Chapter 2
Progressive Type-II Censoring: Distribution Theory

2.1 Joint Distribution

The general quantile representation for progressively Type-II censored order statistics due to Balakrishnan and Dembińska [96] (see also Balakrishnan and Dembińska [97] and Cramer and Kamps [301]) provides a powerful tool in the derivation of distributional results. Many identities can be obtained first for uniform distributions and then transferred to any particular distribution of interest.

Theorem 2.1.1. Suppose $X_{1:m:n}, \dots, X_{m:m:n}$ and $U_{1:m:n}, \dots, U_{m:m:n}$ are progressively Type-II censored order statistics based on a cumulative distribution function F and a uniform distribution, respectively. Then,

$$(X_{j:m:n})_{1 \le j \le m} \stackrel{\mathsf{d}}{=} \big(F^{\leftarrow}(U_{j:m:n})\big)_{1 \le j \le m}.$$

Proof. Let $X_1, \dots, X_n$ and $U_1, \dots, U_n$ be IID samples from F and a uniform distribution on a probability space $(\Omega, \mathfrak{A}, P)$, respectively. Then,

$$(X_1, \dots, X_n) \stackrel{\mathsf{d}}{=} (F^{\leftarrow}(U_1), \dots, F^{\leftarrow}(U_n))$$

and we conclude that the vector $(X_{1:m:n}, \dots, X_{m:m:n})$ has the same distribution as progressively Type-II censored order statistics based on the sample $F^{\leftarrow}(U_1), \dots, F^{\leftarrow}(U_n)$. Therefore, it is sufficient to prove that these progressively Type-II censored order statistics have the same values as the random variables $\big(F^{\leftarrow}(U_{j:m:n})\big)_{1 \le j \le m}$ for any fixed $\omega \in \Omega$ in the underlying probability space. For brevity, let $u_j = U_j(\omega)$, $1 \le j \le n$, and $u_i^* = U_{i:m:n}(\omega)$, $1 \le i \le m$. Notice that $F^{\leftarrow}$ is an increasing function and that for given numbers $x_1, \dots, x_r$,

$$\min_{1 \le k \le r} F^{\leftarrow}(x_k) = F^{\leftarrow}\Big(\min_{1 \le k \le r} x_k\Big). \tag{2.1}$$

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From the generation process 1.1.3, we find that u_j^* is defined by the minimum of a selection M_j of numbers. Thus,

$$u_j^* = \min_{i \in M_j} u_i, \quad 1 \le j \le m,$$

and we obtain from (2.1) that

$$F^{\leftarrow}(u_j^*) = \min_{i \in M_j} F^{\leftarrow}(u_i), \quad 1 \le j \le m.$$

This yields the desired quantile representation. □

The result can alternatively be proved by using the mixture representation in Theorem 10.1.1 due to Fischer et al. [371]. It is worth mentioning that the quantile representation in Theorem 2.1.1 shows that

$$F^{\mathbf{X}^{\mathscr{R}}} = F^{\mathbf{U}^{\mathscr{R}}} \circ (F^{*m}). \tag{2.2}$$

In particular, any marginal cumulative distribution function can be written in this way. For instance, $F^{X_{r:m:n}} = F^{U_{r:m:n}} \circ F$, $F^{X_{r:m:n}, X_{s:m:n}} = F^{U_{r:m:n}, U_{s:m:n}} \circ (F, F)$, etc.

An important tool in the analysis of progressively Type-II censored order statistics is the joint density function of uniform progressively Type-II censored order statistics which is given in the following theorem. A formal proof is provided in Sect. 10.2 for the more general INID situation.

Theorem 2.1.2. The joint density function of uniform progressively Type-II censored order statistics $U_{1:m:n}, \ldots, U_{m:m:n}$ is given by

$$f^{U_{1:m:n},\ldots,U_{m:m:n}}(\mathbf{u}_m) = \prod_{j=1}^{m} \left[\gamma_j (1 - u_j)^{R_j}\right], \; 0 \le u_1 \le \cdots \le u_m \le 1. \tag{2.3}$$

If F is absolutely continuous, the joint density function of progressively Type-II censored order statistics is given in the following corollary (cf. Cohen [267], Herd [440], and Balakrishnan and Aggarwala [86]). It follows directly from Theorems 2.1.1 and 2.1.2 [see also (2.2)].

Corollary 2.1.3. The joint density function of progressively Type-II censored order statistics $X_{1:m:n}, \ldots, X_{m:m:n}$ based on a cumulative distribution function F with density function f is given by

$$f^{\mathbf{X}^{\mathscr{R}}}(\mathbf{x}_m) = \prod_{j=1}^{m} \left[\gamma_j f(x_j)(1 - F(x_j))^{R_j}\right], \quad x_1 \le \cdots \le x_m. \tag{2.4}$$

Example 2.1.4.

(i) For order statistics, i.e., $m = n$, $\mathscr{R} = (0^{*m})$ and $\gamma_j = n - j + 1$, $1 \le j \le n$, the joint density function is given by

$$f^{X_{1:n},\ldots,X_{n:n}}(\mathbf{x}_n) = n! \prod_{j=1}^{n} f(x_j), \quad x_1 \le \cdots \le x_n \tag{2.5}$$

(cf. Arnold et al. [58] and David and Nagaraja [327]).

(ii) The censoring plan $\mathscr{R} = (R^{*m})$ with equal removal number $R \in \mathbb{N}_0$ is called equi-balanced censoring scheme. Progressively Type-II censored order statistics with such a censoring scheme possess the joint density function

$$\begin{aligned} f^{X_{1:m:n},\ldots,X_{m:m:n}}(\mathbf{x}_m) &= \prod_{j=1}^{m} \left[\gamma_j f(x_j)(1 - F(x_j))^R\right] \\ &= m! \prod_{j=1}^{m} \left[(R+1) f(x_j)(1 - F(x_j))^R\right], \quad x_1 \le \cdots \le x_m. \end{aligned} \tag{2.6}$$

Notice that $\gamma_j = (m - j + 1)(R + 1)$, $1 \le j \le m$ (see also Table 1.2). Defining g by $g(t) = (R + 1) f(t)(1 - F(t))^R$, we find that the density function in (2.6) equals the joint density function of order statistics from a sample of size m and with density function g. Hence, this particular scheme does not lead to a new model. It can be seen simply as an order statistic model from a different distribution. Notice that this distribution is the same as that of the minimum of $R+1$ IID random variables from f. This comment applies also to the models with non-absolutely continuous distribution.

(iii) In the OSP-case with censoring scheme $\mathscr{O}_k$, $k \in \{1, \ldots, m\}$, the joint density function is given by

$$\begin{aligned} f^{\mathbf{X}^{\mathscr{O}_k}}(\mathbf{x}_m) &= \Big[\prod_{j=1}^{m} \left[\gamma_j f(x_j)\right]\Big](1 - F(x_k))^{n-m} \\ &= \frac{n!(m-k)!}{(n-k)!}\Big[\prod_{j=1}^{m} f(x_j)\Big](1 - F(x_k))^{n-m}, \quad x_1 \le \cdots \le x_m. \end{aligned}$$

2.2 On the Connection to Generalized Order Statistics and Sequential Order Statistics

It is obvious from the joint density function of uniform progressively Type-II censored order statistics presented in Theorem 2.1.2 that uniform progressively Type-II censored order statistics can be seen as particular uniform generalized order statistics introduced by Kamps [498, 499] (see also Cramer [285, 288], Cramer and Kamps [300, 301], and Kamps [502]). Commonly, generalized order statistics are parametrized by one of the following sets of parameters which are very similar to those for progressively Type-II censored order statistics given on page 7:

(i) $k, m_1, \ldots, m_{n-1}$,
(ii) $\gamma_1, \ldots, \gamma_n > 0$.

The density function of uniform generalized order statistics is usually given as (see Kamps [498, p. 49])

$$f^{U(1,n,\mathbf{m},k),\ldots,U(n,n,\mathbf{m},k)}(\mathbf{u}_n) = k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big[\prod_{j=1}^{n-1}(1-u_j)^{m_j}\Big](1-u_n)^{k-1},$$
$$0 \le u_1 \le \cdots \le u_n \le 1, \qquad (2.7)$$

where $\mathbf{m} = (m_1, \ldots, m_{n-1})$. Generalized order statistics $X(1,n,\mathbf{m},k), \ldots, X(n,n,\mathbf{m},k)$ based on an arbitrary cumulative distribution function F are defined via the quantile transformation

$$X(j,n,\mathbf{m},k) = F^{\leftarrow}(U(j,n,\mathbf{m},k)), \quad 1 \le j \le n,$$

so that the same comment applies to progressively Type-II censored order statistics from an arbitrary cumulative distribution function using the representation in terms of the quantile function (see, e.g., the density function given in Corollary 2.1.3).

Hence, the joint density function has a similar form as (2.7) (see (2.4) for progressively Type-II censored order statistics). Sometimes, the parameters are suppressed in the notation and (uniform) generalized order statistics are also denoted by $U_{*,1}, \ldots, U_{*,n}$. Notice that m and n are differently used in both models. However, we have the correspondences $R_j = m_j$ and k equals the last γ_j. To be more specific, we consider uniform progressively Type-II censored order statistics with censoring scheme $\mathscr{R}$. Then, they can be seen as uniform generalized order statistics

$$U(1, m, \mathscr{R}_{\rhd m-1}, R_m + 1), \ldots, U(m, m, \mathscr{R}_{\rhd m-1}, R_m + 1),$$

where $\mathscr{R}_{\rhd m-1} = (R_1, \ldots, R_{m-1})$ denotes a right truncated censoring scheme [see (1.6)]. Thus, progressively Type-II censored order statistics are generalized order statistics in distribution wherein some restrictions have to be imposed on the parameters $\gamma_1, \ldots, \gamma_m > 0$ of generalized order statistics as introduced in Cramer and Kamps [301] [see also (1.3)]:

(i) $\gamma_j \in \mathbb{N}$, $j = 1, \ldots, m$,
(ii) $n = \gamma_1 > \cdots > \gamma_m \geq 1$.

Therefore, in the model of progressively Type-II censored order statistics, the parameters $\gamma_1, \ldots, \gamma_m$ are strictly decreasingly ordered positive integers. Although this difference seems to be minor, the picture becomes simpler for progressively Type-II censored order statistics in many cases. In particular, calculations become easier and representations get simpler. An example may be the representation of the marginal density functions (see (2.28) for progressively Type-II censored order statistics and Cramer and Kamps [301] for the density functions of generalized order statistics in terms of Meijer's G-functions). Moreover, it turns out that many results obtained for generalized order statistics are valid for progressively Type-II censored order statistics without imposing further restrictions on the parameters.

Notice that the connection is only of distributional nature, but it can be used in many areas. For instance, results for moments can be directly applied to progressively Type-II censored order statistics. Similar comments apply to characterizations, stochastic orders, reliability properties, inferential results, etc., which are available for generalized order statistics with arbitrary parameters. Therefore, many results can be directly taken from properties of generalized order statistics. We utilize this connection in the following by reformulating the results in terms of progressively Type-II censored order statistics. On the other hand, extensions to generalized order statistics are also possible in many settings.

However, one has to be careful using this connection because many results for generalized order statistics are often obtained only for the so-called m-generalized order statistics. In this case, the parameters satisfy the condition $m_1 = \cdots = m_{n-1}$. For progressively Type-II censored order statistics, this corresponds to the case of an equi-balanced censoring scheme $\mathscr{R} = (R_1, \ldots, R_m) = (R^{*m})$ with $R \in \mathbb{N}_0$.

Moreover, it has to be mentioned that some results are also available in terms of sequential order statistics from some cumulative distribution functions $F_1, \ldots, F_m$. This model has been introduced in Kamps [498] in order to extend the model of k-out-of-m systems (see also Burkschat [230], Cramer [288], and Cramer and Kamps [300]). According to Cramer and Kamps [301], the distribution of sequential order statistics $X_*^{(1)}, \ldots, X_*^{(m)}$ (based on $F_1, \ldots, F_n$) can be represented via quantile-type transformations

$$X_*^{(r)} = F_r^{\leftarrow}(X^{(r)}) \quad \text{with } X^{(r)} = 1 - V_r \overline{F}_r(X_*^{(r-1)}), \quad 1 \leq r \leq m,$$

where $X_*^{(0)} = -\infty$, $F_1, \ldots, F_m$ are cumulative distribution functions with $F_1^{\leftarrow}(1) \leq \cdots \leq F_m^{\leftarrow}(1)$, and $V_1, \ldots, V_m$ are independent random variables with $V_r \sim \mathsf{Beta}(m - r + 1, 1)$, $1 \leq r \leq m$.

As pointed out in Cramer and Kamps [300], sequential order statistics can be seen as generalized order statistics based on F if the cumulative distribution functions

$F_1, \ldots, F_m$ satisfy the proportional hazards relation $\overline{F}_j = \overline{F}^{\alpha_j}$, $1 \le j \le m$, for some continuous cumulative distribution function F and $\alpha_1, \ldots, \alpha_m > 0$. Using this connection, results for sequential order statistics can also be applied to progressively Type-II censored order statistics.

Finally, the distribution of exponential progressively Type-II censored order statistics is connected to the distribution of order statistics from a Weinman multivariate exponential distribution which is an extension of Freund's bivariate exponential distribution (see Block [206], Freund [383], and Weinman [896]). As pointed out by Cramer and Kamps [297] and Cramer and Kamps [300], this connection can also be utilized in the framework of progressively Type-II censored order statistics.

2.3 Results for Particular Population Distributions

2.3.1 Exponential Distributions

In this section, progressively Type-II censored order statistics are based on a two-parameter exponential distribution $\mathsf{Exp}(\mu, \vartheta)$, $\mu \in \mathbb{R}, \vartheta > 0$. From Corollary 2.1.3, we find directly the respective representation of the joint density function in the exponential case.

Corollary 2.3.1. The joint density function of exponential progressively Type-II censored order statistics $Z_{1:m:n}, \ldots, Z_{m:m:n}$ from an $\mathsf{Exp}(\mu, \vartheta)$-distribution is given by

$$f^{\mathbf{Z}^{\mathscr{R}}}(\mathbf{x}_m) = \Big(\prod_{j=1}^{m} \gamma_j\Big) \exp\Big\{-\frac{1}{\vartheta}\sum_{j=1}^{m}(R_j+1)(x_j-\mu)\Big\}, \quad \mu \le x_1 \le \cdots \le x_m. \tag{2.8}$$

The joint density function given in (2.8) yields directly the fundamental result that the normalized spacings of exponential progressively Type-II censored order statistics are IID exponential random variables. This observation is due to Thomas and Wilson [843] (see also Viveros and Balakrishnan [875]). Let

$$S_r^{\mathscr{R}} = \gamma_r (Z_{r:m:n}^{\mathscr{R}} - Z_{r-1:m:n}^{\mathscr{R}}), \quad r = 1, \ldots, m, \tag{2.9}$$

be the (normalized) spacings of $Z_{1:m:n}^{\mathscr{R}}, \ldots, Z_{m:m:n}^{\mathscr{R}}$, where $Z_{0:m:n}^{\mathscr{R}} = \mu$. Moreover, let $\mathbf{S}^{\mathscr{R}} = (S_1^{\mathscr{R}}, \ldots, S_m^{\mathscr{R}})'$ and $\mathbf{Z}^{\mathscr{R}} = (Z_{1:m:n}^{\mathscr{R}}, \ldots, Z_{m:m:n}^{\mathscr{R}})'$. Then,

$$\mathbf{S}^{\mathscr{R}} = T(\mathbf{Z}^{\mathscr{R}} - \mu\mathbb{1}) \tag{2.10}$$

with

$$T = \begin{pmatrix} \gamma_1 & 0 & \cdots & \cdots & 0 \\ -\gamma_2 & \gamma_2 & 0 & \cdots & 0 \\ 0 & -\gamma_3 & \gamma_3 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\gamma_m & \gamma_m \end{pmatrix}.$$

Theorem 2.3.2. The spacings $S_1^{\mathscr{R}}, \dots, S_m^{\mathscr{R}}$ are independently and identically distributed with $S_r^{\mathscr{R}} \sim \mathrm{Exp}(\vartheta)$, $r = 1, \dots, m$.

Proof. Since $\gamma_j > 0$, $1 \le j \le m$, T is a regular matrix with

$$T^{-1} = \begin{pmatrix} 1/\gamma_1 & 0 & \cdots & \cdots & 0 \\ 1/\gamma_1 & 1/\gamma_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1/\gamma_1 & 1/\gamma_2 & \cdots & \cdots & 1/\gamma_m \end{pmatrix} \quad \text{and} \quad \det T = \prod_{j=1}^m \gamma_j .$$

Now, the density transformation theorem yields the density function

$$f^{\mathbf{S}^{\mathscr{R}}}(\mathbf{t}) = \frac{1}{|\det T|} \cdot f^{\mathbf{Z}^{\mathscr{R}}}(T^{-1}\mathbf{t} + \mu\mathbb{1}), \quad \mathbf{t} = (t_1, \dots, t_m). \tag{2.11}$$

Noticing that $\gamma_j - \gamma_{j+1} = R_j + 1$, $1 \le j \le m-1$, and $\gamma_m = R_m + 1$, we find

$$\begin{aligned} \sum_{j=1}^m (R_j + 1)[(T^{-1}\mathbf{t} + \mu\mathbb{1})_j - \mu] &= \underbrace{(\gamma_1 - \gamma_2, \dots, \gamma_{m-1} - \gamma_m, \gamma_m)}_{=\mathbb{1}'T} T^{-1}\mathbf{t} \\ &= \mathbb{1}'\mathbf{t} = \sum_{j=1}^m t_j . \end{aligned}$$

Thus, (2.11) in combination with (2.8) yields the density function $f^{\mathbf{S}^{\mathscr{R}}}(\mathbf{t}) = \exp\{-\frac{1}{\vartheta}\sum_{j=1}^m t_j\}$, $t_1, \dots, t_m \ge 0$. This proves the desired result. □

Theorem 2.3.2 yields the following well-known result for spacings of order statistics due to Sukhatme [826]. It follows from Theorem 2.3.2 by choosing the censoring scheme $\mathscr{R} = (0^{*m})$.

Corollary 2.3.3 (Sukhatme [826]). The spacings $S_{1,n}, \dots, S_{n,n}$ of exponential order statistics $Z_{1:n}, \dots, Z_{n:n}$ from an $\mathrm{Exp}(\mu, \vartheta)$-distribution are independently and identically distributed with $S_{r,n} \sim \mathrm{Exp}(\vartheta)$, $r = 1, \dots, n$.

The preceding result can be extended easily to one-parameter exponential families.

Remark 2.3.4. Suppose the cumulative distribution function F_θ of a one-parameter exponential family is given by

$$F_\theta(x) = 1 - \mathrm{e}^{-\eta(\theta)d(x)}, \quad x \in (\alpha, \omega), \tag{2.12}$$

with $-\infty \le \alpha < \omega \le \infty$, $d(\alpha+) = \lim_{x\to\alpha+} d(x) = 0$, $d(\omega-) = \lim_{x\to\omega-} d(x) = \infty$, where d is nondecreasing and differentiable and η is positive and twice differentiable. Then, the random variables $\gamma_j(d(X_{j:m:n}) - d(X_{j-1:m:n}))$, $1 \le j \le m$, are IID exponential random variables with mean $1/\eta(\theta)$ (see, for example, Cramer and Kamps [300]). This family is also discussed in the context of Fisher information in Sect. 9.1.3.

The exponential family defined via (2.12) can be characterized by the property of hazard rate factorization, i.e., by $\lambda_\theta(x) = \eta(\theta)d'(x)$. It includes, for instance, the exponential distribution (scale parameter), the extreme value distribution (location parameter), the Weibull distribution (scale parameter), and the Pareto distribution (shape parameter). Characterizations of distribution in terms of the Fisher information are given by, e.g., Hofmann et al. [445], Zheng [941], and Gertsbakh and Kagan [396].

Remark 2.3.5. Bairamov and Eryılmaz [78] discussed minimal and maximal (non-normalized) spacings for exponential progressively Type-II censored order statistics, i.e.,

$$S_1^{\star\mathscr{R}} = \frac{1}{\gamma_1} S_1^{\mathscr{R}} = Z_{1:m:n}^{\mathscr{R}},$$

$$S_j^{\star\mathscr{R}} = \frac{1}{\gamma_j} S_j^{\mathscr{R}} = Z_{j:m:n}^{\mathscr{R}} - Z_{j-1:m:n}^{\mathscr{R}}, \quad j = 2, \ldots, m.$$

In particular, they were interested in the random indicators η and ν with

$$S_\nu^{\star\mathscr{R}} = \min_{1\le j\le m} S_j^{\star\mathscr{R}}, \quad S_\eta^{\star\mathscr{R}} = \max_{1\le j\le m} S_j^{\star\mathscr{R}}.$$

Clearly, Theorem 2.3.2 implies $S_j^{\star\mathscr{R}} \sim \mathrm{Exp}(\vartheta/\gamma_j)$, $1 \le j \le m$. Bairamov and Eryılmaz [78] obtained expressions for the joint probability mass function as well as for the marginal probability mass functions of ν and η. For instance, for $k = 1, \ldots, m$ and an underlying $\mathrm{Exp}(\vartheta)$-distribution, $S_j^{\star\mathscr{R}}$, $1 \le j \le m$, are independent random variables. This directly leads to the expressions

$$P(\nu = k) = \frac{\gamma_k}{\sum_{j=1}^m \gamma_j},$$

$$P(\eta = k) = \gamma_k \int_0^\infty \prod_{j=1, j\ne k}^m \left(1 - \mathrm{e}^{-\gamma_j t}\right)\mathrm{e}^{-\gamma_k t}\, dt,$$

which are independent of the scale parameter ϑ. Moreover, the joint and marginal cumulative distribution functions of the maximal spacing can be obtained. For $0 < x < y$, we get

$$P(S_\nu^{\star\mathscr{R}} \le x, S_\eta^{\star\mathscr{R}} \le y) = \prod_{j=1}^{m} \big(1 - \mathrm{e}^{-\gamma_j y/\vartheta}\big) - \prod_{j=1}^{m} \big(\mathrm{e}^{-\gamma_j x/\vartheta} - \mathrm{e}^{-\gamma_j y/\vartheta}\big),$$

$$P(S_\nu^{\star\mathscr{R}} \le x) = 1 - \exp\Big\{-\Big(\sum_{j=1}^{m} \gamma_j\Big)x/\vartheta\Big\},$$

$$P(S_\eta^{\star\mathscr{R}} \le y) = \prod_{j=1}^{m} \big(1 - \mathrm{e}^{-\gamma_j y/\vartheta}\big).$$

From (2.10), we find the following representation of exponential progressively Type-II censored order statistics in terms of the spacings:

$$\mathbf{Z}^{\mathscr{R}} = T^{-1}\mathbf{S}^{\mathscr{R}} + \mu\mathbb{1} \quad \text{or} \quad Z_{r:m:n}^{\mathscr{R}} = \mu + \sum_{j=1}^{r} \frac{1}{\gamma_j} S_j^{\mathscr{R}}, 1 \le r \le m. \tag{2.13}$$

Thus, we can write exponential progressively Type-II censored order statistics as a weighted sum of independent exponential random variables. This expression will be very useful in deriving marginal distributions, moments, recurrence relations, etc.

Moreover, (2.13) yields an interesting representation of progressively Type-II censored order statistics. In particular, we have from Theorem 2.1.1 in the exponential case

$$Z_{r:m:n} \stackrel{\mathrm{d}}{=} \mu - \vartheta \log(1 - U_{r:m:n}), \quad 1 \le r \le m,$$

or, equivalently, with $F_{\exp}(t) = 1 - \mathrm{e}^{-(t-\mu)/\vartheta}, t \ge \mu$,

$$F_{\exp}(Z_{r:m:n}) \stackrel{\mathrm{d}}{=} U_{r:m:n}, \quad 1 \le r \le m.$$

From (2.13), we note that

$$F_{\exp}(Z_{r:m:n}) = 1 - \prod_{j=1}^{r} \Big(\mathrm{e}^{-S_j^{\mathscr{R}}/\vartheta}\Big)^{1/\gamma_j}$$

with $S_j^{\mathscr{R}}/\vartheta \sim \mathsf{Exp}(1)$. Thus, $U_j = \mathrm{e}^{-S_j^{\mathscr{R}}/\vartheta}, 1 \le j \le m$, are independent uniformly distributed random variables. This yields the representation

$$U_{r:m:n} \stackrel{\mathsf{d}}{=} 1 - \prod_{j=1}^{r} U_j^{1/\gamma_j}, \quad 1 \le r \le m,$$

of $U_{r:m:n}$ as a product of independent random variables. Combining this expression with the quantile representation from Theorem 2.1.1, we arrive at the following theorem (see also Cramer and Kamps [301]).

Theorem 2.3.6. Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be progressively Type-II censored order statistics from an arbitrary cumulative distribution function F and $U_1, \ldots U_m \stackrel{\text{iid}}{\sim} \mathsf{U}(0,1)$. Then,

$$X_{r:m:n} \stackrel{\mathsf{d}}{=} F^{\leftarrow}\Big(1 - \prod_{j=1}^{r} U_j^{1/\gamma_j}\Big), \quad 1 \le r \le m. \tag{2.14}$$

Sometimes, the following representation in terms of exponential progressively Type-II censored order statistics is useful which is immediate from Theorem 2.1.1 and the above theorem.

Corollary 2.3.7. Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be progressively Type-II censored order statistics from an arbitrary cumulative distribution function F and $Z_{1:m:n}, \ldots,$ $Z_{m:m:n}$ be progressively Type-II censored order statistics with the same censoring scheme. Moreover, let $Z_1, \ldots, Z_m \stackrel{\text{iid}}{\sim} \mathsf{Exp}(1)$ and $\Psi(x) = F^{\leftarrow}(1 - \mathsf{e}^{-x})$, $x \ge 0$. Then,

$$X_{r:m:n} \stackrel{\mathsf{d}}{=} F^{\leftarrow}\left(1 - \mathsf{e}^{-Z_{r:m:n}}\right) \stackrel{\mathsf{d}}{=} \Psi\Big(\sum_{j=1}^{r} \frac{1}{\gamma_j} Z_j\Big), \quad 1 \le r \le m.$$

The above representation can be simplified for order statistics. In this particular setup, we find the following result which shows that uniform order statistics are beta distributed. Therefore, order statistics have been called transformed beta variables (see, e.g., Blom [208]).

Corollary 2.3.8. For uniform order statistics $U_{1:n}, \ldots, U_{n:n}$, we have $U_{r:n} \sim \mathsf{Beta}(r, n-r+1)$, $1 \le r \le n$.

Proof. By definition, we have $\gamma_j = n - j + 1$, $1 \le j \le n$. Thus, we obtain for uniform order statistics

$$1 - U_{r:n} \stackrel{\mathsf{d}}{=} \prod_{j=1}^{r} U_j^{1/\gamma_j}, \quad U_j^{1/\gamma_j} \sim \mathrm{Beta}(n - j + 1, 1).$$

Using a result of Rao [738] (see also Jambunathan [477], Kotlarski [545], Fan [359], and Johnson et al. [484, p. 257]) we get

$$\prod_{j=1}^{r} U_j^{1/\gamma_j} \sim \mathsf{Beta}(n-r+1, r).$$

Hence, $U_{r:n}$ has a $\mathsf{Beta}(r, n-r+1)$-distribution. □

A simple representation also holds for one-step censoring plans.

Corollary 2.3.9. Let $\mathscr{O}_k$, $1 \le k \le m$, be a one-step censoring plan. Then,

$$U_{r:m:n} \stackrel{\mathsf{d}}{=} U_{r:n}, \quad 1 \le r \le k,$$
$$U_{r:m:n} \stackrel{\mathsf{d}}{=} 1 - (1 - U_{k:n}) \cdot (1 - \widetilde{U}_{r-k:m-k}), \quad k+1 \le r \le m,$$

where $\widetilde{U}_{r-k:m-k}$ denotes the $(r-k)$th order statistic in a sample $\widetilde{U}_1, \ldots, \widetilde{U}_{m-k}$ from a uniform distribution and independent of $U_1, \ldots, U_n$. Thus, the distribution of $1 - U_{r:m:n}$ is given by the distribution of a product of independent beta random variables with parameters $(n-k+1, k)$ and $(m-r+1, r-k)$, respectively.

Proof. From Table 1.2, we have $\gamma_j = n-j+1$, $1 \le j \le k$, and $\gamma_j = m-j+1$, $k+1 \le j \le m$. Thus, for $1 \le r \le k$, we obtain from Corollary 2.3.8 that $U_{r:m:n} \stackrel{\mathsf{d}}{=} U_{r:n} \sim \mathsf{Beta}(r, n-r+1)$. Let $r > k$. Then, the product $\prod_{j=1}^{r} U_j^{1/\gamma_j}$ equals

$$\prod_{j=1}^{k} U_j^{1/(n-j+1)} \prod_{j=k+1}^{r} U_j^{1/(m-j+1)} = \prod_{j=1}^{k} U_j^{1/(n-j+1)} \prod_{j=1}^{r-k} U_j^{1/(m-k-j+1)}.$$

The first product has a $\mathsf{Beta}(n-k+1, k)$ distribution, while the second one has a $\mathsf{Beta}(m-r+1, r-k)$ distribution. By the independence of the factors, we obtain the desired result. □

Since $1 - U_{j:\nu} \stackrel{\mathsf{d}}{=} U_{\nu-j+1:\nu}$, the result of Corollary 2.3.9 can be expressed as

$$\begin{aligned} U_{r:m:n} &\stackrel{\mathsf{d}}{=} U_{r:n}, \quad 1 \le r \le k, \\ U_{r:m:n} &\stackrel{\mathsf{d}}{=} 1 - U_{n-k+1:k} \cdot \widetilde{U}_{m-r+1:m-k}, \quad k+1 \le r \le m. \end{aligned} \tag{2.15}$$

This representation can be used for simulation purposes (see Algorithm 8.1.8).

2.3.2 Reflected Power Distribution and Uniform Distribution

Theorem 2.3.6 has some interesting implications to generalized Pareto distributions (see Definition A.1.11). In particular, we find for reflected power distributions with $F^{\leftarrow}(t) = 1 - (1-t)^{1/\beta}$, $t \in (0, 1)$, the following identity.

Corollary 2.3.10. Let $X_{1:m:n}, \dots, X_{m:m:n}$ be progressively Type-II censored order statistics from a reflected power function distribution RPower(β), $\beta > 0$. Then,

$$X_{r:m:n} \stackrel{\mathrm{d}}{=} 1 - \prod_{j=1}^{r} U_j^{1/(\beta\gamma_j)}, \quad 1 \le r \le m.$$

For $\beta = 1$, this yields the representation for the uniform distribution, i.e.,

$$U_{r:m:n} \stackrel{\mathrm{d}}{=} 1 - \prod_{j=1}^{r} U_j^{1/\gamma_j}, \quad 1 \le r \le m. \tag{2.16}$$

Using this representation, we can easily derive the following result. For $\beta = 1$, it can be found in Balakrishnan and Aggarwala [86].

Corollary 2.3.11. Let $X_{1:m:n}, \dots, X_{m:m:n}$ be progressively Type-II censored order statistics from a reflected power function distribution RPower(β). Then, with $X_{0:m:n} = 0$,

$$V_j = \left(\frac{1 - X_{j:m:n}}{1 - X_{j-1:m:n}}\right)^{\beta}, \quad 1 \le j \le m,$$

are independent random variables with $V_j \stackrel{\mathrm{d}}{=} U_j^{1/\gamma_j} \sim$ Beta$(\gamma_j, 1)$, $1 \le j \le m$.

For $\beta = 1$, we have, with $U_{0:m:n} = 0$,

$$V_j = \frac{1 - U_{j:m:n}}{1 - U_{j-1:m:n}}, \quad 1 \le j \le m,$$

to be independent random variables with $V_j \stackrel{\mathrm{d}}{=} U_j^{1/\gamma_j} \sim$ Beta$(\gamma_j, 1)$, $1 \le j \le m$.

In the uniform case, this yields the following well-known result of Malmquist [633].

Corollary 2.3.12. Let $U_{1:n}, \dots, U_{n:n}$ be order statistics from a uniform distribution. Then, with $U_{0:n} = 0$,

$$V_j = \frac{1 - U_{j:n}}{1 - U_{j-1:n}}, \quad 1 \le j \le n,$$

are independent random variables with $V_j \sim$ Beta$(n - j + 1, 1)$, $1 \le j \le n$.

2.3.3 Pareto Distributions

For Pareto distributions Pareto(α), we find with $F^{\leftarrow}(t) = (1-t)^{-1/\alpha}$, $t \in (0,1)$, the following result.

Corollary 2.3.13. Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be progressively Type-II censored order statistics from a Pareto distribution Pareto(α), $\alpha > 0$. Then,

$$X_{r:m:n} \stackrel{\mathrm{d}}{=} \prod_{j=1}^{r} U_j^{-1/(\alpha\gamma_j)}, \quad 1 \le r \le m.$$

Using this representation we can easily derive the following result (see also Balakrishnan and Aggarwala [86, p. 24]).

Corollary 2.3.14. Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be progressively Type-II censored order statistics from a Pareto distribution Pareto(α). Then, with $X_{0:m:n} = 1$,

$$W_j = \frac{X_{j:m:n}^{\alpha}}{X_{j-1:m:n}^{\alpha}}, \quad 1 \le j \le m,$$

are independent random variables with $W_j \stackrel{\mathrm{d}}{=} U_j^{-1/\gamma_j} \sim$ Pareto(γ_j), $1 \le j \le m$.

In the case of order statistics, this yields a well-known result for Pareto distributions which was first mentioned by Malik [632]. Further references are Huang [458], Arnold [49], and Johnson et al. [483].

2.3.4 Progressive Withdrawal and Dual Generalized Order Statistics

It is a well-known property of order statistics, $X_{j:n}$, $1 \le j \le n$, from a symmetric distribution (symmetric about 0), that

$$X_{j:n} \stackrel{\mathrm{d}}{=} -X_{n-j+1:n}, \quad 1 \le j \le n, \tag{2.17}$$

or that, jointly,

$$(X_{1:n}, \ldots, X_{n:n}) \stackrel{\mathrm{d}}{=} (-X_{n:n}, \ldots, -X_{1:n}); \tag{2.18}$$

see, for example, David and Nagaraja [327] and Arnold et al. [58]. Thus, the negatives of the order statistics are once again distributed as order statistics from the same symmetric distribution. It will therefore be natural to see whether a similar connection holds for progressively Type-II censored order statistics from a symmetric distribution as it will facilitate the handling of these random variables (see, e.g., Sect. 7.4). In the case of order statistics, the result in (2.18) is easily observed by considering the joint density function of order statistics given in Example 2.1.4. Using the representation in (2.5) and the fact that $f(x) = f(-x)$, $x \in \mathbb{R}$, the joint density function can be rewritten in the desired form. A similar argument for the identity (2.17) using the marginal density function has been employed in Balakrishnan and Aggarwala [86]. Alternatively, we may use the quantile representation of order statistics given in Theorem 2.1.1. Using the identity $F(x) = 1 - F(-x)$, $x \in \mathbb{R}$, for the cumulative distribution function for symmetric distributions, we get $F^{\leftarrow}(t) = -F^{\leftarrow}(1-t)$, $t \in (0,1)$. This implies for $1 \le r \le n$

$$-X_{r:n} \stackrel{\mathsf{d}}{=} -F^{\leftarrow}(U_{r:n}) = F^{\leftarrow}(1 - U_{r:n}) \stackrel{\mathsf{d}}{=} F^{\leftarrow}(U_{n-r+1:n}) \stackrel{\mathsf{d}}{=} X_{n-r+1:n},$$

where we have used Corollary 2.3.8 and that $1 - X \sim \mathsf{Beta}(\beta, \alpha)$ holds for a $\mathsf{Beta}(\alpha, \beta)$-distributed random variable X.

However, in working with the progressively Type-II censored order statistics, we begin with the joint distribution of all m progressively Type-II censored order statistics. As before, let $X_{1:m:n}^{\mathscr{R}}, \ldots, X_{m:m:n}^{\mathscr{R}}$ denote the sample of progressively Type-II censored order statistics of size m obtained from a random sample of size n with censoring scheme $\mathscr{R} = (R_1, \ldots, R_m)$ from a symmetric distribution. Multiplying each random variable by -1 we get the decreasingly ordered sample

$$Y_1 = -X_{m:m:n}^{\mathscr{R}}, \ldots, Y_m = -X_{1:m:n}^{\mathscr{R}}.$$

It follows from the quantile representation in Theorem 2.3.6 and the quantile function

$$F_{-X}^{\leftarrow}(t) = -F^{\leftarrow}(1-t), \quad t \in (0,1),$$

that

$$Y_r \stackrel{\mathsf{d}}{=} -F^{\leftarrow}\Big(1 - \prod_{j=1}^{m-r+1} U_j^{1/\gamma_j}\Big) = F_{-X}^{\leftarrow}\Big(\prod_{j=1}^{m-r+1} U_j^{1/\gamma_j}\Big), \quad 1 \le r \le m.$$

This representation tells us that $Y_1, \ldots, Y_m$ are connected to the so-called dual generalized order statistics introduced by Burkschat et al. [234]. Moreover, we get the following expression for the joint density function:

$$f^{Y_1,\ldots,Y_m}(\mathbf{t}_m) = \prod_{j=1}^{m} \left[\gamma_j f(t_j) F^{R_j}(t_j)\right], \quad t_1 \le \cdots \le t_m.$$

As a result, the negatives of the progressively Type-II censored order statistics are generally not jointly distributed as progressively Type-II censored order statistics. Further results and applications can be found in Balakrishnan and Aggarwala [86, p. 71–81] and Burkschat et al. [234].

2.4 Marginal Distributions

Using the results of the preceding sections, explicit representations for the marginal distributions of progressively Type-II censored order statistics can be established. First, we notice that a right censored progressively Type-II censored sample can be seen as progressively Type-II censored order statistics from the same distribution with a modified censoring scheme. Thus, right censored progressively censored samples can always be seen as a complete progressively censored sample with a modified censoring scheme. In particular, we have the following result (see Balakrishnan and Aggarwala [86]).

Theorem 2.4.1. Let $X_{1:m:n}, \dots, X_{m:m:n}$ be progressively Type-II censored order statistics from a cumulative distribution function F with censoring scheme $\mathscr{R} = (R_1, \dots, R_m)$.

Then, for $1 \le r \le m$, the right censored sample $X_{1:m:n}^{\mathscr{R}}, \dots, X_{r:m:n}^{\mathscr{R}}$ can be seen as a complete sample of progressively Type-II censored order statistics $X_{1:r:n}^{\mathscr{R}_r}, \dots, X_{r:r:n}^{\mathscr{R}_r}$ from the same population with censoring scheme $\mathscr{R}_r = (R_1, \dots, R_{r-1}, \gamma_r - 1)$.

Proof. The iterative construction of progressively Type-II censored order statistics presented in Procedure 1.1.3 yields directly the above property. The only property that has to be shown is the particular structure of the censoring scheme. But, according to the construction process 1.1.3, γ_r denotes the number of items in the experiment before the rth failure. Thus, stopping the experiment after the rth failure is equivalent to removing the remaining $\gamma_r - 1$ units.

Alternatively, the iterative construction in Theorem 2.3.6 can be used for this purpose. □

Thus, right censoring of progressively Type-II censored samples results in the same model with a reduced number of observations and a modified censoring scheme. In particular, we can apply the preceding results and obtain, for example, the joint marginal density function of $X_{1:m:n}, \dots, X_{r:m:n}$ as

$$f^{X_{1:m:n},\dots,X_{r:m:n}}(\mathbf{x}_r) = \prod_{j=1}^{r-1} \left[\gamma_j f(x_j)(1-F(x_j))^{R_j}\right] \gamma_r f(x_r)(1-F(x_r))^{\gamma_r - 1},$$

$$x_1 \le \dots \le x_r.$$

In particular, Theorem 2.4.1 illustrates that any progressively Type-II censored order statistic can be seen as a maximal progressively Type-II censored order statistic with an appropriately chosen censoring scheme.

In order to calculate the marginal distributions, we consider the exponential case first. The presentation of the following results uses the notation proposed in Kamps and Cramer [503]. An alternative but equivalent representation has been established in Balakrishnan et al. [132] using the integral identity (2.31) (see also Balakrishnan [84] and Nagaraja [667]). This representation has also been exploited in many papers.

2.4.1 Exponential Distribution

The marginal distributions of exponential progressively Type-II censored order statistics can be derived using the sum representation (2.13). This expression shows that we are interested in finding distributions of sums of independent but not necessarily identically distributed exponential random variables. An important point in the following derivations is that the γ's cannot be equal. Such problems have been considered earlier by, e.g., Likeš [597] and Kamps [497] (see also Johnson et al. [483, p. 552]). This type of distribution is called hyperexponential distribution or generalized Erlang distribution (see Johnson and Kotz [482, p. 222]). A review on this topic including various applications of hyperexponential distributions is provided by Botta et al. [217]. This yields directly the following result.

Theorem 2.4.2 (Kamps and Cramer [503]). Let $Z_{1:m:n}, \ldots, Z_{m:m:n}$ be standard exponential progressively Type-II censored order statistics. Then,

$$F^{Z_{r:m:n}}(t) = 1 - \Big(\prod_{i=1}^{r} \gamma_i\Big) \sum_{j=1}^{r} \frac{1}{\gamma_j} a_{j,r} \mathrm{e}^{-\gamma_j t}, \quad t > 0, \tag{2.19}$$

where $a_{j,r} = \prod_{\substack{i=1\\ i\neq j}}^{r} \frac{1}{\gamma_i - \gamma_j}$, $1 \le j \le r \le n$. The density function of $Z_{r:m:n}$ is given by

$$f^{Z_{r:m:n}}(t) = \Big(\prod_{j=1}^{r} \gamma_j\Big) \sum_{j=1}^{r} a_{j,r} \mathrm{e}^{-\gamma_j t}, \quad t > 0. \tag{2.20}$$

Schenk [782] considered multiply censored samples of progressively Type-II censored order statistics (see also Cramer [287]). He derived expressions for the corresponding density functions. His derivations are based on the Markov property of the thinned sample $Z_{k_1:m:n}, \ldots, Z_{k_\ell:m:n}$ with $1 \le k_1 < k_2 < \cdots < k_\ell \le m$ (see Sect. 2.5.1). Using the sum representation (2.13), we find

$$Z_{k_2:m:n} = \sum_{j=1}^{k_2} \frac{1}{\gamma_j} S_j^{\mathscr{R}} = Z_{k_1:m:n} + \sum_{j=k_1+1}^{k_2} \frac{1}{\gamma_j} S_j^{\mathscr{R}}.$$

Thus, the cumulative distribution function of $Z_{j_2:m:n}$, given $Z_{j_1:m:n} = s$, is given by $P\big(\sum_{j=k_1+1}^{k_2} \frac{1}{\gamma_j} S_j^{\mathscr{R}} \le t - s\big)$, $t \ge s$. Hence, the density function follows from (2.20) as

$$f^{Z_{k_2:m:n}|Z_{k_1:m:n}}(t|s) = \Big(\prod_{j=k_1+1}^{k_2} \gamma_j\Big) \sum_{j=k_1+1}^{k_2} a_{j,k_2}^{(k_1)} \mathrm{e}^{-\gamma_j(t-s)}, \quad t > s > 0,$$

where $a_{j,k_2}^{(k_1)} = \prod_{\substack{\nu=k_1+1\\ \nu\neq j}}^{k_2} \frac{1}{\gamma_\nu - \gamma_j}$. Combining these expressions, we arrive at the joint density function of two exponential progressively Type-II censored order statistics given in Kamps and Cramer [503] ($t > s > 0$):

$$\begin{aligned} f^{Z_{k_1:m:n},Z_{k_2:m:n}}(s,t) &= f^{Z_{k_1:m:n}}(s) f^{Z_{k_2:m:n}|Z_{k_1:m:n}}(t|s) \\ &= \Big(\prod_{j=1}^{k_2} \gamma_j\Big) \sum_{i=1}^{k_1} \sum_{j=k_1+1}^{k_2} a_{i,k_1} a_{j,k_2}^{(k_1)} \mathrm{e}^{-\gamma_j(t-s)} \mathrm{e}^{-\gamma_i s}. \end{aligned}$$

A repeated application of the preceding result yields the joint density function of the multiply censored sample:

$$\begin{aligned} &f^{Z_{k_1:m:n},\dots,Z_{k_\ell:m:n}}(x_{k_1},\dots,x_{k_\ell}) \\ &\quad = \prod_{i=1}^{\ell} \Big[\Big(\prod_{j=k_{i-1}+1}^{k_i} \gamma_j\Big) \sum_{j=k_{i-1}+1}^{k_i} a_{j,k_i}^{(k_{i-1})} \mathrm{e}^{-\gamma_j(x_{k_i} - x_{k_{i-1}})}\Big], \end{aligned} \tag{2.21}$$

where $k_0 = 0$, $0 = x_0 \le x_{k_1} \le \cdots \le x_{k_\ell}$, and

$$a_{j,k_i}^{(k_{i-1})} = \prod_{\substack{\nu=k_{i-1}+1\\ \nu\neq j}}^{k_i} \frac{1}{\gamma_\nu - \gamma_j}, \quad k_{i-1}+1 \le j \le k_i, 1 \le i \le \ell. \tag{2.22}$$

This result can be directly applied to a general progressively Type-II censored sample $X_{r+1:m:n}^{\mathscr{R}_{\lhd r}}, \dots, X_{m:m:n}^{\mathscr{R}_{\lhd r}}$ with the left truncated censoring scheme $\mathscr{R}_{\lhd r} = (R_{r+1},\dots,R_m) \in \mathscr{C}_{m-r,n-r}^{m-r}$. The corresponding density function is given by

$$f^{Z_{r+1:m:n},\dots,Z_{m:m:n}}(x_{r+1},\dots,x_m)$$
$$= \binom{n}{r}\Big(\prod_{j=r+1}^{m}\gamma_j\Big)(1-\mathrm{e}^{-x_{r+1}})^r \exp\Big\{-\sum_{j=r+1}^{m}(R_j+1)x_j\Big\},$$
$$0 \le x_{r+1} \le \cdots \le x_m. \tag{2.23}$$

2.4.2 Uniform Distribution

Due to its importance, it is useful to present the representations of the marginal density functions and cumulative distribution functions for uniform progressively Type-II censored order statistics. They can be taken directly from Theorem 2.4.2 using a quantile transformation.

Corollary 2.4.3. Let $U_{1:m:n},\dots,U_{m:m:n}$ be uniform progressively Type-II censored order statistics. Then, for $1 \le r \le m$,

$$F^{U_{r:m:n}}(t) = 1 - \Big(\prod_{i=1}^{r}\gamma_i\Big)\sum_{j=1}^{r}\frac{1}{\gamma_j}a_{j,r}(1-t)^{\gamma_j}, \quad t \in [0,1].$$

The density function of $U_{r:m:n}$ is given by

$$f^{U_{r:m:n}}(t) = \Big(\prod_{j=1}^{r}\gamma_j\Big)\sum_{j=1}^{r}a_{j,r}(1-t)^{\gamma_j-1}, \quad t \in [0,1]. \tag{2.24}$$

2.4.3 General Distributions

Using the quantile transformation result 2.1.1, the preceding results can be directly applied to arbitrary distributions. For brevity, we present only the expressions in the univariate and bivariate case. From Theorem 2.4.2, we obtain the following result.

Corollary 2.4.4. Let $X_{1:m:n},\dots,X_{m:m:n}$ be progressively Type-II censored order statistics from a cumulative distribution function F. Then, for $1 \le r \le m$,

$$F^{X_{r:m:n}}(t) = 1 - \Big(\prod_{i=1}^{r}\gamma_i\Big)\sum_{j=1}^{r}\frac{1}{\gamma_j}a_{j,r}(1-F(t))^{\gamma_j}, \quad t \in \mathbb{R}. \tag{2.25}$$

From (2.25), we find with $t \to -\infty$ the identity

$$1 = \Big(\prod_{i=1}^{r} \gamma_i\Big) \sum_{j=1}^{r} \frac{1}{\gamma_j} a_{j,r}. \tag{2.26}$$

Applying this identity and writing $F_{1:\gamma_j} = 1 - (1-F)^{\gamma_j}$, we get a representation of the cumulative distribution function in terms of distributions of minima as

$$F^{X_{r:m:n}}(t) = \Big(\prod_{i=1}^{r} \gamma_i\Big) \sum_{j=1}^{r} \frac{1}{\gamma_j} a_{j,r} F_{1:\gamma_j}(t). \tag{2.27}$$

Noticing that for order statistics the identity

$$\Big(\prod_{i=1}^{r} \gamma_i\Big) \frac{1}{\gamma_j} a_{j,r} = (-1)^{r-j} \binom{n}{j-1} \binom{n-j}{r-j}$$

holds, we find

$$F_{r:n}(t) = \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{n-r} \binom{n}{j} F_{1:j}(t).$$

This identity is given, for instance, in David and Nagaraja [327, p. 46] and Arnold et al. [58, p. 113] in terms of moments. For moments of order statistics, this result is due to Srikantan [822].

For absolutely continuous distributions, we have the following representation of the density function.

Corollary 2.4.5. Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be progressively Type-II censored order statistics from an absolutely continuous cumulative distribution function F with density function f. Then, for $1 \le r \le m$,

$$f^{X_{r:m:n}}(t) = f(t) \Big(\prod_{i=1}^{r} \gamma_i\Big) \sum_{j=1}^{r} a_{j,r} (1 - F(t))^{\gamma_j - 1}, \quad t \in \mathbb{R}. \tag{2.28}$$

For $1 \le k_1 < k_2 \le m$ and $t > s$, the bivariate density function is given by

$$\begin{aligned} &f^{X_{k_1:m:n}, X_{k_2:m:n}}(s,t) \\ &= f(s) f(t) \Big(\prod_{j=1}^{k_2} \gamma_j\Big) \sum_{i=1}^{k_1} \sum_{j=k_1+1}^{k_2} a_{i,k_1} a_{j,k_2}^{(k_1)} \Big[\frac{1-F(t)}{1-F(s)}\Big]^{\gamma_j - 1} (1 - F(s))^{\gamma_i - 2}. \end{aligned} \tag{2.29}$$

Remark 2.4.6. For order statistics, the representations given above simplify to the well-known expressions

$$F_{r:n}(t) = \sum_{j=r}^{n} \binom{n}{j} F^{j}(t)(1-F(t))^{n-j}, \quad t \in \mathbb{R},$$
$$f_{r:n}(t) = r\binom{n}{r} F^{r-1}(t)(1-F(t))^{n-r} f(t), \quad t \in \mathbb{R} \tag{2.30}$$

(see, e.g., Arnold et al. [58], David and Nagaraja [327]).

For the cumulative distribution function and $x_1 \leq x_2$, the expression (2.29) yields by integration the following representation which holds for any baseline cumulative distribution function F:

$$F^{X_{k_1:m:n},X_{k_2:m:n}}(x_1,x_2) = F^{X_{k_1:m:n}}(x_1)$$
$$-\Big(\prod_{j=1}^{k_2}\gamma_j\Big)\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_2} \frac{a_{i,k_1}a_{j,k_2}^{(k_1)}}{\gamma_j(\gamma_i-\gamma_j)}\overline{F}^{\gamma_j}(x_2)\Big[1-\overline{F}^{\gamma_i-\gamma_j}(x_1)\Big].$$

The result can be established by using the relation

$$F^{X_{k_1:m:n},X_{k_2:m:n}}(x_1,x_2) = F^{X_{k_1:m:n}}(x_1) - P(X_{k_1:m:n} \leq x_1, X_{k_2:m:n} > x_2).$$

Assuming uniform progressively Type-II censored order statistics and using (2.29), the probability on the right-hand side reads

$$c_{k_2-1}\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_2} a_{i,k_1}a_{j,k_2}^{(k_1)} \int_0^{x_1}\int_{x_2}^{1}(1-t)^{\gamma_i-\gamma_j-1}(1-s)^{\gamma_j-1}ds\,dt$$
$$= c_{k_2-1}\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_2} \frac{a_{i,k_1}a_{j,k_2}^{(k_1)}}{\gamma_j(\gamma_i-\gamma_j)}(1-x_2)^{\gamma_j}\Big[1-(1-x_1)^{\gamma_i-\gamma_j}\Big].$$

Using the quantile transformation and (2.2), we arrive at the desired representation. Notice that, for $x_2 < x_1$, one has $F^{X_{k_1:m:n},X_{k_2:m:n}}(x_1,x_2) = F^{X_{k_2:m:n}}(x_2)$.

Multiply Censored Progressively Type-II Censored Order Statistics

Similar results can be established for multiply censored samples (see (2.21) and, for generalized order statistics, Cramer [287]). The joint density function of progressively Type-II censored order statistics $X_{k_1:m:n},\ldots,X_{k_\ell:m:n}$ is given by

$$f^{X_{k_1:m:n},\dots,X_{k_\ell:m:n}}(x_{k_1},\dots,x_{k_\ell})$$
$$=\prod_{i=1}^{\ell}\Big[\frac{f(x_{k_i})}{1-F(x_{k_i})}\Big(\prod_{j=k_{i-1}+1}^{k_i}\gamma_j\Big)\sum_{j=k_{i-1}+1}^{k_i}a_{j,k_i}^{(k_i-1)}\Big(\frac{1-F(x_{k_i})}{1-F(x_{k_{i-1}})}\Big)^{\gamma_j}\Big],$$

where $k_0 = 0$, $x_0 = -\infty$, $x_{k_1} \le \cdots \le x_{k_\ell}$. For order statistics, the expression simplifies to that given in Kong [543].

In the special case of a general progressively Type-II censored sample $X_{r+1:m:n}^{\mathscr{R}_{\lhd r}},\dots,X_{m:m:n}^{\mathscr{R}_{\lhd r}}$ with the left truncated censoring scheme $\mathscr{R}_{\lhd r} = (R_{r+1},\dots,R_m) \in \mathscr{C}_{m-r,n-r}^{m-r}$, the corresponding joint density function is given by ([see 86, p. 10] and (2.21) for the exponential distribution)

$$f(x_{r+1},\dots,x_m)=\binom{n}{r}F^r(x_{r+1})\Big(\prod_{j=r+1}^{m}\gamma_j f(x_j)(1-F(x_j))^{R_j}\Big),$$
$$x_{r+1}\le\cdots\le x_m.$$

An Important Recurrence Relation

The preceding results yield the following connection between cumulative distribution functions and density functions.

Corollary 2.4.7. For $r \in \{1,\dots,m-1\}$,

$$F^{X_{r:m:n}}(t)-F^{X_{r+1:m:n}}(t)=\frac{1}{\gamma_{r+1}}(1-F(t))f^{U_{r+1:m:n}}(F(t)),\quad t\in\mathbb{R}.$$

Proof. From (2.25), we obtain

$$F^{X_{r:m:n}}(t)=1-\Big(\prod_{i=1}^{r}\gamma_i\Big)\sum_{j=1}^{r}\frac{1}{\gamma_j}a_{j,r}(1-F(t))^{\gamma_j}.$$

Since $(\gamma_{r+1}-\gamma_j)a_{j,r+1}=a_{j,r}$, $1\le j\le r$, we get

$$F^{X_{r:m:n}}(t)-F^{X_{r+1:m:n}}(t)$$
$$=\Big(\prod_{i=1}^{r}\gamma_i\Big)\Big(a_{r+1,r+1}(1-F(t))^{\gamma_{r+1}}+\sum_{j=1}^{r}a_{j,r+1}(1-F(t))^{\gamma_j}\Big)$$
$$=\frac{1}{\gamma_{r+1}}(1-F(t))f^{U_{r+1:m:n}}(F(t)),\quad t\in\mathbb{R},$$

which yields the desired result. □

For order statistics, the above relation simplifies to

$$F^{X_{r:n}}(t) = F^{X_{r+1:n}}(t) + \frac{1}{n-r+1}(1-F(t))f^{U_{r+1:n}}(F(t))$$
$$= F^{X_{r+1:n}}(t) + \binom{n}{r}F^r(t)(1-F(t))^{n-r},$$

which can be found in David and Shu [328].

An Alternative Approach to Derive the Marginals

Balakrishnan et al. [132] presented an alternative approach to derive the marginals of progressively Type-II censored order statistics. They tackled the problem through an explicit evaluation of the resulting integrals as shown in Lemma 1 in Balakrishnan et al. [132]. Using the notation introduced in (2.22), a version of this lemma adapted to the present setting is as follows.

Lemma 2.4.8. Let $\mathscr{R} = (R_1, \ldots, R_{r+1})$ be a censoring scheme, $r \geq k+1 \geq 1$. Then, for a cumulative distribution function F with density function f and $t \leq y$, the following identity holds:

$$\int_t^y \int_t^{x_r} \cdots \int_t^{x_{k+2}} \prod_{i=k+1}^{r} \Big[f(x_i)\overline{F}^{R_i}(x_i) \Big] dx_{k+1} \ldots dx_r$$
$$= \sum_{j=k+1}^{r+1} a_{j,r+1}^{(k)} \overline{F}(t)^{\gamma_{k+1}-\gamma_j} \overline{F}(y)^{\gamma_j - \gamma_{r+1}}. \quad (2.31)$$

For $y \to \infty$, we get

$$\int_t^\infty \int_t^{x_r} \cdots \int_t^{x_{k+2}} \prod_{i=k+1}^{r} \Big[f(x_i)\overline{F}^{R_i}(x_i) \Big] dx_{k+1} \ldots dx_r$$
$$= a_{r+1,r+1}^{(k)} \overline{F}(t)^{\gamma_{k+1}-\gamma_{r+1}}.$$

This integral representation will be helpful in many settings. For instance, it will be used later to derive the power function of precedence-type tests under Lehmann alternative [see (21.9)].

Connection of Marginals to Interpolation Polynomials

Cramer [289] established a connection of one-dimensional marginal density functions and cumulative distribution functions to divided differences and Lagrangian interpolation polynomials (see, e.g., Neumaier [679]). For $t \geq 0$, let

$$h_t : [0,\infty) \longrightarrow [0,\infty) \quad \text{be defined by } h_t(x) = t^x, \quad x \geq 0. \tag{2.32}$$

Then, the divided differences $h_t[x_j,\ldots,x_\nu]$ of order $\nu - j$ at $x_1 > \cdots > x_m$ are defined by

$$\begin{aligned} h_t[x_j] &= h_t(x_j), \\ h_t[x_j,\ldots,x_\nu] &= \frac{h_t[x_{j+1},\ldots,x_\nu] - h_t[x_j,\ldots,x_{\nu-1}]}{x_\nu - x_j}, \end{aligned}$$

for $1 \leq j < \nu \leq m$. Then,

$$f^{U_{r:m:n}}(t) = (-1)^{r-1} h_{1-t}[\gamma_1 - 1,\ldots,\gamma_r - 1], \quad t \in [0,1].$$

Cramer [289] showed that the survival function of $X_{r:m:n}$ can be written as a specific Lagrangian interpolation polynomial $\mathcal{P}_r^{F(t)}$ evaluated at the point zero

$$\begin{aligned} \mathcal{P}_r^{F(t)}(0) &= 1 - F^{X_{r:m:n}}(t) = \sum_{j=1}^{r} \frac{\prod\limits_{\nu=1,\nu\neq j}^{r} (0-\gamma_\nu)}{\prod\limits_{\nu=1,\nu\neq j}^{r} (\gamma_j - \gamma_\nu)} \overline{F}^{\gamma_j}(t) \\ &= \sum_{j=1}^{r} \frac{\prod\limits_{\nu=1,\nu\neq j}^{r} (0-\gamma_\nu)}{\prod\limits_{\nu=1,\nu\neq j}^{r} (\gamma_j - \gamma_\nu)} h_{1-F(t)}(\gamma_j). \end{aligned}$$

Thus, $\mathcal{P}_r^{F(t)}$ interpolates the function $h_{1-F(t)}$ given in (2.32) at the points $\gamma_1,\ldots,\gamma_r$. Precisely, the evaluation of the polynomial at zero is an extrapolation since zero does not belong to the range of the γ_j's. In particular, this shows that the cumulative distribution function of a progressively Type-II censored order statistic can be understood as a Lagrangian interpolation polynomial.

2.5 Conditional Distributions

2.5.1 *Markov Property*

From the joint density function of progressively Type-II censored order statistics given in (2.4), the Markov property can be easily derived for an absolutely continuous cumulative distribution function F. However, this property holds even for continuous cumulative distribution function.

Theorem 2.5.1. Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be progressively Type-II censored order statistics from a continuous cumulative distribution function F.

Then, $X_{1:m:n}, \ldots, X_{m:m:n}$ form a Markov chain with transition probabilities $(2 \leq r \leq m)$

$$P(X_{r:m:n} \leq t | X_{r-1:m:n} = s) = 1 - \left(\frac{1-F(t)}{1-F(s)}\right)^{\gamma_r}, \quad s \leq t \text{ with } F(s) < 1.$$

Proof. First, we consider the uniform distribution. According to representation (2.16), we find

$$U_{r:m:n} = 1 - U_r^{1/\gamma_r}(1 - U_{r-1:m:n}), \quad 2 \leq r \leq m. \tag{2.33}$$

Using the independence of U_j, $1 \leq j \leq m$, the progressively Type-II censored order statistics $U_{1:m:n}, \ldots, U_{r:m:n}$ form a Markov chain with $(s \leq t < 1)$

$$P(U_{r:m:n} \leq t | U_{r-1:m:n} = s) = P\left(U_r^{1/\gamma_r} \geq \frac{1-t}{1-s}\right) = 1 - \left(\frac{1-t}{1-s}\right)^{\gamma_r}.$$

Using the properties of the quantile function in Lemma A.2.2 and Theorem 2.3.6, we obtain from the continuity of F and (2.33) that

$$\begin{aligned} X_{r:m:n} &= F^{\leftarrow}(U_{r:m:n}) = F^{\leftarrow}(1 - U_r^{1/\gamma_r}[1 - U_{r-1:m:n}]) \\ &= F^{\leftarrow}(1 - U_r^{1/\gamma_r}[1 - F(F^{\leftarrow}(U_{r-1:m:n}))]) \\ &= F^{\leftarrow}(1 - U_r^{1/\gamma_r}[1 - F(X_{r-1:m:n})]). \end{aligned} \tag{2.34}$$

Therefore, the independence of U_r and $X_{1:m:n}, \ldots, X_{r-1:m:n}$ yields for $s_1 \leq \cdots \leq s_{r-1} \leq t$ with $F(s_{r-1}) < 1$, the conditional cumulative distribution function

$$\begin{aligned} &P(X_{r:m:n} \leq t | X_{j:m:n} = s_j, j = 1, \ldots, r-1) \\ &\quad = P(F^{\leftarrow}(1 - U_r^{1/\gamma_r}[1 - F(s_{r-1})]) \leq t) = P\left(U_r^{1/\gamma_r} \geq \frac{1-F(t)}{1-F(s_{r-1})}\right). \end{aligned}$$

This proves the desired result. □

As proved by Balakrishnan and Dembińska [96], the above result does not hold for noncontinuous distributions (see also Tran [854]). More details on the noncontinuous case are provided in Sect. 2.8. For order statistics, the Markov property is a well-known property (see Arnold et al. [58] and David and Nagaraja [327]). Obviously, Theorem 2.5.1 can be extended to the following result (see Balakrishnan and Aggarwala [86, p. 15]).

Theorem 2.5.2. Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be progressively Type-II censored order statistics from a continuous cumulative distribution function F. Then, conditional on $X_{r-1:m:n} = x_{r-1}$, the random variables $X_{r:m:n}, \ldots, X_{m:m:n}$ are progressively Type-II censored order statistics from a left truncated cumulative distribution function

$$G_{x_{r-1}}(y) = \frac{F(y) - F(x_r)}{1 - F(x_{r-1})}, \quad x_{r-1} \le y, F(x_{r-1}) < 1, \tag{2.35}$$

and left (truncated) censoring scheme $\mathscr{R}_{\lhd r-1} = (R_r, \ldots, R_m)$.

Proof. Applying (2.34) to the random variables $X_{r:m:n}, \ldots, X_{m:m:n}$, we find

$$\begin{aligned} X_{r:m:n} &= F^{\leftarrow}(1 - U_r^{1/\gamma_r}[1 - F(X_{r-1:m:n})]), \\ &\vdots \\ X_{m:m:n} &= F^{\leftarrow}\Big(1 - \prod_{j=r}^{m} U_j^{1/\gamma_j}[1 - F(X_{r-1:m:n})]\Big). \end{aligned}$$

Since the quantile function of the truncated cumulative distribution function in (2.35) is given by

$$G_{x_{r-1}}^{\leftarrow}(t) = F^{\leftarrow}(1 - (1-t)(1 - F(x_{r-1}))), \quad t \in (0,1),$$

we find for $\ell = r, \ldots, m$,

$$X_{\ell:m:n} = F^{\leftarrow}\Big(1 - \prod_{j=r}^{\ell} U_j^{1/\gamma_j}[1 - F(X_{r-1:m:n})]\Big) = G_{X_{r-1:m:n}}^{\leftarrow}\Big(1 - \prod_{j=r}^{\ell} U_j^{1/\gamma_j}\Big).$$

Therefore, conditional on $X_{r-1:m:n} = x_{r-1}$, Theorem 2.3.6 yields the assertion. Notice that the parameters $\gamma_r, \ldots, \gamma_m$ yield the left truncated censoring scheme $\mathscr{R}_{\lhd r-1} = (R_r, \ldots, R_m)$. □

For order statistics, the corresponding result is due to Scheffé and Tukey [781] (see also Arnold et al. [58, p. 25–26] and David and Nagaraja [327]). Horn and Schlipf [450] utilized this representation to develop an efficient algorithm for the generation of Type-II doubly censored data.

The Markov property yields the following factorization of the density function of $\mathbf{X}^{\mathscr{R}}$. For $2 \le r \le m$, the conditional density function $f_{r|r-1:m:n}(\cdot|s)$ of $X_{r:m:n}$, given $X_{r-1:m:n} = s$, is defined by

$$f_{r|r-1:m:n}(t|s) = \begin{cases} \gamma_r \frac{f(t)}{1-F(s)} \cdot \left(\frac{1-F(t)}{1-F(s)}\right)^{\gamma_r - 1}, & s \le t \text{ with } F(s) < 1, \\ f_{r:m:n}(t), & \text{otherwise.} \end{cases}$$

Of course, $f_{1|0:m:n} = f_{1:m:n}$ is the marginal density function of $X_{1:m:n}$.

Corollary 2.5.3. The density function of a progressively Type-II censored sample $\mathbf{X}^{\mathscr{R}}$ factorizes as follows:

$$f^{\mathbf{X}^{\mathscr{R}}}(\mathbf{x}_m) = \prod_{r=1}^{m} f_{r|r-1:m:n}(x_r|x_{r-1}), \quad x_1 \le \cdots \le x_m. \tag{2.36}$$

According to Cramer [287], a conditional P^F-density function exists provided that the population cumulative distribution function F is continuous. It is given by

$$f_{r|r-1:m:n}(t|s) = \frac{\gamma_r}{1 - F(s)} \cdot \left(\frac{1 - F(t)}{1 - F(s)}\right)^{\gamma_r - 1} \quad P^F \text{a.e.} \tag{2.37}$$

2.5.2 Distributions of Generalized Spacings

The preceding results can be used to calculate the density functions of generalized spacings (so-called subranges or contrasts), i.e., of the (r, s)-spacing

$$S_{r,s}^{\star\mathscr{R}} = X_{r:m:n} - X_{s:m:n}, \quad 1 \le s < r \le m. \tag{2.38}$$

From Lemma 3 of Kamps and Cramer [503], we get the following expressions:

$$\begin{aligned} f^{S_{r,s}^{\star\mathscr{R}}}(w) &= \prod_{j=1}^{s} \gamma_j \int_{\mathbb{R}} \Big(\sum_{i=r+1}^{s} a_{i,s}^{(r)} \Big(\frac{1 - F(v + w)}{1 - F(v)}\Big)^{\gamma_i} \Big) \\ &\quad \times \Big(\sum_{i=1}^{r} a_{i,r} (1 - F(v))^{\gamma_i} \Big) \frac{f(v)}{1 - F(v)} \, \frac{f(v + w)}{1 - F(v + w)} \, dv, \\ F^{S_{r,s}^{\star\mathscr{R}}}(w) &= 1 - \int_{\mathbb{R}} H_{r,s}\Big(\frac{1 - F(v + w)}{1 - F(v)}\Big) dF^{X_{r:m:n}}(v), \end{aligned} \tag{2.39}$$

where the function H is defined by $H_{r,s}(z) = \left(\prod_{j=r+1}^{s} \gamma_j\right) \sum_{i=r+1}^{s} a_{i,s}^{(r)} \frac{1}{\gamma_i} z^{\gamma_i}$, $z \in [0, 1]$ [see also (2.29)].

2.5.3 Block Independence of Progressively Type-II Censored Order Statistics

In this section, we study the distribution of a progressively Type-II censored sample randomly divided into two blocks by a threshold $T \in \mathbb{R}$.

Lemma 2.5.4. For a fixed time T, let D denote the number of progressively Type-II censored order statistics that do not exceed T, i.e.,

$$D = \sum_{j=1}^{m} \mathbb{1}_{(-\infty,T]}(X_{j:m:n}). \tag{2.40}$$

Then,

$$\begin{aligned} P(D = 0) &= (1 - F(T))^n, \\ P(D = d) &= \Big(\prod_{i=1}^{d} \gamma_i\Big) \sum_{j=1}^{d+1} a_{j,d+1}(1 - F(T))^{\gamma_j}, \quad d = 1, \dots, m-1, \\ P(D = m) &= F^{X_{m:m:n}}(T) = 1 - \Big(\prod_{i=1}^{m} \gamma_i\Big) \sum_{j=1}^{m} \frac{1}{\gamma_j} a_{j,m}(1 - F(T))^{\gamma_j}. \end{aligned} \tag{2.41}$$

Proof. For $d = 0$, we have $P(D = 0) = P(X_{1:m:n} > T) = (1 - F(T))^n$.
Let $d \in \{1, \dots, m-1\}$. From the definition of D, we have

$$\begin{aligned} P(D = d) =& P(X_{d:m:n} \le T < X_{d+1:m:n}) \\ =& P(X_{d:m:n} \le T) - P(X_{d+1:m:n} \le T) = F^{X_{d:m:n}}(T) - F^{X_{d+1:m:n}}(T). \end{aligned}$$

From Corollary 2.4.7, we conclude for $d \in \{1, \dots, m-1\}$,

$$F^{X_{d:m:n}}(T) - F^{X_{d+1:m:n}}(T) = \frac{1}{\gamma_{d+1}}(1 - F(T)) f^{U_{d+1:m:n}} F(T)). \tag{2.42}$$

An application of (2.24), i.e.,

$$f^{U_{d+1:m:n}}(t) = \Big(\prod_{i=1}^{d+1} \gamma_i\Big) \sum_{j=1}^{d+1} a_{j,d+1}(1 - t)^{\gamma_j - 1} \quad t \in (0, 1),$$

proves the desired result. The case $d = m$ follows from $P(D = m) = P(X_{m:m:n} \le T) = F^{X_{m:m:n}}(T)$ and (2.25). □

In the case of order statistics, the distribution of D simplifies considerably. In particular, we have (see Iliopoulos and Balakrishnan [469]) that D has a binomial distribution with parameters n and $F(T)$, i.e.,

$$P(D = d) = \binom{n}{d} F^d(T)(1 - F(T))^{n-d}, \quad d = 0, \dots, n.$$

This follows directly from (2.42) in the above proof.

The following conditional independence result for progressively Type-II censored order statistics is due to Iliopoulos and Balakrishnan [469].

Theorem 2.5.5. Let $d \in \{1, \dots, m-1\}$, F be a cumulative distribution function, $\mathscr{R} = (R_1, \dots, R_m)$ be a censoring scheme, and $\mathbf{K}_d = (K_1, \dots, K_d)$ be a discrete random vector on the Cartesian product $\times_{j=1}^d \{0, \dots, R_j\}$ with probability mass function

$$p^{\mathbf{K}_d}(\mathbf{k}_d) = \frac{1}{P(D = d)} F^{\eta_1(d)}(T)(1 - F(T))^{n-\eta_1(d)} \prod_{i=1}^d \frac{\gamma_i}{\eta_i(d)} \binom{R_i}{k_i},$$

where $\eta_i(d) = \sum_{j=i}^d (k_j + 1)$, $1 \le i \le d$.

Conditional on $D = d$, the two random vectors $(X_{1:m:n}^{\mathscr{R}}, \dots, X_{d:m:n}^{\mathscr{R}})$ and $(X_{d+1:m:n}^{\mathscr{R}}, \dots, X_{m:m:n}^{\mathscr{R}})$ are independent with

$$(X_{1:m:n}^{\mathscr{R}}, \dots, X_{d:m:n}^{\mathscr{R}}) \stackrel{\mathsf{d}}{=} \left(V_{1:d:\kappa_d}^{\mathscr{K}}, \dots, V_{d:d:\kappa_d}^{\mathscr{K}}\right)$$

$$(X_{d+1:m:n}^{\mathscr{R}}, \dots, X_{m:m:n}^{\mathscr{R}}) \stackrel{\mathsf{d}}{=} \left(W_{1:m-d:\gamma_d}, \dots, W_{m-d:m-d:\gamma_d}\right)$$

where $\mathscr{K} = \mathbf{K}_d$, $\kappa_d = \sum_{j=1}^d (1 + K_j)$, and

(i) $V_1, \dots, V_n$ are IID random variables with the right truncated cumulative distribution function F_T given by

$$F_T(t) = \frac{F(t)}{F(T)}, \quad t \le T,$$

(ii) $W_1, \dots, W_{\gamma_d}$ are IID random variables with left truncated cumulative distribution function G_T

$$G_T(t) = 1 - \frac{1 - F(t)}{1 - F(T)}, \quad t \ge T.$$

Remark 2.5.6. Notice that the sample size κ_d for the progressively Type-II censored order statistics $V_{1:d:\kappa_d}^{\mathscr{K}}, \dots, V_{d:d:\kappa_d}^{\mathscr{K}}$ is a random variable. This representation means that the distribution of $(X_{1:m:n}^{\mathscr{R}}, \dots, X_{d:m:n}^{\mathscr{R}})$, given $D = d$, is a mixture of distributions of progressively Type-II censored order statistics with mixing distribution $p^{\mathscr{K}}$. It is well known that the right truncation of progressively

Type-II censored order statistics does not result in progressively Type-II censored order statistics from the corresponding right truncated distribution (see, for example, Balakrishnan and Aggarwala [86]). This is due to the fact that those observations (progressively) censored before T could have values larger than T.

Proof of Theorem 2.5.5. First, notice that we can restrict ourselves to the uniform distribution replacing T by $F(T)$. Then, notice that the conditional density function of $\mathbf{U}^{\mathscr{R}} = (U_{1:m:n}, \ldots, U_{m:m:n})$, given $D = d$, is given by

$$f^{\mathbf{U}^{\mathscr{R}}|D=d}(\mathbf{x}_m) = \frac{1}{P(D=d)} f^{\mathbf{U}^{\mathscr{R}}}(\mathbf{x}_m) \mathbb{1}_{[x_d, x_{d+1})}(F(T)).$$

Decomposing the joint density function of $U_{1:m:n}, \ldots, U_{m:m:n}$, we obtain

$$\begin{aligned} f^{\mathbf{U}^{\mathscr{R}}}(\mathbf{x}_m) &= \prod_{i=1}^{m} \gamma_i \prod_{j=1}^{m} (1-x_i)^{R_i} \\ &= \left[\prod_{i=1}^{d} \gamma_i \prod_{i=1}^{d} (1-x_i)^{R_i}\right] \times \left[\prod_{i=d+1}^{m} \gamma_i \prod_{i=d+1}^{m} (1-x_i)^{R_i}\right] \\ &= \left(\prod_{i=1}^{d} \gamma_i\right) h_1(\mathbf{x}_d) \cdot h_2(x_{d+1}, \ldots, x_m). \end{aligned}$$

Now, we consider h_1. Let $f_{\mathbf{k}_d}$ denote the joint density function of progressively Type-II censored order statistics $X_{1:m:n}, \ldots, X_{d:m:n}$ from a right truncated uniform distribution (at $F(T)$) and censoring scheme $\mathbf{k}_d = (k_1, \ldots, k_d)$. Then, by using the binomial theorem, we find for $0 < x_1 < \cdots < x_d \le F(T)$,

$$\begin{aligned} h_1(\mathbf{x}_d) &= \prod_{i=1}^{d} (1-x_i)^{R_i} = \prod_{i=1}^{d} (1 - F(T) + F(T) - x_i)^{R_i} \\ &= \prod_{i=1}^{d} \left\{ \sum_{k_i=0}^{R_i} \binom{R_i}{k_i} (F(T) - x_i)^{k_i} (1-F(T))^{R_i - k_i} \right\} \\ &= \prod_{i=1}^{d} \left\{ \sum_{k_i=0}^{R_i} \binom{R_i}{k_i} F^{k_i+1}(T)(1-F(T))^{R_i-k_i} \frac{1}{F(T)} \left(1 - \frac{x_i}{F(T)}\right)^{k_i} \right\} \\ &= \sum_{k_1=0}^{R_1} \cdots \sum_{k_d=0}^{R_d} f_{\mathbf{k}_d}(\mathbf{x}_d) F^{\eta_1(d)}(T)(1-F(T))^{n-\gamma_{d+1}-\eta_1(d)} \left\{ \prod_{i=1}^{d} \frac{1}{\eta_i(d)} \binom{R_i}{k_i} \right\} \end{aligned}$$

$$= \frac{P(D=d)}{(1-F(T))^{\gamma_{d+1}} \prod_{i=1}^{d} \gamma_i} \sum_{k_1=0}^{R_1} \cdots \sum_{k_d=0}^{R_d} f_{\mathbf{k}_d}(\mathbf{x}_d) p^{\mathbf{K}_d}(\mathbf{k}_d).$$

On the other hand, from $\gamma_{d+1} = \sum_{j=d+1}^{m}(R_i + 1)$, we obtain for $F(T) < x_{d+1} < \cdots < x_m$,

$$\begin{aligned} h_2(x_{d+1},\ldots,x_m) &= (1-F(T))^{\gamma_{d+1}} \prod_{i=d+1}^{m} \gamma_i \prod_{i=d+1}^{m} \frac{1}{1-F(T)} \left(\frac{1-x_i}{1-F(T)}\right)^{R_i} \\ &= (1-F(T))^{\gamma_{d+1}} g_{R_{d+1},\ldots,R_m}(x_{d+1},\ldots,x_m), \end{aligned}$$

where $g_{R_{d+1},\ldots,R_m}$ denotes the joint density function of progressively Type-II censored order statistics from the left truncated uniform distribution (at $1 - F(T)$) with left truncated censoring scheme $\mathscr{R}_{\lhd d} = (R_{d+1},\ldots,R_m)$. Combining all the results, we find

$$\begin{aligned} &f^{U_{1:m:n},\ldots,U_{m:m:n}|D=d}(\mathbf{x}_m) \\ &\quad = \left\{ \sum_{k_1=0}^{R_1} \cdots \sum_{k_d=0}^{R_d} f_{\mathbf{k}_d}(\mathbf{x}_d) p^{\mathbf{K}_d}(\mathbf{k}_d) \right\} g_{R_{d+1},\ldots,R_m}(x_{d+1},\ldots,x_m). \end{aligned}$$

The factorization of the density function yields the independence result. Finally, the joint density functions $f_{\mathbf{k}_d}$ and $g_{R_{d+1},\ldots,R_m}$ yield the claimed distributions. □

In the case of order statistics, the above theorem simplifies. In particular, we find from $R_1 = \cdots = R_d = 0$ that $p^{\mathbf{K}_d}$ is a one-point distribution in (0^{*d}). Thus, the corresponding result due to Iliopoulos and Balakrishnan [469] is given in the following corollary.

Corollary 2.5.7. Let $d \in \{1,\ldots,n-1\}$ and F be a cumulative distribution function. Conditional on $D = d$, the random vectors $(X_{1:n},\ldots,X_{d:n})$ and $(X_{d+1:n},\ldots,X_{n:n})$ are mutually independent with

$$\begin{aligned} (X_{1:n},\ldots,X_{d:n}) &\stackrel{\mathsf{d}}{=} (V_{1:d},\ldots,V_{d:d}), \\ (X_{d+1:n},\ldots,X_{n:n}) &\stackrel{\mathsf{d}}{=} (W_{1:n-d},\ldots,V_{n-d:n-d}). \end{aligned}$$

The distributions of $V_1,\ldots,V_d$ and $W_1,\ldots,W_{n-d}$ are as given in Theorem 2.5.5.

Finally, it has to be mentioned that the result of Theorem 2.5.5 can be extended to multiple cut-points $-\infty \equiv T_0 < T_1 < \cdots < T_k$. Instead of D, the random vector $(D_1,\ldots,D_k)$ is considered, where D_j counts the number of progressively

Type-II censored order statistics in the interval $(T_{j-1}, T_j]$. Further details are given in Iliopoulos and Balakrishnan [469].

2.5.4 Dependence Structure of Progressively Type-II Censored Order Statistics

In this section, we focus on the notion of multivariate total positivity. This property of a density function is important in many areas including reliability theory, since it implies association of the components of the corresponding random vector. This notion of dependence was introduced by Esary et al. [354].

Definition 2.5.8. An n-dimensional real valued random vector $\mathbf{X}$ is associated if

$$\mathrm{Cov}(g(\mathbf{X}), h(\mathbf{X})) \geq 0$$

for every pair of increasing functions $g, h : \mathbb{R}^n \longrightarrow \mathbb{R}$.

This definition has some interesting implications (cf. Szekli [829, Chap. 3]). An important feature due to Esary et al. [354] is that any subset of associated random variables is associated as well. For instance, this implies that two associated random variables are positively correlated. Furthermore, association of a random vector $(X_1, \ldots, X_n)'$ implies the inequality

$$P(X_1 > x_1, \ldots, X_n > x_n) \geq \prod_{i=1}^{n} P(X_i > x_i), \quad x_1, \ldots, x_n \in \mathbb{R}. \tag{2.43}$$

Although association is a desirable feature of random variables, it is often difficult to verify. A more restrictive property which implies association, but is often easy to verify, is multidimensional total positivity.

Definition 2.5.9 (Karlin and Rinott [510]). A density function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is MTP_2 (multidimensional totally positive) if

$$f(\mathbf{x})f(\mathbf{y}) \leq f(\mathbf{x} \wedge \mathbf{y})f(\mathbf{x} \vee \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

A random vector $\mathbf{X}$ is said to be MTP_2 if its density function is MTP_2.

The basic properties of multidimensional total positivity are derived in Karlin and Rinott [510]. In our setup, it is important that the indicator function $\mathbb{1}_{\mathbb{R}^n_{\leq}}(\cdot)$ is MTP_2 and that a product of the form

$$\left(\prod_{i=1}^{n} f_i(x_i)\right) g(\mathbf{x}_n)$$

has this property provided that f_i are nonnegative and that g is MTP_2. Since the joint density function of progressively Type-II censored order statistics has this structure, it immediately yields the following result well known for order statistics from absolutely continuous distributions (cf. Karlin and Rinott [510]). The MTP_2 property for discrete order statistics was established by Rüschendorf [761].

Theorem 2.5.10 (Cramer [287]). Let F be a continuous cumulative distribution function. Then, progressively Type-II censored order statistics $X^{\mathscr{R}}_{1:m:n}, \ldots, X^{\mathscr{R}}_{m:m:n}$ based on F with censoring scheme $\mathscr{R}$ are MTP_2. In particular, covariances are nonnegative, i.e., $\mathrm{Cov}(X^{\mathscr{R}}_{j:m:n}, X^{\mathscr{R}}_{r:m:n}) \geq 0$ for all $1 \leq j, r \leq n$.

Obviously, the MTP_2 property implies nonnegative covariances. For order statistics, nonnegative correlation was first claimed by Bickel [201]. Further, it should be noted that the MTP_2 property implies association of the progressively Type-II censored order statistics (see, e.g., Cohen and Sackrowitz [275]). However, as pointed out in Cramer and Lenz [303], the MTP_2 property does not hold in general if the assumption of identical distribution is dropped for the data $X_1, \ldots, X_n$. Nevertheless, association still holds (see Sect. 10.3).

From Theorem 2.5.10, if follows for progressively Type-II censored order statistics that any marginal distribution of at least two (different) progressively Type-II censored order statistics has the MTP_2 property (cf. Karlin and Rinott [510, Proposition 3.2 for the general result on associated random variables]). This feature of progressively Type-II censored order statistics has many interesting implications concerning the dependence structure of progressively Type-II censored order statistics. As mentioned above, it implies association of progressively Type-II censored order statistics which means that all the covariances are nonnegative. It implies inequality (2.43) giving a lower bound for the multivariate survival function in terms of the univariate survival functions.

Burkschat [229] has studied the dependence structure of spacings of generalized order statistics. His results can be directly applied to spacings of progressively Type-II censored order statistics [cf. (2.9)]

$$S^{\star\mathscr{R}}_j = X^{\mathscr{R}}_{j:m:n} - X^{\mathscr{R}}_{j-1:m:n}, j = 2, \ldots, m, \quad S^{\star\mathscr{R}}_1 = X^{\mathscr{R}}_{1:m:n}. \tag{2.44}$$

First, the notion of conditionally increasing in sequence is discussed which is defined as follows.

Definition 2.5.11. A random vector $\mathbf{X} = (X_1, \ldots, X_m)$ is said to be conditionally increasing/decreasing in sequence (CIS/DIS) if

$$P(X_j > t_j | \mathbf{X}_{j-1} = \mathbf{x}_{j-1})$$

are increasing/decreasing in $\mathbf{x}_{j-1} = (x_1, \ldots, x_{j-1})$ for any $j \in \{2, \ldots, m\}$.

Burkschat [229] proved the following result.

Theorem 2.5.12. Let F be IFR (DFR). Then, the vector of spacings $\mathbf{S}^{\star\mathscr{R}}$ is DIS (CIS).

Furthermore, he showed that spacings of progressively Type-II censored order statistics have the MTP_2 property when the baseline distribution satisfies some additional conditions. A similar result is available for the MMR_2 property (see Burkschat [229, Theorem 2.9]).

Theorem 2.5.13. Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be progressively Type-II censored order statistics from an absolutely continuous cumulative distribution function F with censoring scheme $\mathscr{R}$ and hazard rate λ_F. Moreover, let the density function f be positive on the support (α, ∞) of F. Then, provided that

(i) f is log-convex on (α, ∞) or
(ii) F is DFR and λ_F is log-convex on (α, ∞),

the vector of spacings $\mathbf{S}^{\star\mathscr{R}}$ is MTP_2.

2.6 Basic Recurrence Relations

The expression for the marginal cumulative distribution function given in (2.25) can be used to establish a generalization of the triangle rule for cumulative distribution functions of order statistics due to Cole [278] in the continuous case (see also David and Joshi [325] and David and Nagaraja [327, p. 44]):

$$F_{r:n-1} = \frac{r}{n} F_{r+1:n} + \Big(1 - \frac{r}{n}\Big) F_{r:n}, \quad 1 \le r \le n-1. \tag{2.45}$$

In order to prove the required result, we use the following lemma.

Lemma 2.6.1. Let $Z_1, \ldots, Z_{r+1}$ be IID exponential random variables and $\gamma_1 > \cdots > \gamma_{r+1} > 0$. Then, for $t \in \mathbb{R}$,

$$P\Big(\sum_{j=2}^{r+1} \frac{1}{\gamma_j} Z_j \le t\Big) = \Big(1 - \frac{\gamma_{r+1}}{\gamma_1}\Big) P\Big(\sum_{j=1}^{r+1} \frac{1}{\gamma_j} Z_j \le t\Big) + \frac{\gamma_{r+1}}{\gamma_1} P\Big(\sum_{j=1}^{r} \frac{1}{\gamma_j} Z_j \le t\Big).$$

Proof. To prove the above recurrence relation, we make use of the Laplace transform of the exponential random variables Z_j/γ_j:

$$\mathscr{L}_{Z_j/\gamma_j}(t) = E(\mathrm{e}^{-tZ_j/\gamma_j}) = \frac{\gamma_j}{t + \gamma_j}, \quad t > -\gamma_j, 1 \le j \le r+1.$$

Noticing that

$$\frac{\gamma_1 - \gamma_{r+1}}{(\gamma_1 + t)(\gamma_{r+1} + t)} = \frac{1}{\gamma_{r+1} + t} - \frac{1}{\gamma_1 + t},$$

we find for the Laplace transform $\mathscr{L}_{r+1}$ of $\sum_{j=1}^{r+1} \frac{1}{\gamma_j} Z_j$, for $t > -\gamma_{r+1}$,

$$\begin{aligned}(\gamma_1 - \gamma_{r+1})\mathscr{L}_{r+1}(t) &= (\gamma_1 - \gamma_{r+1}) \prod_{j=1}^{r+1} \frac{\gamma_j}{t + \gamma_j} \\ &= \gamma_1 \prod_{j=2}^{r+1} \frac{\gamma_j}{t + \gamma_j} - \gamma_{r+1} \prod_{j=1}^{r} \frac{\gamma_j}{t + \gamma_j} \\ &= \gamma_1 \tilde{\mathscr{L}}_r(t) - \gamma_{r+1} \mathscr{L}_r(t),\end{aligned}$$

where $\mathscr{L}_r$ and $\tilde{\mathscr{L}}_r$ are the Laplace transforms of $\sum_{j=1}^{r} \frac{1}{\gamma_j} Z_j$ and $\sum_{j=2}^{r+1} \frac{1}{\gamma_j} Z_j$, respectively. A simple rearrangement yields the identity

$$\tilde{\mathscr{L}}_r = \Big(1 - \frac{\gamma_{r+1}}{\gamma_1}\Big)\mathscr{L}_{r+1} + \frac{\gamma_{r+1}}{\gamma_1}\mathscr{L}_r,$$

which proves the result. □

Theorem 2.6.2. Marginal cumulative distribution functions of progressively Type-II censored order statistics from an arbitrary cumulative distribution function F and with censoring scheme $\mathscr{R}$ satisfy the recurrence relation

$$F_{r:m-1:n-R_1-1}^{(R_2,\dots,R_m)} = \Big(1 - \frac{\gamma_{r+1}}{n}\Big) F_{r+1:m:n}^{\mathscr{R}} + \frac{\gamma_{r+1}}{n} F_{r:m:n}^{\mathscr{R}}, \quad 1 \le r \le m-1. \tag{2.46}$$

Proof. From the representation in Corollary 2.3.7, we conclude that it is sufficient to consider exponential progressively Type-II censored order statistics. But, the corresponding identity for exponential progressively Type-II censored order statistics follows directly from Lemma 2.6.1 and (2.13) (see also Corollary 2.3.7). □

Remark 2.6.3.

(i) Relation (2.46) was first established by Kamps and Cramer [503] using density representations;
(ii) $(R_2, \dots, R_m) = \mathscr{R}_{\lhd 1}$ is a left truncated censoring scheme. For order statistics, this yields directly the classical triangle rule (2.45);
(iii) It is easy to see that the right-hand side of (2.46) is a convex combination of cumulative distribution functions with probabilities $1 - \frac{\gamma_{r+1}}{n}$ and $\frac{\gamma_{r+1}}{n}$. Notice that $\frac{\gamma_{r+1}}{n}$ is the probability that a particular choice of random variables remains in the experiment after the rth censoring step;

(iv) The above identities hold for density functions and moments as well (provided they exist):
(v) Obviously, the representation in Lemma 2.6.1 holds also for other choices of γ_1 and γ_{r+1}. Thus, we can obtain other identities by selecting other γ's.

A similar relation has been established by Balakrishnan et al. [137] for bivariate marginals. For $1 \le r < s < m-1$, they obtained

$$F^{\mathscr{R}_{\lhd 1}}_{r,s:m-1:n-R_1-1} = \left(1 - \frac{\gamma_{r+1}}{n}\right) F^{\mathscr{R}}_{r+1,s+1:m:n} + \frac{\gamma_{r+1} - \gamma_{s+1}}{n} F^{\mathscr{R}}_{r,s+1:m:n} + \frac{\gamma_{s+1}}{n} F^{\mathscr{R}}_{r,s:m:n}. \quad (2.47)$$

This shows that the bivariate cumulative distribution function $F^{\mathscr{R}_{\lhd 1}}_{r,s:m-1:n-R_1-1}$ can be written as a convex combination of three bivariate cumulative distribution functions from a progressively censored sample with censoring plan $\mathscr{R}$. This quadruple rule extends a result for order statistics due to Srikantan [822] (see also David and Joshi [325]):

$$F_{r,s:n-1} = \frac{r}{n} F_{r+1,s+1:n} + \frac{s-r}{n} F_{r,s+1:n} + \left(1 - \frac{s}{n}\right) F_{r,s:n}.$$

Related results are given by Govindarajulu [409] and Balasubramanian and Beg [164].

2.7 Shape of Density Functions

We now present unimodality properties of progressively Type-II censored order statistics established in Cramer [286]. A cumulative distribution function F is said to be unimodal with a mode η if F is convex on $(-\infty, \eta)$ and concave on (η, ∞). In particular, we consider the stronger concept of log-concavity. This approach extends well-known results for order statistics which are summarized in Dharmadhikari and Joag-dev [339]. For instance, it is proved that progressively Type-II censored order statistics based on a strongly unimodal cumulative distribution function F are strongly unimodal (cf. Barlow and Proschan [167], Huang and Ghosh [460]).

2.7.1 Log-Concavity of Uniform Progressively Type-II Censored Order Statistics

Cramer et al. [313] proved by an induction argument similar to that given in Balakrishnan et al. [129, Lemma 2.7] that uniform generalized order statistics are unimodal. Since uniform progressively Type-II censored order statistics are a particular model of generalized order statistics, this implies the desired result. Considering the notion of strong unimodality, the respective result is strengthened and the proof is simplified.

A cumulative distribution function F is said to be strongly unimodal if the convolution of F with any unimodal cumulative distribution function G is unimodal (cf. Ibragimov [468] and Hájek and Šidák [429]). It can be seen (cf. Dharmadhikari and Joag-dev [339]) that the set of strongly unimodal cumulative distribution functions is closed under convolutions and weak limits and that any degenerate cumulative distribution function is strongly unimodal. A fundamental result of Ibragimov [468] says that nondegenerate strongly unimodal distributions are absolutely continuous with a log-concave density function (cf. Dharmadhikari and Joag-dev [339, Theorem 1.9, Lemma 1.4]).

Definition 2.7.1. Let $m \in \mathbb{N}$. A nonnegative function $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ is said to be log-concave if $\log g$ is concave, i.e., $\log g(\lambda t + (1-\lambda)z) \geq \lambda \log g(t) + (1-\lambda)\log g(z)$ for all $t, z \in \mathbb{R}^m$ and $\lambda \in [0,1]$.

The result of Ibragimov [468] links this property of the density function with strong unimodality of the corresponding cumulative distribution function (cf. Dharmadhikari and Joag-dev [339, Theorem 1.10]). If F is a nondegenerate cumulative distribution function, then F is strongly unimodal iff F is absolutely continuous and its density function f is log-concave. At this point, it has to be mentioned that log-concavity of the density f is equivalent to the property that f is a Pólya frequency function of order 2 (for brevity, we write f PF_2). Pólya frequency functions of order 2 are functions such that $K(x,y) = f(x-y)$ is totally positive of order 2 (cf. Karlin [509]). Barlow and Proschan [168, p. 76] proved that the PF_2 property of a cumulative distribution function F is equivalent to its IFR property. In Lemma 5.8, they established that a strongly unimodal cumulative distribution function has the IFR property.

In order to prove that the cumulative distribution function of a uniform progressively Type-II censored order statistic is unimodal, it is shown that the associated density function is log-concave. The following theorem shows that the joint density of uniform progressively Type-II censored order statistics is log-concave.

Theorem 2.7.2. The joint density function $f^{\mathbf{U}^{\mathscr{R}}}$ of uniform progressively Type-II censored order statistics is log-concave.

Proof. The joint density function of $U_{1:m:n}, \ldots, U_{m:m:n}$ can be expressed as $f^{\mathbf{U}^{\mathscr{R}}}(\mathbf{u}_m) = c\prod_{j=1}^m g_j(u_j)$, $0 \leq u_1 \leq \cdots \leq u_m < 1$, where $g_j(t) = (1-t)^{R_j}$,

$t \in [0, 1]$, are log-concave functions, $1 \le j \le m$. Since a product of log-concave functions is log-concave, $f^{\mathbf{U}^{\mathscr{R}}}$ is log-concave. □

The preceding theorem is now applied to the marginal distributions of uniform progressively Type-II censored order statistics. We make use of the following lemma which was established independently by Prékopa [729] and Brascamp and Lieb [219] (see also Eaton [345]).

Lemma 2.7.3. Let $m, n \in \mathbb{N}$ and $f : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}$ be a log-concave density function. Let $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ be defined by $g(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$, $\mathbf{x} \in \mathbb{R}^m$. Then, g is log-concave on $\mathbb{R}^m$.

Corollary 2.7.4. Suppose $U_{1:m:n}, \dots, U_{m:m:n}$ are uniform progressively Type-II censored order statistics. Then, any marginal density is log-concave. In particular, $F^{U_{r:m:n}}$ is strongly unimodal and, thus, unimodal, $1 \le r \le m$.

2.7.2 *The Shape of Densities of Uniform Progressively Type-II Censored Order Statistics*

In addition to the unimodality and log-concavity properties, the shape of the density functions of uniform progressively Type-II censored order statistics can be classified. The following result is taken from Bieniek [203] who established it in the more general case of generalized order statistics.

Theorem 2.7.5. The density functions of uniform progressively Type-II censored order statistics have the following shapes:

(i) $f^{U_{1:m:n}}$ is constant for $n = 1$, linear decreasing for $n = 2$, and convex decreasing for $n \ge 3$;
(ii) Let $m \ge 2$. $f^{U_{2:m:n}}$ is
  (a) linear increasing for $\gamma_2 = 1$ and $\gamma_1 = n = 2$;
  (b) concave increasing for $\gamma_2 = 1$ and $\gamma_1 = n \ge 3$;
  (c) concave increasing–decreasing for $\gamma_2 = 2$;
  (d) concave increasing, concave decreasing, and convex decreasing for $\gamma_2 \ge 3$;
(iii) For $m \ge r \ge 3$, $f^{U_{r:m:n}}$ is
  (a) convex increasing for $\gamma_r = 1$ and $\gamma_{r-1} = 2$;
  (b) convex–concave increasing for $\gamma_r = 1$ and $\gamma_{r-1} \ge 3$;
  (c) convex increasing, concave increasing, and concave decreasing for $\gamma_r = 2$;
  (d) convex increasing, concave increasing, concave decreasing, and convex decreasing for $\gamma_r \ge 3$.

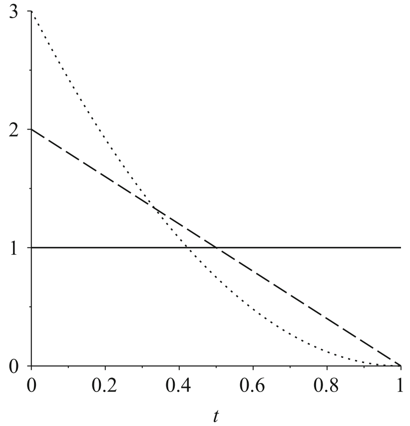

Fig. 2.1 Plots of $f^{U_{1:m:n}}$ for $n = 1$ (*solid line*), $n = 2$ (*dashed line*), and $n = 3$ (*dotted line*)

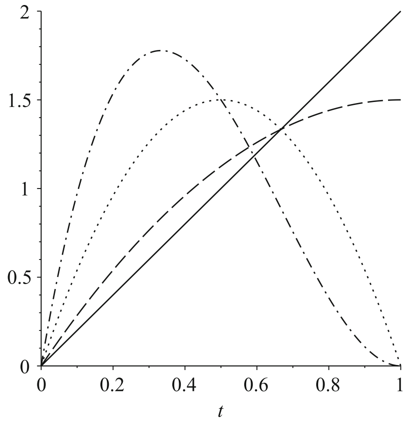

Fig. 2.2 Plots of $f^{U_{2:m:n}}$ for $\gamma_2 = 1, n = 2$ (*solid line*) $\gamma_2 = 1, n = 3$ (*dashed line*), $\gamma_2 = 2, n = 3$ (*dotted line*), and $\gamma_2 = 3, n = 4$ (*dashed–dotted line*)

Figures 2.1, 2.2, and 2.3 illustrate the shapes of density functions of uniform progressively Type-II censored order statistics given in (2.24).

In order to prove these shapes, Bieniek [203] established a variation diminishing property of density functions of uniform progressively Type-II censored order statistics. This property is well known for density functions of uniform order statistics, i.e., Bernstein polynomials (see Schoenberg [786]). In particular, let $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m \setminus \{0\}$. Then, he proved that the number of zeros of any linear combination

$$H_{\mathbf{a}} = \sum_{j=1}^{m} a_j f^{U_{j:m:n}} \tag{2.48}$$

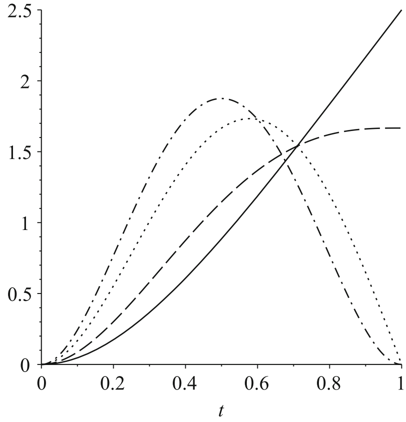

Fig. 2.3 Plots of $f^{U_{3:m:n}}$ for $\gamma_3 = 1, \gamma_2 = 2, n = 5$ (*solid line*) $\gamma_3 = 1, \gamma_2 = 4, n = 5$ (*dashed line*), $\gamma_3 = 2, \gamma_2 = 4, n = 5$ (*dotted line*), and $\gamma_3 = 3, \gamma_2 = 4, n = 5$ (*dashed–dotted line*)

in the unit interval (0, 1) does not exceed the number of sign changes $S^-(\mathbf{a})$ in the sequence $(a_1, \ldots, a_r)$ (after deleting the zeroes in $\mathbf{a}$). Denoting by $Z(f)$ the number of zeroes in (0, 1), the result is given as follows.

Theorem 2.7.6. For any censoring scheme $\mathscr{R}$ and $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m \setminus \{0\}$,

$$Z(H_\mathbf{a}) \leq S^-(\mathbf{a}) \leq m - 1.$$

Moreover, Bieniek [203] obtained for progressively Type-II censored order statistics that the sign of H_a close to zero (1) is determined by the sign of the first (last) nonzero element of $\mathbf{a}$. The result for order statistics has been established by Gajek and Rychlik [386].

2.7.3 Unimodality and Log-Concavity of Progressively Type-II Censored Order Statistics Based on F

For exponential distribution, strong unimodality is obvious.

Theorem 2.7.7. Any marginal density function of progressively Type-II censored order statistics based on an exponential distribution is log-concave. Moreover, the one-dimensional cumulative distribution functions are strongly unimodal.

Huang and Ghosh [460] presented a proof that the cumulative distribution function of an order statistic is strongly unimodal provided that the underlying cumulative distribution function F is strongly unimodal. The same property was obtained earlier by Barlow and Proschan [167, Theorem 7.2] in terms of PF_2

functions (which is an equivalent formulation of log-concavity of the density function). This result has been extended to progressively Type-II censored order statistics by Cramer [286]. Chen et al. [254, Theorem 2.1] present a more general result in terms of generalized order statistics that covers the log-concavity property as a special case. For a vector $\boldsymbol{\tau} = (\tau_1, \dots, \tau_{m-1})$ with $0 \le \tau_j \le j$, let

$$\mathbf{V}(\boldsymbol{\tau}) = (X_{1:m:n}, X_{2:m:n} - X_{\tau_1:m:n}, \dots, X_{m:m:n} - X_{\tau_{m-1}:m:n}),$$

where $X_{0:m:n} = 0$.

Theorem 2.7.8 (Chen et al. [254]). Let $X_{1:m:n}, \dots, X_{m:m:n}$ be progressively Type-II censored order statistics based on a cumulative distribution function F with log-concave density function f. Then, $\mathbf{V}(\boldsymbol{\tau})$, and, thus, each subvector of $\mathbf{V}(\boldsymbol{\tau})$, has a log-concave density function.

Choosing $\boldsymbol{\tau} = (0^{*m-1})$, the following result due to Cramer [286] is included as a special case.

Corollary 2.7.9. Let $X_{1:m:n}, \dots, X_{m:m:n}$ be progressively Type-II censored order statistics based on a strongly unimodal cumulative distribution function F. Then, any marginal density function is log-concave and $F^{X_{r:m:n}}$ is strongly unimodal, $1 \le r \le m$.

Remark 2.7.10. As a consequence of Corollary 2.7.9, progressively Type-II censored order statistics based on strongly unimodal cumulative distribution functions are strongly unimodal and, therefore, unimodal. Examples for strongly unimodal cumulative distribution functions include the following distributions: exponential, normal, truncated normal, Laplace, and particular Weibull and gamma distributions. Further examples are presented in Hájek and Šidák [429, Table 1, p. 16] and Barlow and Proschan [168, p. 79].

Theorem 2.7.8 includes also results for generalized p-spacings of progressively Type-II censored order statistics $X_{p+j:m:n} - X_{j:m:n}$. For instance, it extends a result of Misra and van der Meulen [651] for p-spacings of order statistics. For completeness, we present the result for spacings.

Corollary 2.7.11. Let $X_{1:m:n}, \dots, X_{m:m:n}$ be progressively Type-II censored order statistics based on a strongly unimodal cumulative distribution function F. Then, the vector $\mathbf{S}^{\star\mathscr{R}}$ of spacings has a log-concave density function.

Alam [31] proves that order statistics based on an absolutely continuous cumulative distribution function F with density function f are unimodal if the reciprocal function $1/f$ is convex. Note that concavity of $\log f$ implies convexity of $1/f$. As pointed out by Huang and Ghosh [460], the Cauchy distribution has the above property, but it is not strongly unimodal. In the next theorem, Alam's [31] result is extended to progressively Type-II censored order statistics. A generalization to generalized order statistics is available in Cramer [285] and Alimohammadi and Alamatsaz [39].

Theorem 2.7.12 (Cramer [286]). Let $X_{1:m:n}, \dots, X_{m:m:n}$ be progressively Type-II censored order statistics based on an absolutely continuous cumulative distribution function F with density function f.

Then, convexity of $1/f$ implies unimodality of $F^{X_{r:m:n}}$, $1 \le r \le m$.

Remark 2.7.13. Theorem 2.7.12 provides an alternative proof for the unimodality of the cumulative distribution function of a uniform progressively Type-II censored order statistic. Since the density of the standard uniform distribution is constant on $(0, 1)$ and, therefore, its reciprocal is trivially convex, the theorem leads directly to the unimodality of the respective cumulative distribution function. Further examples are normal, exponential, gamma, and Cauchy distributions.

2.8 Discrete Progressively Type-II Censored Order Statistics

Progressively Type-II censored order statistics from noncontinuous distributions have been studied in Balakrishnan and Dembińska [95,96,97]. The results are based on the quantile representation of progressively Type-II censored order statistics which has been established in Theorem 2.1.1 (see Balakrishnan and Dembińska [96, 97]). For discrete distribution, the quantile representation yields directly the following probability mass function. An extensive discussion of discrete order statistics can be found in Arnold et al. [58, Chap. 3].

Theorem 2.8.1. The joint probability mass function of discrete progressively Type-II censored order statistics $X^{\mathscr{R}}_{1:m:n}, \dots, X^{\mathscr{R}}_{m:m:n}$ from a discrete cumulative distribution function F with support $\mathbb{D}$ is given by

$$P(X^{\mathscr{R}}_{j:m:n} = x_j, 1 \le j \le m) = \int_{\mathcal{A}} f^{\mathbf{U}^{\mathscr{R}}}(\mathbf{u}_m) d\,\mathbf{u}_m, \quad \mathbf{x}_m \in \mathbb{D}^m, \tag{2.49}$$

where

$$\mathcal{A} = \{\mathbf{u}_m \mid u_1 \le \dots \le u_m, F(x_j-) < u_j \le F(x_j), j = 1, \dots, m\}.$$

For discrete order statistics, the corresponding integral representation of the joint probability mass function can be found in Arnold et al. [58, p. 46]. Obviously, a similar representation holds for marginal probability mass functions by replacing $f^{\mathbf{U}^{\mathscr{R}}}$ by the corresponding marginal density function of uniform progressively Type-II censored order statistics. In particular, it follows that the one-dimensional marginal cumulative distribution functions given in (2.25) hold also for discrete parents.

Balakrishnan and Dembińska [96] pointed out that discrete progressively Type-II censored order statistics do not form a Markov chain when the support contains at least three points. The same results has been established independently in the more general setting of generalized order statistics in Tran [854] (see also Cramer and Tran [307]). This work contains also expressions of density function w.r.t. the product measure $\bigotimes_{i=1}^{m} P^F$ for arbitrary discontinuous cumulative distribution functions F.

As in Gan and Bain [391], the concept of tie-runs is applied to obtain a simple expression for the joint probability mass function in the discrete case.

Definition 2.8.2. Let $x_1, \dots, x_r \in \mathbb{R}$ with $x_1 \le \dots \le x_r$. Then, $x_1, \dots, x_r$ is said to have k tie-runs with lengths $\tau_1, \dots, \tau_k$ if

$$x_1 = \dots = x_{\tau_1} < x_{\tau_1+1} = \dots = x_{\tau_1+\tau_2} < \cdots$$
$$\cdots < x_{\tau_1+\dots+\tau_{k-1}+1} = \dots = x_{\tau_1+\dots+\tau_k}$$

with $\sum_{j=1}^{k} \tau_j = r$.

Furthermore, we introduce the lexicographic distribution function to F as given in Arnold et al. [57] and Reiss [750, p. 34].

Definition 2.8.3. Let $F : \mathbb{R} \longrightarrow [0,1]$ be a cumulative distribution function. The function

$$L_F : \begin{cases} \mathbb{R} \times [0,1] \longrightarrow [0,1] \\ (x,u) \qquad \longmapsto F(x-) + u[F(x) - F(x-)] \end{cases}$$

is said to be lexicographic distribution function of F.

This yields an alternative representation of the density function of progressively Type-II censored order statistics.

Theorem 2.8.4 (Tran [854], Cramer and Tran [307]). Let $X_{1:m:n}, \dots, X_{m:m:n}$ be progressively Type-II censored order statistics from an arbitrary cumulative distribution function F, and let A_F denote the set of points of discontinuity of F, i.e.,

$$A_F = \{x \in \mathbb{R} \, : \, F(x) - F(x-) > 0\}.$$

For $x_1 \le \dots \le x_r$, let $\tau_1, \dots, \tau_k \in \mathbb{N}$ denote the lengths of tie-runs in this sequence.

Then, the $\bigotimes_{j=1}^{r} P^F$-joint density function of the first r progressively Type-II censored order statistics is given by

$$f^{\mathbf{X}_r^{\mathscr{R}}}(\mathbf{x}_r) = \prod_{j=1}^{r} \gamma_j \int_{[0,1]^r} \mathbb{1}_{[0,1]^r_{\le}}(L_F(x_1,u_1), \dots, L_F(x_r,u_r))$$
$$\times \Big[\prod_{j=1}^{r} (1 - L_F(x_j,u_j))^{R_j}\Big] d\lambda^r(\mathbf{u}_r) \qquad (2.50)$$

$$= \prod_{j=1}^{r} \gamma_j \mathbb{1}_{\mathbb{R}^r_{\leq}}(\mathbf{x}_r)\Big[\prod_{j\in I_k^c} \frac{(1-F(x_{l_j}))^{\gamma_{l_{j-1}+1}-\gamma_{l_j+1}-\tau_j}}{\tau_j!}\Big]$$

$$\times\Big[\prod_{j\in I_k}(F(x_{l_j})-F(x_{l_j}-))^{-\tau_j}$$

$$\times \int_{\mathcal{B}_j(x_{l_j})} \prod_{l=l_{j-1}+1}^{l_j} (1-z_l)^{R_l}\, dz_{l_{j-1}+1}\cdots dz_{l_j}\Big],$$

where $l_j = \sum_{i=1}^{j} \tau_i$, $j = 1,\ldots,k$, $l_0 = 0$, $\gamma_{r+1} = 0$, and $I_k = \{j \in \{1,\ldots,k\} \mid x_{l_j} \in A_F\}$, $I_k^c = \{1,\ldots,k\} \setminus I_k$, and, for $t \in \mathbb{R}$,

$$\mathcal{B}_j(t) = \{\mathbf{v}_{\tau_j} : F(t-) \leq v_1 \leq \cdots \leq v_{\tau_j} \leq F(t)\} \subseteq [0,1]^{\tau_j}_{\leq}, \quad j \in I_k.$$

For order statistics, (2.50) yields a representation due to Arnold et al. [57, Lemma 2.1], i.e.,

$$f^{X_{1:n},\ldots,X_{r:n}}(\mathbf{x}_r)$$
$$= \frac{n!}{(n-r)!} \int_{[0,1]^r} \mathbb{1}_{[0,1]^r_{\leq}}(L_F(x_1,u_1),\ldots,L_F(x_r,u_r))\overline{L}_F^{n-r}(x_r,u_r)d\lambda^r(\mathbf{u}_r).$$

Remark 2.8.5. The representations in Theorem 2.8.4 simplify when the values $x_1,\ldots,x_r$ are restricted to either continuity or discontinuity points.

For $x_j \notin A_F$, $j = 1,\ldots,r$, and $x_1 < \cdots < x_r$, then $I_k = \emptyset$ and $I_k^c = \{1,\ldots,k\}$. Then, a representation of the $\bigotimes_{j=1}^{r} P^F$-density function results which is similar to that known in the continuous case [see (2.4)]:

$$f^{X_{1:m:n},\ldots,X_{r:m:n}}(\mathbf{x}_r) = \prod_{j=1}^{r-1}\big[\gamma_j(1-F(x_j))^{R_j}\big]\gamma_r(1-F(x_r))^{\gamma_r-1}.$$

For $x_j \in A_F$, $j = 1,\ldots,r$, and $x_1 < \cdots < x_r$, then

$$f^{X_{1:m:n},\ldots,X_{r:m:n}}(\mathbf{x}_r) = \Big[\prod_{j=1}^{r}\Big(F(x_j)-F(x_j-)\Big)^{-1}\Big]\int_{B_r} f^{\mathbf{U}_r^{\mathscr{R}}}(\mathbf{u}_r)d\lambda^r(\mathbf{u}_r),$$

where $B_r = \times_{j=1}^{r}[F(x_j-),F(x_j)]$. This expression is seen to coincide with (2.49) for $r = m$ noticing that $f^{X_{1:m:n},\ldots,X_{r:m:n}}$ is given w.r.t. the product measure $\bigotimes_{j=1}^r P^F$. Multiplying $f^{X_{1:m:n},\dots,X_{r:m:n}}$ with $\prod_{j=1}^r P(X_j = x_j) = \prod_{j=1}^r \big(F(x_j) - F(x_j-)\big)$ leads to (2.49). Since B_r is a Cartesian product, one gets

$$
\begin{aligned}
f^{X_{1:m:n},\dots,X_{r:m:n}}(\mathbf{x}_r) &= \prod_{j=1}^r \left[\frac{\gamma_j}{F(x_j) - F(x_j-)} \int_{F(x_j-)}^{F(x_j)} (1-u_j)^{R_j}\, du_j)\right] \\
&= \left[\prod_{j=1}^{r-1} \frac{\gamma_j [\overline{F}^{R_j+1}(x_j-) - \overline{F}^{R_j+1}(x_j)]}{(R_j+1)[\overline{F}(x_j-) - \overline{F}(x_j)]}\right] \frac{\overline{F}^{\gamma_r}(x_r-) - \overline{F}^{\gamma_r}(x_r)}{\overline{F}(x_r-) - \overline{F}(x_r)}. \qquad (2.51)
\end{aligned}
$$

Example 2.8.6. Suppose F is the cumulative distribution function of a geometric distribution with parameter $p \in (0,1)$ and support $\mathbb{N}$. Then, $\overline{F}(t) = (1-p)^t$, $t \in \mathbb{N}_0$, and (2.51) simplifies to

$$
\begin{aligned}
f^{X_{1:m:n},\dots,X_{r:m:n}}(\mathbf{x}_r) &= \Big(\prod_{j=1}^{r-1} \frac{\gamma_j}{R_j+1}(1-p)^{R_j(x_j-1)}\Big)(1-p)^{(\gamma_r-1)(x_r-1)} h(p) \\
&= \Big(\prod_{j=1}^{r-1} \frac{\gamma_j}{R_j+1}\Big)(1-p)^{\sum_{j=1}^{r-1} R_j(x_j-1) + (\gamma_r-1)(x_r-1)} h(p),
\end{aligned}
$$

where $h(p) = p^{-r}\big(\prod_{j=1}^{r-1}[1-(1-p)^{R_j+1}]\big)[1-(1-p)^{\gamma_r}]$.

Finally, we present a result on the Markovian structure due to Tran [854] (see also Cramer and Tran [307]). It extends a result of Rüschendorf [761] for order statistics from arbitrary cumulative distribution function F. It shows that progressively Type-II censored order statistics from discontinuous cumulative distribution function form a Markov chain if the ties are taken into account.

Theorem 2.8.7. Let $\mathbf{X}^{\mathscr{R}}$ be the random vector of progressively Type-II censored order statistics. For $1 \le r \le m$ and $\mathbf{x}_r = (x_1, \dots, x_r) \in \mathbb{R}_{\le}^r$, let $\tau_1, \dots, \tau_k \in \mathbb{N}$ denote the lengths of the occurring ties in $\mathbf{x}_r$. Moreover, let $\tilde{\tau}_r : \mathbb{R}_{\le}^r \longrightarrow \{1, \dots, r\}$ be a map defined by

$$
\tilde{\tau}_r = \tilde{\tau}_r(\mathbf{x}_r) = \sum_{j=1}^r \mathbb{1}_{\{x_r\}}(x_j), \quad \mathbf{x}_r \in \mathbb{R}_{\le}^r,
$$

and let $T_r = \tilde{\tau}_r\big(X_{1:m:n}, \dots, X_{r:m:n}\big)$. Then,

(i) the joint $\bigotimes_{j=1}^r (P^F \otimes \sum_{l=1}^r \varepsilon_l)$-density of progressively Type-II censored order statistics $X_{1:m:n}, \dots, X_{r:m:n}$ and $T_1, \dots, T_r$ is given by

$$h(x_1, t_1, \ldots, x_r, t_r) = \mathbb{1}_{\{(\tilde{\tau}_1(\mathbf{x}_r),\ldots,\tilde{\tau}_r(\mathbf{x}_r))\}}(\mathbf{t}_r) f^{X_{1:m:n},\ldots,X_{r:m:n}}(\mathbf{x}_r),$$
$$t_1, \ldots, t_r, \in \mathbb{N},\ x_1, \ldots, x_r \in \mathbb{R},$$

where ε_l denotes the probability measure associated with the degenerate distribution in l, $l = 1, \ldots, r$;

(ii) $(X_{j:m:n}, T_j)_{1\le j\le m}$ forms a Markov chain.

Remark 2.8.8.

(i) If F is continuous, then $(T_j)_{1\le j\le n}$ equals $(1, \ldots, 1)$ P^F a.e. This yields the Markovian property of $(X_{j:m:n})_{1\le j\le m}$ as given in Theorem 2.5.1.

(ii) Tran [854] and Balakrishnan and Dembińska [96] showed that progressively Type-II censored order statistics (or, more generally, generalized order statistics) do not form a Markov chain when the support of F has at least three points (see also Balakrishnan and Dembińska [95]). For order statistics, this result is due to Nagaraja [661] (see also Arnold et al. [58]). For further details on the dependence structure of order statistics, we refer to Arnold et al. [57] and Nagaraja [662, 665].

2.9 Exceedances

Bairamov and Eryılmaz [78] addressed the problem of exceedance statistics. Consider progressively Type-II censored order statistics $X_{1:m:n}, \ldots, X_{m:m:n}$ from a continuous cumulative distribution function F and an IID sample $Y_1, \ldots, Y_k$ from a continuous cumulative distribution function G. Then, for $1 \le r < s \le m$, the statistics $V^{(k)}_{r:m:n}$ and $W^{(k)}_{r,s:m:n}$ are defined as

$$V^{(k)}_{r:m:n} = \sum_{i=1}^{k} \mathbb{1}_{(-\infty, X_{r:m:n})}(Y_i), \quad W^{(k)}_{r,s:m:n} = \sum_{i=1}^{k} \mathbb{1}_{(X_{r:m:n}, X_{s:m:n})}(Y_i),$$

i.e., the number of Y's not exceeding $X_{r:m:n}$ and included in the interval $(X_{r:m:n}, X_{s:m:n})$, respectively. Notice that $W^{(k)}_{r,s:m:n} = V^{(k)}_{s:m:n} - V^{(k)}_{r:m:n}$. Clearly,

$$EV^{(k)}_{r:m:n} = P(Y_1 \le X_{r:m:n}) = \int [1 - F_{r:m:n}(t)]dG(t) = p^{\mathscr{R}}_r, \quad \text{say.}$$

Using arguments as in Bairamov [75] and Bairamov and Eryılmaz [77], Bairamov and Eryılmaz [78] showed that

$$\lim_{k\to\infty} \frac{1}{k} V^{(k)}_{r:m:n} \xrightarrow{\mathsf{d}} P^{G(X_{r:m:n})}, \quad \lim_{k\to\infty} \frac{1}{k} W^{(k)}_{r:m:n} \xrightarrow{\mathsf{d}} P^{G(X_{s:m:n})-G(X_{r:m:n})}.$$

For $F = G$, the probability $p_r^{\mathscr{R}}$ can be written as

$$p_r^{\mathscr{R}} = \int_0^1 [1 - F^{U_{r:m:n}}(t)]dt = EU_{r:m:n} = 1 - \prod_{j=1}^{r} \frac{\gamma_j}{\gamma_j + 1}$$

(see Theorem 7.2.3). Since $\sum_{i=1}^{k} \mathbb{1}_{(-\infty,u)}(F(Y_i))$ is $\mathsf{bin}(k,u)$-distributed, the exact distribution of $V_{r:m:n}^{(k)}$ under the hypothesis $F = G$ is given by

$$P(V_{r:m:n}^{(k)} = j) = \binom{k}{j}\Big(\prod_{i=1}^{r}\gamma_i\Big)\sum_{i=1}^{r} a_{i,r}\mathsf{B}(j+1, \gamma_i + k - j), \quad j \in \{0, \dots, k\}.$$

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