral point group $O_h$. The icosahedral groups $I$ and $I_h$ with 60 and 120 operations, respectively, have ten noncollinear $C_3$ axes and six noncollinear $C_5$ axes and are also distinguished by the presence or absence of reflection planes.

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3. If the object does not belong to either a linear group or a polyhedral group, then does it have proper or improper axes of rotation (i.e. $C_n$ or $S_n$)?

If no axes of either type are found, the group is either $C_s$, $C_i$, or $C_1$ with 2, 2, and 1 operations, respectively, depending whether the object has a plane of symmetry ($\sigma$), and inversion center ($i$), or neither. The $C_1$ designation corresponds to an object with no symmetry at all and therefore to a symmetry point group containing only one element, namely the identity element $E$.

![Diagram of axes in octahedron and cube](image)

**Figure 2-1:** The multiple rotation axes in the octahedron and cube, both of which have $O_h$ symmetry. In the octahedron the $C_4$ axes pass through opposite pairs of vertices, the $C_3$ axes pass through the midpoints (•) of opposite pairs of faces, and the $C_2$ axes pass through opposite pairs of edges. In the cube, the $C_3$ axes pass through opposite pairs of vertices, the $C_4$ axes pass through midpoints (•) of opposite pairs of faces, and the $C_2$ axes pass through opposite pairs of edges.

(4) Does the object have an *even*-order improper rotation axis $S_{2n}$ but no planes of symmetry or any proper rotation axis other than one collinear with the improper rotation axis?
The presence of an even-order improper rotation axis \( S_{2n} \) without any noncollinear proper rotation axes or any reflection planes indicates the symmetry point group \( S_{2n} \) with \( 2n \) operations.

(5) If the object does not belong to the linear point groups, the polyhedral point groups, or the point groups \( C_s, C_n, C_1, \) or \( S_{2n} \), then look for the highest order rotation axis. Call the highest order rotation axis \( C_n \).

(6) Are there \( n \) \( C_2 \) axes lying in a plane perpendicular to the \( C_n \) axis? If there are \( n \) \( C_2 \) axes lying in a plane perpendicular to the \( C_n \) axis then the object belongs to one of the symmetry point groups \( D_n, D_{nh} \), or \( D_{nd} \) with \( 2n \), \( 4n \), and \( 4n \) operations, respectively, depending on whether there are no planes of symmetry, a horizontal plane of symmetry \( (\sigma_h) \), or \( n \) vertical planes of symmetry \( (\sigma_v) \), respectively. If there are no \( C_2 \) axes lying in a plane perpendicular to the \( C_n \) axis, then the object belongs to one of the symmetry point groups \( C_n, C_{nv}, C_{nh} \), with \( n, 2n, \) and \( 2n \) operations, respectively, depending upon whether there are no planes of symmetry, \( n \) vertical planes of symmetry \( (\sigma_v) \), or a horizontal plane of symmetry \( (\sigma_h) \), respectively. If there are only \( C_2 \) axes, then a unique \( C_2 \) axis is chosen as the "reference axis" if there is any ambiguity as to which \( C_2 \) axis to choose.

2.3 Regular Polyhedra

The symmetries of regular polyhedra are important in the theory of the solution of algebraic equations. The symmetries of the five regular Platonic solids, namely the tetrahedron, octahedron, cube, icosahedron, and dodecahedron (Figures 2-1 and 2-2) were already recognized by the ancient Greeks. The fact that there are only five regular geometric solids in contrast to an infinite number of polygons must have been a major revelation to the ancient thinkers. The reason for only five regular geometric solids can be readily seen by the following arguments.\(^8\),\(^9\)

In order to have the solid angles required for forming closed polyhedra, a minimum of three edges must meet at a vertex. In this connection the number of edges meeting at a vertex is called the degree of the vertex. In the case of completely regular polyhedra the configuration of edges meeting at each vertex must be identical. If three equal-length edges meet at every vertex and if they join each other in equilateral triangular faces, the result is a tetrahedron. If instead, four or five equilateral triangles meet at each vertex, the result is an octahedron or an icosahedron, respectively. All of these regular polyhedra with equilateral triangular faces may be considered to have their vertices on finite surfaces with positive curvature such as the sphere. If six equilateral triangles meet at a vertex, an infinite flat two-dimensional lattice is obtained with a familiar hexagon structure with its vertices on an infinite surface with zero curvature. Such planar structures are called tessellations (Figure 2–3). More than six equilateral triangles meet at a vertex only with puckering and thus cannot give a regular solid. The vertices of such configurations can only be placed on infinite surfaces with negative curvature. Thus there are only three regular polyhedra with (equilateral) triangular faces, namely the tetrahedron,

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Figure 2–2: The regular tetrahedron, icosahedron, and dodecahedron. The regular octahedron and cube are shown in Figure 2–1.

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octahedron, and icosahedron. Such polyhedra with triangular faces are called *deltahedra* since their faces are shaped like the Greek letter *delta*, \( \Delta \).

![Hexagonal tessellation of triangles](image1.png)

**Hexagonal tessellation of triangles**

![Tessellation of squares ("checkerboard plane")](image2.png)

**Tessellation of squares ("checkerboard plane")**

![Tessellation of hexagons](image3.png)

**Tessellation of hexagons**

**Figure 2–3:** Examples of tessellations on a flat plane.

Additional regular solids can be formed from square or regular pentagonal faces. Vertices where three squares meet generate a cube whereas vertices where four squares meet generate a checkerboard plane (Figure 2–3). Vertices where three regular pentagons meet generate a regular dodecahedron; regular pentagons cannot form a tessellation. Regular hexagons with 120° angles can only form a tessellation and regular polygons with more than six sides have angles larger than 120° and thus cannot meet with two other equivalent polygons to form a solid angle at all.

The properties of the regular polyhedra are summarized in Table 2–2.
The regular polyhedra provide examples of pairs of dual polyhedra. In this connection a given polyhedron \( P \) can be converted into its dual \( P^* \) by locating the centers of the faces of \( P^* \) at the vertices of \( P \) and the vertices of \( P^* \) above the centers of the faces of \( P \). Two vertices in the dual \( P^* \) are connected by an edge when the corresponding faces in \( P \) share an edge. The process of dualization has the following properties:

1. The numbers of vertices and edges in a pair of dual polyhedra \( P \) and \( P^* \) satisfy the relationships \( v^* = f, e^* = e, f^* = v \).
2. Dual polyhedra have the same symmetry elements and thus belong to the same symmetry point group.
3. Dualization of the dual of a polyhedron leads to the original polyhedron.
4. The degree of a vertex in a polyhedron corresponds to the number of edges in the corresponding face in its dual.

Examples of pairs of dual polyhedra among the regular polyhedra are the octahedron/cube dual pair with \( O_h \) symmetry (Figure 2–4) and the icosahedron/dodecahedron dual pair with \( I_h \) symmetry. The tetrahedron with \( T_d \) symmetry is self-dual (i.e., dual to itself). For example, a degree 3 vertex of a polyhedron corresponds to a triangular face in its dual.

**Table 2–2: Properties of the Regular Polyhedra**

<table>
<thead>
<tr>
<th>Polyhedron</th>
<th>Face Type</th>
<th>Vertex Degrees</th>
<th>Number of Edges</th>
<th>Number of Faces</th>
<th>Number of Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>Triangle</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Octahedron</td>
<td>Triangle</td>
<td>4</td>
<td>12</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>Cube</td>
<td>Square</td>
<td>3</td>
<td>12</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td>Pentagon</td>
<td>3</td>
<td>30</td>
<td>12</td>
<td>20</td>
</tr>
<tr>
<td>Icosahedron</td>
<td>Triangle</td>
<td>5</td>
<td>30</td>
<td>20</td>
<td>12</td>
</tr>
</tbody>
</table>

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The regular icosahedron and its dual, the regular dodecahedron, have the $I_h$ symmetry point group with 120 elements. Its pure rotation subgroup, $I$, with 60 elements is isomorphic with the alternating group $A_5$. The symmetry point group of the regular tetrahedron, $T_d$, is a subgroup of $I_h$ of index 5 but not a normal subgroup. Nevertheless, the ability to partition an object of icosahedral symmetry into five equivalent objects of tetrahedral or octahedral symmetry ($O_h = T_d \times C_2$) is an essential part of the Kiepert algorithm for solution of the general quintic equation (Chapter 6).

There are several ways of visualizing the partition of an object of icosahedral symmetry into five equivalent objects of at least tetrahedral symmetry thereby separating the effect of the fivefold axis of the icosahedron from that of its twofold and threefold symmetry elements. The 20 vertices of the dual of the icosahedron, namely the regular dodecahedron, can be partitioned into five sets of four vertices, each corresponding to a regular tetrahedron. Using the same idea Klein\textsuperscript{11} partitions the 30 edges of an icosahedron into five sets of six edges each by the following method (Figure 2–5):

(1) A straight line is drawn from the midpoint of each edge through the center of the icosahedron to the midpoint of the opposite edge;

(2) The resulting 15 straight lines are divided into five sets of three mutually perpendicular straight lines.

Each of these five sets of three mutually perpendicular straight lines resembles a set of Cartesian coordinates and defines a regular octahedron. The construction is the dual of a construction depicted by Du Val\textsuperscript{12} in color in which a regular dodecahedron is partitioned into five equivalent cubes. Alternatively, the 30 edge midpoints of an icosahedron can be used to form an icosidodecahedron having 30 vertices, 60 edges, 32 faces, and retaining icosahedral symmetry. The 30 vertices of this icosidodecahedron can then be partitioned into five sets of six vertices, each corresponding to a regular octahedron (Figure 2–6).

\textbf{Figure 2–5:} A regular icosahedron with its 30 edges partitioned into five sets of six edges each so that the midpoints of the edges in each set form a regular octahedron. The five sets of six edges each are indicated by the following types of lines: ________________

\textbf{Figure 2–6:} The icosidodecahedron formed from the 30 edge-midpoints of a regular icosahedron and the partitioning of its 30 vertices into five sets of six vertices, each corresponding to regular octahedra. The six vertices in one of these sets are indicated by spades (♠).

2.4 Permutation Groups

The previous sections apply the concepts of group theory to symmetry point groups such as those describing the symmetry of readily visualized polyhedra. The concepts of group theory, of course, can be applied to more abstract sets such as the permutations of a set $X$ of $n$ objects, which, for example, may be the $n$ roots of an algebraic equation of degree $n$. A set of permutations of $n$ objects (including the identity permutation) with the structure of a group is called a permutation group of degree $n$.\footnote{N. L. Biggs, \textit{Finite Groups of Automorphisms}, Cambridge University Press, London, 1971.} Let $G$ be a permutation group acting on the set $X$. Let $g$ be any operation in $G$ and $x$ be any object in set $X$. The subset of $X$ obtained by the action of all operations in $G$ on $x$ is called the orbit of $x$. The operations in $G$ leaving $x$ fixed is called the stabilizer of $x$; it is a subgroup of $G$ and may be abbreviated as $G_x$. A transitive permutation group has only one orbit containing all objects of the set $X$. Sites permuted by a transitive permutation group are thus equivalent. Transitive permutation groups represent permutation groups of the highest symmetry and thus play a special role in permutation group theory.

The maximum number of distinct permutations of $n$ objects is $n!$. The corresponding group is called the symmetric group of degree $n$ and is traditionally designated as $S_n$ (not to be confused with the designation $S_n$ for an improper rotation of order $n$ in Section 2.2). The symmetric group $S_n$ is obviously the highest symmetry permutation group of degree $n$. All permutation groups of degree $n$ must be a subgroup of the corresponding symmetric group $S_n$.

Let us now consider the structure of permutation groups. In this connection a permutation $P_n$ of $n$ objects can be described by a $2 \times n$ matrix of the following general type where the top row represents site labels and the bottom row represents object labels:

$$P_n = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ p_1 & p_2 & p_3 & \ldots & p_n \end{pmatrix}$$  \hfill (2.4–1)
The numbers \( p_1, p_2, p_3, \ldots, p_n \) can be taken to run through the integers 1, 2, 3, \ldots, \( n \) in some sequence. For a given \( n \) there are \( n! \) possible different \( P_n \) matrices. The matrix \( P_n^0 \) in which the bottom row \( p_1, p_2, p_3, \ldots, p_n \) has the integers in the natural order 1, 2, 3, \ldots, \( n \) (i.e., the bottom row of \( P_n^0 \) is identical to the top row) can be taken to represent a reference configuration corresponding to the identity element in the corresponding permutation group.

Permutations can be classified as odd or even permutations based on how many pairs of numbers in the bottom row of the matrix \( P_n \) are out of their natural order. Alternatively, if the interchange of a single pair of numbers is called a transposition, the parity of a permutation corresponds to the parity of the number of transpositions. Thus a permutation which is obtained by an odd number of transpositions from the reference configuration is called an odd permutation and a permutation which is obtained by an even number of transpositions from the reference configuration is called an even permutation. The identity permutation corresponding to the reference configuration has zero transpositions and is therefore an even permutation by this definition.

A group can be defined relating the \( P_n \) matrices for a given \( n \). First redefine the rows of \( P_n \) so that the top row represents the reference configuration \( P_n^0 \) and the bottom row represents the object labels in any of the \( n! \) possible permutations of the \( n \) objects. These permutations form a group of order \( n! \) with the permutation leaving the reference configuration unchanged (i.e., that represented by \( P_n^0 \) as so redefined) corresponding to the identity element, \( E \). This permutation group is the symmetric group, \( S_n \), of order \( n! \) as defined above.

Now consider the nature of the operations in a symmetric permutation group \( S_n \). These operations are permutations of labels which can be written as a product of cycles which operate on mutually exclusive sets of labels, e.g.

\[
\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 1 & 3 & 6 \end{pmatrix} = (1 \ 2 \ 4)(3 \ 5)(6)
\]  
(2.4–2)

The cycle structure of a given permutation in the symmetric group \( S_n \) can be represented by a sequence of indexed variables, i.e., \( x_1 x_2 x_3 \) for the
permutation in equation 2.4–2. A characteristic feature of the symmetric permutation group $S_n$ for all $n$ is that all permutations having the same cycle structure come from the same conjugacy class. Furthermore, no two permutations with different cycle structures can belong to the same conjugacy class. Therefore, for the symmetric permutation group $S_n$ (but not necessarily for any of its subgroups) the cycle structures of permutations are sufficient to define their conjugacy classes. Furthermore, the number of conjugacy classes of the symmetric group $S_n$ corresponds to the number of different partitions of $n$ where a partition of $n$ is defined as a set of positive integers $i_1,i_2,\ldots,i_k$ whose sum is $n$ (equation 2.4–3).

$$\sum_{j=1}^{k} i_j = n$$  \hspace{1cm} (2.4–3)

An alternative presentation of conjugacy class information for the symmetric groups $S_n$ is given by their cycle indices. A cycle index $Z(S_n)$ for a symmetric permutation group $S_n$ is a polynomial of the following form:

$$Z(S_n) = \sum_{i=1}^{c} a_i x_1^{c_1} x_2^{c_2} \ldots x_n^{c_n}$$  \hspace{1cm} (2.4–4)

In equation 2.4–4 $c =$ number of conjugacy classes (i.e., partitions of $n$ by equation 2.4–3), $a_i =$ number of operations of $S_n$ in conjugacy class $i$, $x_j =$ dummy variable referring to cycles of length $j$, and $c_{ij} =$ exponent indicating the number of cycles of length $j$ in class $i$. These parameters in the cycle indices of the symmetric groups $S_n$ must satisfy the following relationships:

(1) Each of the $n!$ permutations of $S_n$ must be in some class, i.e.,

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Beyond the Quartic Equation

\[ \sum_{i=1}^{i=c} a_i = n! \quad (2.4-5) \]

(2) Each of the \( n \) objects being permuted must be in some cycle of each permutation in \( S_n \) (counting, of course, fixed points of cycles of length 1 represented by \( x_1^{c1} \)), i.e.,

\[ \sum_{j=1}^{j=n} j c_{ij} = n \text{ for } 1 \leq i \leq c \quad (2.4-6) \]

The relationship between symmetry point groups and permutation groups as well as some important concepts can be illustrated by the symmetry point groups of the tetrahedron and trigonal bipyramid and the corresponding permutation groups on their vertices (Figure 2.7). The permutation group of the tetrahedron vertices is a transitive group since all four vertices are equivalent. However, the permutation group of the trigonal bipyramid vertices is an intransitive group partitioning its five vertices into two orbits, namely the two axial vertices (A in Figure 2.7) and the three equatorial vertices (E in Figure 2.7).

\[ \text{Tetrahedron} \]

\[ \text{Trigonal Bipyramid} \]

Figure 2.7: The tetrahedron and the trigonal bipyramid showing the proper and improper rotation axes (Note that \( S_4 \) refers to an improper rotation axis of order 4 rather than the symmetric permutation group on four objects). The axial (A) and equatorial vertices (E) of the trigonal bipyramid are also shown.
Let us now consider some of the properties of the symmetric permutation groups, \( S_n \). Any symmetric permutation group \( S_n \) has a normal subgroup of index 2 (and thus of order \( n! / 2 \)) consisting of only the permutations of even parity (which necessarily includes the identity permutation). This special subgroup of \( S_n \) is called the \textit{alternating group} and is designated as \( A_n \). The symmetric permutation group \( S_4 \) is isomorphic with the full tetrahedron symmetry point group of order \( 4! = 24 \), \( T_d \), (Figure 2.7) whereas the alternating permutation group \( A_4 \) is isomorphic with the tetrahedral proper rotation subgroup \( T \) of order \( 4! / 2 = 12 \). The corresponding chain of normal subgroups for \( S_4 \) is \( C_1 \triangleleft C_2 \triangleleft D_2 \triangleleft A_4 \equiv T \triangleleft S_4 \equiv T_d \) with the quotient groups \( C_2 / C_1 = C_2 \); \( D_2 / C_2 = C_2 \); \( A_4 / D_2 = C_3 \); \( S_4 / A_4 = C_2 \).

The problem of solving the general quintic equation relates to the fact that the alternating group \( A_5 \) is a simple group. This can be proved by showing that \( A_5 \) has no normal subgroups and that \( A_5 \) is the only normal subgroup of \( S_5 \). The proof of this fact can be related to the symmetries of the icosahedron (e.g., Figure 2.5). Thus the 60 permutations of \( A_5 \) can be partitioned into the following classes:

1. The identity;
2. 24 cycles of period 5 such as \((12345)\), each of these corresponding to a rotation through \( 2\pi k / 5 \) \((k = 1, 2, 3, 4)\) around a diameter passing through opposite vertices of the icosahedron;
3. 20 cycles of period 3 such as \((123)\), each of these corresponding to a rotation through \( \pm 2\pi / 3 \) around a diameter passing through midpoints of opposite faces of the icosahedron;
4. 15 double transpositions of period 2 such as \((12)(34)\) corresponding to half turns around a diameter passing through midpoints of opposite edges of the icosahedron.

A normal subgroup of a group \( G \) consists of \textit{entire} conjugacy classes of \( G \). For this reason the order of a \textit{normal} subgroup of \( A_5 \) must be of the form \( 1 + 24n_5 + 20n_3 + 15n_2 \) where \( n_5 \), \( n_3 \), and \( n_2 \) are each 0 or 1 (but not all 0 nor all 1). But this number must be a factor of 60 and thus must be 30, 20, or 15, which is clearly impossible. Thus the alternating group \( A_5 \) cannot have any normal subgroups.
A similar argument can be used to show that $A_5$ is the only normal subgroup of $S_5$. Furthermore all $A_n$ ($n \geq 6$) can also be shown to be simple groups.

The transitive groups of low degrees are particularly significant in the theory of permutation groups and the theory of solution of algebraic equations. The transitive groups of degrees up to eleven have been tabulated and their properties are given in detail.\(^{18}\) All of the transitive permutation groups of degree $\leq 7$ are listed in Table 2–3.

The following points about Table 2–3 are of interest:

1. In the groups $C_n$ the order is the same as the number of classes; therefore these groups are Abelian.
2. The groups $C_3, C_4, A_4 = T, C_5, A_5 = I$, and $A_6$ contain only even operations.
3. The groups $M_5$ and $M_7$ are metacyclic groups of degrees 5 and 7, respectively. For prime $n$, metacyclic groups of order $n(n + 1)$ are the largest permutation groups of degree $n$ which are soluble groups. A metacyclic group is defined in terms of two generators $s$ and $t$ and the relationships $s^p = t^{p-1} = E$ and $t^{-1}st = s^r$ where $p$ is a prime (5 in the case of $M_5$) and $r$ is a primitive root (mod $p$) which is 2 when $p = 5$.\(^\text{3}\)
4. The dihedral group $D_3$ can be a transitive permutation group of either degree 3 or 6. Similarly, the tetrahedral rotation group $T$ of order 24 can be a transitive permutation group of either degree 4 (e.g., the faces of a tetrahedron) where it is the alternating group $A_4$ or degree 6 (e.g., the edges of a tetrahedron). In addition, the icosahedral rotation group $I$ of order 60 can be a transitive permutation group of either degree 5 where it is the alternating group $A_5$ (see Section 2.3 and Figures 2–5 and 2–6 for more details) or degree 6 permuting the six diameters of the icosahedron connecting pairs of antipodal vertices.

Table 2–3: Transitive Permutation Groups of Degrees Less than Eight

<table>
<thead>
<tr>
<th>Group</th>
<th>Degree</th>
<th>Order</th>
<th>Number of Classes</th>
<th>Special Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>C₃</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>Even, Abelian</td>
</tr>
<tr>
<td>D₃ ≅ S₃</td>
<td>3</td>
<td>6</td>
<td>3</td>
<td>Dihedral, Symmetric</td>
</tr>
<tr>
<td>C₄</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>Abelian</td>
</tr>
<tr>
<td>D₂</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>Even, Abelian</td>
</tr>
<tr>
<td>D₂ᵈ</td>
<td>4</td>
<td>8</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>A₄ ≅ T</td>
<td>4</td>
<td>12</td>
<td>4</td>
<td>Even</td>
</tr>
<tr>
<td>S₄ ≅ T₉</td>
<td>4</td>
<td>24</td>
<td>5</td>
<td>Symmetric</td>
</tr>
<tr>
<td>C₅</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>Even, Abelian</td>
</tr>
<tr>
<td>D₅</td>
<td>5</td>
<td>10</td>
<td>4</td>
<td>Dihedral</td>
</tr>
<tr>
<td>M₅</td>
<td>5</td>
<td>20</td>
<td>5</td>
<td>Metacyclic</td>
</tr>
<tr>
<td>A₅ ≅ I</td>
<td>5</td>
<td>60</td>
<td>5</td>
<td>Even, Simple</td>
</tr>
<tr>
<td>S₅</td>
<td>5</td>
<td>120</td>
<td>7</td>
<td>Symmetric, Not I₉!</td>
</tr>
<tr>
<td>C₆</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>Abelian</td>
</tr>
<tr>
<td>D₃</td>
<td>6</td>
<td>6</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>D₆</td>
<td>6</td>
<td>12</td>
<td>3</td>
<td>Dihedral</td>
</tr>
<tr>
<td>A₄ ≅ T</td>
<td>6</td>
<td>12</td>
<td>4</td>
<td>Even</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>18</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>24</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>24</td>
<td>5</td>
<td>Even</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>24</td>
<td>5</td>
<td>Isomorphic to T₉</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>36</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>36</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>48</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>L(2,5) ≅ I</td>
<td>6</td>
<td>60</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>72</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>120</td>
<td>7</td>
<td>Isomorphic to S₅</td>
</tr>
<tr>
<td>A₆</td>
<td>6</td>
<td>360</td>
<td>7</td>
<td>Even, Simple</td>
</tr>
<tr>
<td>S₆</td>
<td>6</td>
<td>720</td>
<td>11</td>
<td>Symmetric</td>
</tr>
<tr>
<td>C₇</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>Even, Abelian</td>
</tr>
<tr>
<td>D₇</td>
<td>7</td>
<td>14</td>
<td>5</td>
<td>Dihedral</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>21</td>
<td>5</td>
<td>Even</td>
</tr>
<tr>
<td>M₇</td>
<td>7</td>
<td>42</td>
<td>7</td>
<td>Metacyclic</td>
</tr>
<tr>
<td>L(3,2)</td>
<td>7</td>
<td>168</td>
<td>6</td>
<td>Even, Simple</td>
</tr>
<tr>
<td>A₇</td>
<td>7</td>
<td>2520</td>
<td>9</td>
<td>Even, Simple</td>
</tr>
<tr>
<td>S₇</td>
<td>7</td>
<td>5040</td>
<td>15</td>
<td>Symmetric</td>
</tr>
</tbody>
</table>
The usual type of convex polyhedra, including the regular polyhedra discussed in Section 2.3, can be represented as points on the surface of a sphere. The same sphere, taken as a special unit sphere called the *Riemann sphere*, can also be used to represent the complex numbers \( z = a + bi \) thereby providing a method for generating special polynomials associated with a given polyhedron. First represent the complex numbers \( z = a + bi \) by points on the \((x,y)\) plane with the \(x\)-coordinate corresponding to \(a\) and the \(y\)-coordinate corresponding to \(b\); such a plane is called an *Argand plane* (Figure 2–8a). The Argand plane can correspond to the equatorial plane of the *Riemann sphere*, (Figure 2–8b) in which the north pole is \(\infty\) and the south pole is 0; the \(0-\infty\) axis can be called the *polar axis*.\(^{19}\) The equation of the Riemann sphere is taken to be \(p^2 + q^2 + r^2 = 1\).

\[ \text{Figure 2–8: (a) The Argand plane indicating the point for } z = a + bi; \text{ (b) The Riemann sphere.} \]

The polyhedral polynomials are polynomials whose roots correspond to the locations of the polyhedral vertices, the midpoints of the polyhedral edges, or the midpoints of the polyhedral faces on the surface of the Riemann sphere.

\(^{19}\)F. Klein, *Vorlesungen über das Ikosaeder*, Teubner, Leipzig, 1884, Part I, Chapter II.
Polyhedral polynomials are frequently expressed in terms of *homogeneous variables* so that all terms are the same combined degree in two variables, i.e.,

\[
\sum_{i=1}^{n} a_{n-i}x^i = \sum_{i=1}^{n} a_{n-i}u^iv^{n-i}
\]  \hspace{1cm} (2.5–1)

The use of homogeneous variables corresponds to the substitution \(x = u/v\). The special symmetry of the regular polyhedra corresponds to the vanishing of some special functions of these polyhedral polynomials known as *transvectants* (Section 2.6).

Now consider the projection of points on the Riemann sphere onto its equatorial plane, taken as an Argand plane as noted above (Figure 2–8). A complex number \(z = a + bi\) gives

\[
a = \frac{p}{1 - r}, \quad b = \frac{q}{1 - r}, \quad a + bi = \frac{p + iq}{1 - r}
\]  \hspace{1cm} (2.5–2)

Solving for \(p\), \(q\), and \(r\) gives

\[
p = \frac{2a}{1 + a^2 + b^2}, \quad q = \frac{2b}{1 + a^2 + b^2}, \quad r = -\frac{1 + a^2 + b^2}{1 + a^2 + b^2}
\]  \hspace{1cm} (2.5–3)

Every rotation of the Riemann sphere around its center corresponds to a linear substitution

\[
z' = \frac{\alpha z + \beta}{\gamma z + \delta}
\]  \hspace{1cm} (2.5–4)

For a rotation of the sphere by an angle \(\theta\) where \(p, q, r\) and \(-p, -q, -r\) remain constant,

\[
z' = \frac{(v + iu)z - (t - is)}{(t + is)z + (v - iu)}
\]  \hspace{1cm} (2.5–5)

where \(s = p \sin(\theta/2),\ t = q \sin(\theta/2),\ u = r \sin(\theta/2),\ and\ v = \cos(\theta/2)\) so that

\[
s^2 + t^2 + u^2 + v^2 = 1
\]  \hspace{1cm} (2.5–6)
For a rotation about the polar axis this reduces to

\[ z' = e^{i\theta} z \]  \hspace{1cm} (2.5–7)

Now consider the vertices of a regular octahedron and a regular icosahedron as points on the surface of such a Riemann sphere oriented such that the north pole \( (z = \infty) \) is one of the vertices in each case. This leads to the following homogeneous polynomials for these regular polyhedra in these orientations where \( z \) is taken to be \( u/v \):

(a) Octahedron \((O_h \text{ symmetry})\):
- Vertices: \( \tau = uv(u^4 - v^4) \) \hspace{1cm} (2.5–8a)
- Edges: \( \chi = u^{12} - 33u^8v^4 - 33u^4v^8 + v^{12} \) \hspace{1cm} (2.5–8b)
- Faces: \( W = u^8 + 14u^4v^4 + v^8 \) \hspace{1cm} (2.5–8c)

(b) Icosahedron \((I_h \text{ symmetry})\):
- Vertices: \( f = uv(u^{10} + 11u^5v^5 - v^{10}) \) \hspace{1cm} (2.5–9a)
- Edges: \( T = u^{30} + 522u^{25}v^5 - 10,005 u^{20}v^{10} - 10,005 u^{15}v^{20} - 522 u^5v^{25} + v^{30} \) \hspace{1cm} (2.5–9b)
- Faces: \( H = -u^{20} + 228 u^5v^{15} - 494 u^{10}v^{10} - 228 u^5v^{15} - v^{20} \) \hspace{1cm} (2.5–9c)

The roots of these polyhedral polynomials correspond to the locations of the vertices, edge midpoints, and face midpoints on the Riemann sphere. Their degrees are equal to the numbers of corresponding elements (vertices, edges, or faces).

The following features are of interest concerning the polyhedral polynomials such as those in equations 2.5–8 and 2.5–9:

(1) The vertices of a regular polygon with \( n \) vertices in the equatorial plane with one vertex at \( z = 1 \) correspond to the \( n \) roots of unity.

(2) Derivation of the vertex polynomial of the octahedron, \( \tau(u,v) \), is particularly obvious. Orientation of an octahedron so that two of the vertices are at the poles \( (z = 0, \infty) \) and one vertex is at \( z = 1 \) (Figure 2–9) forces the other three vertices to be at \( z = -1, i, \text{ and } -i \) so that

\[ \tau = z(z - 1)(z + 1)(z - i)(z + i) = z(z^4 - 1) = uv(u^4 - v^4) \]  \hspace{1cm} (2.5–10)

after converting to homogeneous variables by \( z = u/v \).

---

Figure 2-9: Orientation of the regular octahedron inside the Riemann sphere.

(3) The vertex and face polynomials of a polyhedron correspond to the face and vertex polynomials, respectively, of its dual. Thus the vertex polynomial $\tau$ (equation 2.5-8a) and face polynomial $W$ (equation 2.5-8c) of the octahedron corresponds to the face polynomial and vertex polynomial, respectively, of its dual, namely the cube.

(4) A cube can be composed of two mutually dual tetrahedra, whose edges correspond to face diagonals of the cube (Figure 2-10). A single tetrahedron in this orientation has the following polynomial functions:

- **Vertices:** $\Phi = u^4 + 2\sqrt{-3} u^2 v^2 + v^4$ \hspace{1cm} (2.5-11a)
- **Edges:** $\tau = uv(u^4 - v^4)$ \hspace{1cm} (2.5-11b)
- **Faces:** $\Psi = u^4 - 2\sqrt{-3} u^2 v^2 + v^4$ \hspace{1cm} (2.5-11c)

Note that the tetrahedron edge polynomial (equation 2.5-11b) is the same as the octahedron vertex polynomial (equation 2.5-8a) since the midpoints of the six edges of the tetrahedron in this orientation are in the same locations ($z = 0, \infty, \pm 1, \pm i$) as the six vertices of the standard octahedron. Furthermore, the face polynomial $\Psi$ of the tetrahedron is the same as the vertex polynomial of its dual. Since the four vertices of the tetrahedron and the four vertices of its dual together make the eight vertices of a cube, the product of the vertex functions of the two tetrahedra equals the vertex function of the cube made by the two tetrahedra $\Phi \Psi = W$. 
Beyond the Quartic Equation

Figure 2-10: Decomposition of a cube into two tetrahedra, whose vertices are indicated by A and B. The edges of the A tetrahedron are shown to the extent that they are visible.

(5) The special symmetries of the regular deltahedra lead to the following identities:

\[
\begin{align*}
\text{Tetrahedron: } & \quad 12\sqrt{3} \tau^2 - \Phi^3 + \Psi^3 \equiv 0 \quad \text{(degree 12)} \\
\text{Octahedron: } & \quad 108 \tau^4 - W^3 + \chi^2 \equiv 0 \quad \text{(degree 24)} \\
\text{Icosahedron: } & \quad 1728 f^5 - H^3 - T^2 \equiv 0 \quad \text{(degree 60)}
\end{align*}
\]

(2.5-12a) (2.5-12b) (2.5-12c)

Note that the degrees of the left side of the identities (2.5-12) in \((u,v)\) correspond to the orders of the pure rotation groups of the corresponding deltahedra, which are isomorphic to the permutation groups relevant to the solution of algebraic equations, i.e., \(O = S_4\) for the quartic equation and \(I = A_5\) for solution of the quintic equation. In this connection the icosahedral identity (equation 2.5-12c) will be seen to be important for the solution of the quintic equation.

2.6 Transvectants of Polyhedral Polynomials

The polyhedral polynomials in homogeneous form can be related by their transvectants derived from their partial derivatives by means of invariant theory.\(^{21,22}\) In this connection the \(n\)th transvectant of two homogeneous polynomials \(f(x,y)\) and \(g(x,y)\), designated as \((f,g)^n\) is defined by the equation

\[^{21}\text{J. H. Grace and A. Young, } The \ Algebra \ of \ Invariants, \ Cambridge, \ 1903.\]
\[^{22}\text{O. E. Glenn, } A \ Treatise \ on \ the \ Theory \ of \ Invariants, \ Ginn \ and \ Co., \ Boston, \ 1915.\]
\[(f,g)^n = \sum_{k=1}^{n} (-1)^k \frac{n!}{k!(n-k)!} \left( \frac{\partial^n f(x,y)}{\partial x^n \partial y^k} \right) \left( \frac{\partial^n g(x,y)}{\partial x^k \partial y^{n-k}} \right) \] (2.6–1)

Of particular interest are the transvectants of a polynomial with itself, i.e., \((f,f)^n\).

*Odd* transvectants of the type \((f,f)^n (n = 1, 3, 5, 7,...)\) vanish since they have *even* numbers of terms of alternating sign which cancel out completely corresponding to the \((-1)^k\) factor in equation (2.6–1). The first transvectant \((f,g)^1\) is also known as the *functional determinant* of \(f\) and \(g\), since it can be expressed by the determinant

\[
(f,g)^1 = \begin{vmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{vmatrix}
\] (2.6–2).

The second transvectants \((f,g)^2\) are also known as *Hessians* and can also be expressed by the determinant

\[
(f,g)^2 = \begin{vmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 g}{\partial y \partial x} & \frac{\partial^2 g}{\partial y^2}
\end{vmatrix}
\] (2.6–3).

Let \(v, e, f\), and \(v\), be the number of vertices, edges, and faces of regular deltahedra, namely the tetrahedron, octahedron, and icosahedron. The functional determinants and Hessians relate the vertex \((V(u,v)\) of degree \(v\), edge \((E(u,v)\) of degree \(e\), and face \((F(u,v)\) of degree \(f\) functions by the following relationships where \(k_{v,v}\) and \(k_{v,f}\) are integers:

\[
(V,V)^2 = k_{v,v} F \] (2.6–4a)

\[
(V,F)^1 = k_{v,f} E \] (2.6–4b)
The fact that each differentiation step lowers the degree of the polynomial by 1 can be used to relate equation (2.6–4b) to Euler's theorem\(^\text{23}\) as follows:
\[
\text{deg}(V',f)^1 = (v - 1) + (f - 1) = \text{deg}(E) = e
\Rightarrow v + f - 2 = e \text{ corresponding to Euler's theorem.} \\
(2.6–5)
\]
Similarly from equation (2.6–4a)
\[
\text{deg} (V',V)^2 = 2(v - 2) = 2v - 4 = f. \\
(2.6–6).
\]
For a deltahedron in which all faces are triangles
\[
3f = 2e \Rightarrow e = \frac{3}{2}f
\]
so that Euler's theorem becomes
\[
v + f - 2 = \frac{3}{2}f \Rightarrow 2v - 4 = f
\]
(2.6–8)
Note the resemblance of equations (2.6–6) and (2.6–8).

A special feature of the vertex polynomials for regular deltahedra, namely \(\tau\) of equation (2.5–8a) for the octahedron, \(f\) of equation (2.5–9a) for the icosahedron, and \(\Phi\) of equation (2.5–11a) for the tetrahedron is the identical vanishing of their fourth transvectants, i.e., \((\tau,\tau)^4 = 0\), \((ff)^4 = 0\), and \((\Phi,\Phi)^4 = 0\), respectively. This is a special indication of the symmetry of the regular deltahedra.

Figure 2–11 illustrates the procedure for calculating the second and fourth transvectants \((\tau,\tau)^2\) and \((\tau,\tau)^4\) of the octahedral vertex function \(\tau = uv(u^4 - v^4)\). Note that the Hessian \((\tau,\tau)^2\) is a multiple of the corresponding face function, namely \(-25W\) whereas the fourth transvectant \((\tau,\tau)^4\) vanishes identically.

\[ \tau = uv(u^4 - v^4) \quad \frac{\partial}{\partial u} u^5 - 5uv^4 \quad \frac{\partial}{\partial v} = -20uv^3 \quad \frac{\partial}{\partial v} -60uv^2 \quad \frac{\partial}{\partial v} = -120uv = \tau_{vvvv} \]

\[ 5u^4v - v^5 \quad \frac{\partial}{\partial u} 20u^3v = \tau_{uv} \quad \frac{\partial}{\partial v} 60u^2v \quad \frac{\partial}{\partial v} = 120uv = \tau_{uuuu} \]

\[ 5u^4 - 5v^4 = \tau_{uv} \quad 20u^3 \quad \frac{\partial}{\partial u} 60u^2 = \tau_{uuuv} \]

\[ -20v^3 \quad 0 = \tau_{uvvv} \]

\[ -60v^2 = \tau_{uvvvv} \]

\[ (\tau, \tau)^2 = (-20uv^3)(20u^3v) - (5u^4 - 5v^4)^2 = -25(u^8 + 14u^6v^4 + v^8) = -25W \]

\[ (\tau, \tau)^4 = (-120uv)(120uv) - 4(60u^2)(-60v^2) + 0 \equiv 0 \]

**Figure 2-11:** Illustration of the calculation of the second transvectant \((\tau, \tau)^2\) and the fourth transvectant \((\tau, \tau)^4\) of the octahedral vertex function \(\tau = uv(u^4 - v^4)\). Subscripts indicate differentiation variables and \(\partial_u\) and \(\partial_v\) indicate differentiation with respect to \(u\) and \(v\), respectively.
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