Chapter 2
Test Functions

We will now introduce test functions and do so by specializing the testing of \( f \) as in (1.12). If we set \( x = 0 \) and replace \( \phi(y) \) by \( \phi(-y) \), the result of testing \( f \) by means of the weight function \( \phi \) becomes equal to the “integral inner product”

\[
\langle f, \phi \rangle = \int_{\mathbb{R}} f(x) \phi(x) \, dx.
\]  

(For real-valued functions this is in fact an inner product; for complex-valued functions one uses the Hermitian inner product \( \langle f, \overline{\phi} \rangle \).) In Chap. 1 we went on to vary \( \phi \), by translating and rescaling.

The idea behind the definition of distributions is that we consider (2.1) as a function of all possible test functions \( \phi \), in other words, we will be considering the mapping

\[
\text{test } f : \phi \mapsto \int_{\mathbb{R}} f(x) \phi(x) \, dx.
\]

Before we can do so, we first have to specify what functions will be allowed as test functions. The first requirement is that all these functions be complex-valued. Definition 2.5 below, of test functions, refers to compact sets. In this text we will be frequently encountering such sets; therefore we begin by collecting some information on them.

**Definition 2.1.** An open cover of a set \( K \) in \( \mathbb{R}^n \) is a collection \( \mathcal{U} \) of open sets in \( \mathbb{R}^n \) such that their union contains \( K \). That is, for every \( x \in K \) there exists a \( U \in \mathcal{U} \) with \( x \in U \). A subcover is a subcollection \( \mathcal{E} \) of \( \mathcal{U} \) still covering \( K \). In other words, \( \mathcal{E} \subseteq \mathcal{U} \) and \( K \) is contained in the union of the sets \( U \) with \( U \in \mathcal{E} \). The set \( K \) is said to be compact if every open cover of \( K \) has a finite subcover. This concept is applicable in very general topological spaces.

Next, recall the concept of a subsequence of an infinite sequence \( (x(j))_{j \in \mathbb{N}} \). This is a sequence having terms of the form \( y(j) = x(i(j)) \) where \( i(1) < i(2) < \cdots \); in particular, \( \lim_{j \to \infty} i(j) = \infty \). Note that if the sequence \( (x(j))_{j \in \mathbb{N}} \) converges to \( x \), every subsequence of this sequence also converges to \( x \).
Proof. (a) ⇒ (c). We begin by proving that a cube \( B = \prod_{j=1}^{n} I_j \) is compact. Here \( I_j \) denotes a closed interval in \( \mathbb{R} \) of length \( l \), for every \( 1 \leq j \leq n \). Let \( \mathcal{U} \) be an open cover of \( B \); we assume that it does not contain a finite cover of \( B \) and will show that this assumption leads to a contradiction.

When we bisect a closed interval \( I \) of length \( l \), we obtain \( I = I^{(l)} \cup I^{(r)} \), where \( I^{(l)} \) and \( I^{(r)} \) are closed intervals of length \( l/2 \). Consider the cubes of the form \( B' = \prod_{j=1}^{n} I'_j \), where for every \( 1 \leq j \leq n \) we have made a choice \( I'_j = I^{(l)}_j \) or \( I'_j = I^{(r)}_j \). Then \( B \) equals the union of the \( 2^n \) subcubes \( B' \). If it were possible to cover each of these by a finite subcollection \( \mathcal{E} \) of \( \mathcal{U} \), the union of these \( \mathcal{E} \) would be a finite subcollection of \( \mathcal{U} \) covering \( B \), in contradiction to the assumption. We conclude that there is a \( B' \) that is not covered by a finite subcollection of \( \mathcal{U} \).

Applying mathematical induction, we thus obtain a sequence \( (B^{(t)}_{t \in \mathbb{N}}) \) of cubes with the following properties:

(i) \( B^{(1)} = B \) and \( B^{(t)} \subset B^{(t-1)} \) for every \( t \in \mathbb{Z}_{\geq 2} \).
(ii) \( B^{(t)} = \prod_{j=1}^{n} I^{(t)}_j \), where \( I^{(t)}_j \) denotes a closed interval of length \( 2^{-t} l \).
(iii) \( B^{(t)} \) is not covered by a finite subcollection of \( \mathcal{U} \).

From (i) we now have, for every \( j \), \( I^{(t)}_j \subset I^{(t-1)}_j \), that is, the left endpoints \( I^{(t)}_j \) of the \( I^{(t)}_j \), considered as a function of \( t \), form a monotonically nondecreasing sequence in \( \mathbb{R} \). This sequence is bounded; indeed, \( I^{(t)}_j \in I^{(s)}_j \) when \( t \geq s \). As \( t \to \infty \), the sequence therefore converges to an \( l_j \in \mathbb{R} \); we have \( l_j \in I^{(s)}_j \) because \( I^{(s)}_j \) is closed.

Conclusion: the limit point \( l := (l_1, \ldots, l_n) \) belongs to \( B^{(s)} \), for every \( s \in \mathbb{N} \).

Because \( \mathcal{U} \) is a cover of \( B \) and \( l \in B \), there exists a \( U \in \mathcal{U} \) for which \( l \in U \).

Since \( U \) is open, there exists an \( \epsilon > 0 \) such that \( x \in \mathbb{R}^n \) and \( |x_j - l_j| < \epsilon \) for all \( j \) implies that \( x \in U \). Choose \( s \in \mathbb{N} \) with \( 2^{-s} < \epsilon \). Because \( l \in B^{(s)} \), the fact that \( x \in B^{(s)} \) implies that \( |x_j - l_j| \leq 2^{-s} \) for all \( j \); therefore \( x \in U \). As a consequence, \( B^{(s)} \subset U \), in contradiction to the assumption that \( B^{(s)} \) was not covered by a finite subcollection of \( \mathcal{U} \).

Now let \( K \) be an arbitrary bounded and closed subset of \( \mathbb{R}^n \) and \( \mathcal{U} \) an open cover of \( K \). Because \( K \) is bounded, there exists a closed cube \( B \) that contains \( K \). Because \( K \) is closed, the complement \( C := \mathbb{R}^n \setminus K \) of \( K \) is open. The collection \( \mathcal{U} := \mathcal{U} \cup \{C\} \) covers \( K \) and \( C \), and therefore \( \mathbb{R}^n \), and certainly \( B \). In view of the foregoing, \( B \) is covered by a finite subcollection \( \mathcal{E} \) of \( \mathcal{U} \). Removing \( C \) from \( \mathcal{E} \), we obtain a finite subcollection \( \mathcal{E} \) of \( \mathcal{U} \); this covers \( K \). Indeed, if \( x \in K \), there exists
\( U \in \mathcal{E} \) with \( x \in U \). Since \( U \) cannot equal \( C \), we have \( U \in \mathcal{E} \).

(c) \( \Rightarrow \) (b). Suppose that \((x(j))\) is an infinite sequence in \( K \) that has no subsequence converging in \( K \). This means that for every \( x \in K \) there exist an \( \epsilon(x) > 0 \) and an \( N(x) \) for which \( \|x - x(j)\| \geq \epsilon(x) \) whenever \( j > N(x) \). Let

\[
U(x) = \{ y \in K \mid \|y - x\| < \epsilon(x) \}.
\]

The \( U(x) \) with \( x \in K \) form an open cover of \( K \); condition (c) implies the existence of a finite subset \( F \) of \( K \) such that for every \( x \in K \) there is an \( f \in F \) with \( x \in U(f) \). Let \( N \) be the maximum of the \( N(f) \) with \( f \in F \); then \( N \) is well-defined because \( F \) is finite. For every \( j \) we find that an \( f \in F \) exists with \( x(j) \in U(f) \), and therefore \( j \leq N(f) \leq N \). This is in contradiction to the unboundedness of the indices \( j \).

(b) \( \Rightarrow \) (a). Suppose that \( K \) satisfies (b). If \( K \) is not bounded, we can find a sequence \((x(j))_{j \in \mathbb{N}}\) with \( \|x(j)\| \geq j \) for all \( j \). There is a subsequence \((x(j(k)))_{k \in \mathbb{N}}\) that converges and that is therefore bounded, in contradiction to \( \|x(j(k))\| \geq j(k) \geq k \) for all \( k \). In order to prove that \( K \) is closed, suppose \( \lim_{j \to \infty} x(j) = x \) for a sequence \((x(j))\) in \( K \). This contains a subsequence that converges to a point \( y \in K \). But the subsequence also converges to \( x \), and in view of the uniqueness of limits we conclude that \( x = y \in K \).

The preceding theorem contains the \textit{Bolzano–Weierstrass Theorem}, which states that every bounded sequence in \( \mathbb{R}^n \) has a convergent subsequence; see [7, Theorem 1.6.3]. The implication (a) \( \Rightarrow \) (c) is also referred to as the \textit{Heine–Borel Theorem}; see [7, Theorem 1.8.18]. However, linear spaces consisting of functions are usually of infinite dimension. In normed linear spaces of infinite dimension, “compact” is a much stronger condition than “bounded and closed,” while in such spaces (b) and (c) are still equivalent.

As a first application of compactness we obtain conditions that guarantee that disjoint closed sets in \( \mathbb{R}^n \) possess disjoint open neighborhoods; see Lemma 2.3 below and its corollary. To do so, we need some definitions, which are of independent interest.

Introduce the \textit{set of sums} \( A + B \) of two subsets \( A \) and \( B \) of \( \mathbb{R}^n \) by means of

\[
A + B := \{ a + b \mid a \in A, \ b \in B \}.
\]  \hfill (2.2)

It is clear that \( A + B \) is bounded if \( A \) and \( B \) are bounded. Also, \( A + B \) is closed whenever \( A \) is closed and \( B \) compact. Indeed, suppose that the sequence \((c_j)_{j \in \mathbb{N}}\) in \( A + B \) converges in \( \mathbb{R}^n \) to \( c \). One then has \( c_j = a_j + b_j \) for some \( a_j \in A \), \( b_j \in B \). By the compactness of \( B \), a subsequence \((b_{j(k)})_{k \in \mathbb{N}}\) converges to a \( b \in B \). Consequently, the sequence with terms \( a_{j(k)} = c_{j(k)} - b_{j(k)} \) converges to \( a := c - b \) as \( k \to \infty \). Because \( A \) is closed, \( a \) lies in \( A \). The conclusion is that \( c \in A + B \). In particular, \( A + B \) is compact whenever \( A \) and \( B \) are both compact.

An example of two closed subsets \( A \) and \( B \) of \( \mathbb{R} \) for which \( A + B \) is not closed is the pair \( A = \mathbb{Z}_{\leq 0} \) and \( B = \{ n + 1/n \mid n \in \mathbb{Z}_{\geq 2} \} \). Clearly, \( A \) and \( B \) are closed.
and $A + B$ does not contain any integer. On the other hand, for every $m \in \mathbb{Z}$ the numbers $m + 1/n = (m - n) + (n + 1/n)$ belong to $A + B$ if $n \in \mathbb{Z}_{\geq 2}$ and $n > m$, while $m + 1/n$ converges to $m$ as $n \to \infty$.

Furthermore, the distance $d(x, U)$ from a point $x \in \mathbb{R}^n$ to a set $U \subset \mathbb{R}^n$ is defined by

$$d(x, U) = \inf \{ \|x - u\| \mid u \in U \}. \quad (2.3)$$

Note that $d(x, U) = 0$ if and only if $x \in \overline{U}$, the closure of $U$ in $\mathbb{R}^n$. The $\delta$-neighborhood $U_\delta$ of $U$ is given by (see Fig. 2.1)

$$U_\delta = \{ x \in \mathbb{R}^n \mid d(x, U) < \delta \}. \quad (2.4)$$

![Fig. 2.1](image)

**Fig. 2.1** Example of a $\delta$-neighborhood

Observe that $x \in U_\delta$ if and only if a $u \in U$ exists with $\|x - u\| < \delta$. Using the notation $B(u; \delta)$ for the open ball of center $u$ and radius $\delta$, this gives

$$U_\delta = \bigcup_{u \in U} B(u; \delta),$$

which implies that $U_\delta$ is an open set. Also, $B(u; \delta) = \{u\} + B(0; \delta)$, and therefore

$$U_\delta = U + B(0; \delta).$$

Finally, we define $U_{-\delta}$ as the set of all $x \in U$ for which the $\delta$-neighborhood of $x$ is contained in $U$. Note that $U_{-\delta}$ equals the complement of $(\mathbb{R}^n \setminus U)_\delta$ and that consequently, $U_{-\delta}$ is a closed set.

Now we are prepared enough to obtain the following two results on separation of sets.

**Lemma 2.3.** Let $K \subset \mathbb{R}^n$ be compact and $A \subset \mathbb{R}^n$ closed, while $K \cap A = \emptyset$. Then there exists $\delta > 0$ such that $K_\delta \cap A_\delta = \emptyset$.

**Proof.** Assume the negation of the conclusion. Then there exists an element $x(j) \in K_{1/j} \cap A_{1/j}$, for every $j \in \mathbb{N}$. Therefore, one can select $y(j) \in K$ and $a(j) \in A$ satisfying

$$\|y(j) - x(j)\| < \frac{1}{j} \quad \text{and} \quad \|x(j) - a(j)\| < \frac{1}{j} ; \quad \text{so} \quad \|y(j) - a(j)\| < \frac{2}{j}.$$  

By passing to a subsequence, one may assume that the $y(j)$ converge to some $y \in K$ in view of criterion (b) in Theorem 2.2 for compactness. Hence $\|a(j) - y\| \to 0,$
in other words, \( a(j) \to y \) as \( j \to \infty \). Since \( A \) is closed, this leads to \( y \in A \); therefore \( y \in K \cap A \), which is a contradiction. \( \square \)

**Corollary 2.4.** Consider \( K \subset X \subset \mathbb{R}^n \) with \( K \) compact and \( X \) open. Then there exists a \( \delta_0 > 0 \) with the following property. For every \( 0 < \delta \leq \delta_0 \) there is a compact set \( C \) such that

\[
K \subset K_\delta \subset C \subset C_\delta \subset X.
\]

**Proof.** The set \( A = \mathbb{R}^n \setminus X \) is closed and \( K \cap A = \emptyset \). On account of Lemma 2.3 there is \( \delta_0 > 0 \) such that \( K_{3\delta_0} \cap A = \emptyset \). Define \( C = K + B(0; \delta) \). Then \( C \) is compact as the set of sums of two compact sets; further, \( C \subset K_{2\delta} \); hence \( C_\delta \subset K_{3\delta} \subset K_{3\delta_0} \). This leads to \( C_\delta \cap A = \emptyset \), and so \( C_\delta \subset X \). \( \square \)

After this longish intermezzo we next come to the definition of the space of test functions, one of the most important notions in the theory.

**Definition 2.5.** Let \( X \) be an open subset of \( \mathbb{R}^n \). For \( \phi : X \to \mathbb{C} \) the support of \( \phi \), written \( \text{supp} \ \phi \), is defined as the closure in \( X \) of the set of the \( x \in X \) for which \( \phi(x) \neq 0 \). A test function on \( X \) is an infinitely differentiable complex-valued function on \( X \) whose support is a compact subset of \( X \). (That is, \( \text{supp} \ \phi \) is a compact subset of \( \mathbb{R}^n \) and \( \text{supp} \ \phi \subset X \).) The space of all test functions on \( X \) is designated as \( C_0^\infty(X) \). (The subscript \( 0 \) is a reminder of the fact that the function vanishes on the complement of a compact subset, and thus in a sense on the largest part of the space.) It is a straightforward verification that \( C_0^\infty(X) \) is a linear space under pointwise addition and multiplication by scalars of functions. \( \square \)

If we extend \( \phi \in C_0^\infty(X) \) to a function on \( \mathbb{R}^n \) by means of the definition \( \phi(x) = 0 \) for \( x \in \mathbb{R}^n \setminus X \), we obtain a \( C^\infty \) function on \( \mathbb{R}^n \). Indeed, \( \mathbb{R}^n \) equals the union of the open sets \( \mathbb{R}^n \setminus \text{supp} \ \phi \) and \( X \). On both these sets we have that \( \phi \) is of class \( C^\infty \). The support of the extension equals the original support of \( \phi \). Stated differently, we may interpret \( C_0^\infty(X) \) as the space of all \( \phi \in C_0^\infty(\mathbb{R}^n) \) with \( \text{supp} \ \phi \subset X \); with this interpretation we have \( C_0^\infty(U) \subset C_0^\infty(V) \) if \( U \subset V \) are open subsets of \( \mathbb{R}^n \).

In the vast majority of cases the test functions need only be \( k \) times continuously differentiable, with \( k \) finite and sufficiently large. To avoid having to keep track of the degree of differentiability, one prefers to work with \( C_0^\infty \) rather than the space \( C_0^k \) of compactly supported \( C^k \) functions.

The question arises whether the combination of the requirements “compactly supported” and “infinitely differentiable” might not be so restrictive as to be satisfied only by the zero function. Indeed, if we were to replace the requirement that \( \phi \) be infinitely differentiable by the requirement that \( \phi \) be analytic, we would obtain only the zero function. Here we recall that a function \( \phi \) is said to be analytic on \( X \) if for every \( a \in X \), \( \phi \) is given by a power series about \( a \) that is convergent on some neighborhood of \( a \). This implies that \( \phi \) is of class \( C^\infty \) and that the power series of \( \phi \) about \( a \) equals the Taylor series of \( \phi \) at \( a \).
Furthermore, an open set \( X \) in \( \mathbb{R}^n \) is said to be connected if \( X \) is not the union of two disjoint nonempty open subsets of \( X \) (for more details, refer to [7, Sect. 1.9]).

**Lemma 2.6.** Let \( X \) be a connected open subset of \( \mathbb{R}^n \) and \( \phi \) an analytic function on \( X \). Then either \( \phi = 0 \) on \( X \) or \( \text{supp} \phi = X \). In the latter case \( \text{supp} \phi \) is not compact, provided that \( X \) is not empty.

**Proof.** Consider the set \( U = \{ x \in X \mid \phi = 0 \text{ in a neighborhood of } x \} \); this definition implies that \( U \) is open in \( X \). Now select \( x \in X \setminus U \). Since \( \phi \) equals its convergent power series in a neighborhood of \( x \), there exists a (possibly higher-order) partial derivative of \( \phi \), say \( \psi \), with \( \psi(x) \neq 0 \). Because \( \psi \) is continuous, there is a neighborhood \( V \) of \( x \) on which \( \psi \) differs from 0. Hence, \( V \subset X \setminus U \), in other words, \( X \setminus U \) is open in \( X \). From the connectivity of \( X \) we conclude that either \( U = X \), in which case \( \phi = 0 \) on \( X \), or \( U = \emptyset \), and in that case \( \text{supp} \phi = X \). \( \square \)

Next we show that \( C_0^\infty(X) \) is sufficiently rich. We fabricate the desired functions step by step.

**Lemma 2.7.** Define the function \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \) by \( \alpha(x) = e^{\frac{1}{x^2}} \) for \( x > 0 \) and \( \alpha(x) = 0 \) for \( x \leq 0 \). Then \( \alpha \in C^\infty(\mathbb{R}) \) with \( \alpha(x) > 0 \) for \( x > 0 \), and \( \text{supp} \alpha = \mathbb{R}_{\geq 0} \).

**Proof.** The only problem is the differentiability at \( 0 \); see Fig. 2.2. From the power series for the exponential function one obtains, for every \( n \in \mathbb{N} \), the estimate \( e^y \geq \frac{y^n}{n!} \) for all \( y \geq 0 \). Hence

\[
\alpha(x) = \frac{1}{e^{1/x}} \leq \frac{n!}{1/x^n} = n! x^n \quad (x > 0).
\]

This tells us that \( \alpha \) is differentiable at \( 0 \), with \( \alpha'(0) = 0 \).

As regards the higher-order derivatives, we note that for \( x > 0 \) the function \( \alpha \) satisfies the differential equation

\[
\alpha'(x) = \frac{\alpha(x)}{x^2}.
\]

By applying this in the induction step we obtain, with mathematical induction on \( k \),

\[
\alpha^{(k)}(x) = p_k\left(\frac{1}{x}\right) \alpha(x),
\]

where the \( p_k \) are polynomial functions inductively determined by

\[
p_0(y) = 1 \quad \text{and} \quad p_{k+1}(y) = \left(p_k(y) - p_k'(y)\right) y^2.
\]

In particular, \( p_k \) is of degree \( 2k \) and therefore satisfies an estimate of the form

\[
|p_k(y)| \leq c(k) y^{2k} \quad (y \geq 1).
\]

From this we derive the estimate
If we then choose \( n \geq 2k + 2 \), we obtain, with mathematical induction on \( k \), that \( \alpha \in C^k(\mathbb{R}) \) and \( \alpha^{(k)}(0) = 0 \).

**Lemma 2.8.** Let \( \alpha \in C^\infty(\mathbb{R}) \) be as in the preceding lemma. Let \( a \) and \( b \in \mathbb{R} \) with \( a < b \). Define the function \( \beta = \beta_{a,b} \) by

\[
\beta(x) = \beta_{a,b}(x) = \alpha(x-a)\alpha(b-x).
\]

One then has \( \beta \in C^\infty(\mathbb{R}) \) with \( \beta > 0 \) on \( ]a, b[ \) and \( \text{supp } \beta = [a, b] \). Furthermore,

\[
I(\beta) := \int_{\mathbb{R}} \beta(x) \, dx > 0.
\]

The function \( \gamma = \gamma_{a,b} := \frac{1}{I(\beta)} \beta \) has the same properties as \( \beta \) (see Fig. 2.2), while \( \int_{\mathbb{R}} \gamma(x) \, dx = 1 \).

![Fig. 2.2](image.png)  
**Fig. 2.2** Graphs of \( \alpha \) as in Lemma 2.7 on \([0, 1/2]\) and of \( \gamma_{-1,2} \) as in Lemma 2.8, with the scales adjusted

**Lemma 2.9.** Let \( a_j \) and \( b_j \in \mathbb{R} \) with \( a_j < b_j \) and define \( \gamma_{a_j,b_j} \in C^\infty_0(\mathbb{R}) \) as in the preceding lemma, for \( 1 \leq j \leq n \). Write \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). For \( a \) and \( b \in \mathbb{R}^n \), define the function \( \Gamma_{a,b} : \mathbb{R}^n \to \mathbb{R} \) by (see Fig. 2.3)

\[
\Gamma_{a,b}(x) = \prod_{j=1}^n \gamma_{a_j,b_j}(x_j).
\]

Then we have

\[
\Gamma_{a,b} \in C^\infty(\mathbb{R}^n), \quad \Gamma_{a,b} > 0 \quad \text{on} \quad \prod_{j=1}^n [a_j, b_j],
\]

\[
\text{supp } \Gamma_{a,b} = \prod_{j=1}^n [a_j, b_j], \quad \int_{\mathbb{R}^n} \Gamma_{a,b}(x) \, dx = 1.
\]

For a complex number \( c \), the notation \( c \geq 0 \) means that \( c \) is a nonnegative real number. For a complex-valued function \( f \), \( f \geq 0 \) means that \( f(x) \geq 0 \) for every \( x \).
in the domain space of $f$. If $g$ is another function, one writes $f \geq g$ or $g \leq f$ if $f - g \geq 0$.

**Corollary 2.10.** For every point $p \in \mathbb{R}^n$ and every neighborhood $U$ of $p$ in $\mathbb{R}^n$ there exists a $\phi \in C_0^\infty(\mathbb{R}^n)$ with the following properties:

(a) $\phi \geq 0$ and $\phi(p) > 0$.
(b) $\text{supp } \phi \subset U$.
(c) $\int_{\mathbb{R}^n} \phi(x) \, dx = 1$.

By superposition and taking limits of the test functions thus constructed we obtain a wealth of new test functions. For example, consider $\phi$ as in Corollary 2.10 and set

$$\phi_\epsilon(x) := \frac{1}{\epsilon^n} \phi\left(\frac{1}{\epsilon} x\right).$$

(2.5)

Further, let $f$ be an arbitrary function in $C_0(\mathbb{R}^n)$, the space of all continuous functions on $\mathbb{R}^n$ with compact support; these are easily constructed in abundance. By straightforward generalization of Lemma 1.6 to $\mathbb{R}^n$, the functions $f_\epsilon := f * \phi_\epsilon$ converge uniformly on $\mathbb{R}^n$ to $f$, as $\epsilon \downarrow 0$. The $f_\epsilon$ are test functions, in other words,

$$f_\epsilon \in C_0^\infty(\mathbb{R}^n),$$

(2.6)

as one can see from Lemma 2.18 below. Consequently, for every $f \in C_0(\mathbb{R}^n)$ there exists a family of functions in $C_0^\infty(\mathbb{R}^n)$ that converges to $f$ uniformly on compact subsets. We say that $C_0^\infty(\mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n)$; see Definition 8.3 below for the general definition of dense sets.

**Lemma 2.11.** For every $a \in \mathbb{R}^n$ and $r > 0$ there exists $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$\text{supp } \phi \subset B(a; 2r), \quad 0 \leq \phi \leq 1, \quad \phi = 1 \text{ on } B(a; r).$$

**Proof.** By translation and rescaling we see that it is sufficient to prove the assertion for $a = 0$ and $r = 1$. By Lemma 2.8 we can find $\beta \in C^\infty(\mathbb{R})$ such that $\beta > 0$ on $[1, 3]$ and $\text{supp } \beta = [1, 3]$, while $I = \int_1^3 \beta(x) \, dx > 0$. Hence we may write
\[ \eta(x) := \frac{1}{T} \int_{x}^{3} \beta(t) \, dt. \]

Then \( \eta \in C^\infty(\mathbb{R}), 0 \leq \eta \leq 1, \) while \( \eta = 1\) on \([-\infty, 1]\) and \( \eta = 0\) on \([3, \infty[.\)

Now set \( \phi(x) = \eta(\|x\|^2) = \eta(x_1^2 + \cdots + x_n^2). \)

We now review notation that will be needed for Definition 2.13 and Lemma 2.18 below, among other things. In this text we use the following notation for higher-order derivatives. A multi-index is a sequence

\[ \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n \]

of \( n \) nonnegative integers. The sum

\[ |\alpha| := \sum_{j=1}^{n} \alpha_j \]

is called the order of the multi-index \( \alpha. \) For every multi-index \( \alpha \) we write

\[ \partial_x^{\alpha} := \frac{\partial^{\alpha}}{\partial x^\alpha} := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad \text{where} \quad \partial_j := \frac{\partial}{\partial x_j}. \quad (2.7) \]

Furthermore, we use the shorthand notation

\[ \partial^{\alpha} = \frac{\partial^{\alpha}}{\partial x^\alpha} \]

when we want to differentiate only with respect to the variables \( x_j. \) The crux is that the Theorem on the interchangeability of the order of differentiation (see for instance [7, Theorem 2.7.2]), which holds for functions sufficiently often differentiable, allows us to write every higher-order derivative in the form (2.7); also refer to the introduction to Chap. 6. Finally, in the case of \( n = 1, \) we define \( \partial \) as \( \partial^{(1)}. \)

**Remark 2.12.** In (2.7) we defined the partial derivatives \( \partial^{\alpha} f \) of arbitrary order of a function \( f \) depending on an arbitrary number of variables. For the \( k \)-th-order derivatives of the product \( fg \) of two functions \( f \) and \( g \) that are \( k \) times continuously differentiable, we have Leibniz’s formula:

\[ \partial^{\alpha} (fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \partial^\beta g, \quad (2.8) \]

for \( |\alpha| = k. \) Here \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) are multi-indices, while \( \beta \leq \alpha \) means that for every \( 1 \leq j \leq n \) one has \( \beta_j \leq \alpha_j. \) The \( n \)-dimensional binomial coefficients in (2.8) are given by

\[ \binom{\alpha}{\beta} := \prod_{j=1}^{n} \binom{\alpha_j}{\beta_j}, \quad \text{where} \quad \binom{p}{q} = \frac{p!}{(p-q)!q!}. \]
for \( p \) and \( q \in \mathbb{Z} \) with \( 0 \leq q \leq p \). Formula (2.8) is obtained with mathematical induction on the order \( k = |\alpha| \) of differentiation, using Leibniz’s rule

\[
\partial_j (fg) = g \partial_j f + f \partial_j g
\]

(2.9)
in the induction step.

Definition 2.5 is supplemented by the following, which introduces a notion of convergence in the infinite-dimensional linear space \( C_0^\infty(X) \):

**Definition 2.13.** Let \( \phi_j \) and \( \phi \in C_0^\infty(X) \), for \( j \in \mathbb{N} \) and \( X \) an open subset of \( \mathbb{R}^n \). The sequence \( (\phi_j)_{j \in \mathbb{N}} \) is said to converge to \( \phi \) in the space \( C_0^\infty(X) \) of test functions as \( j \to \infty \), notation

\[
\lim_{j \to \infty} \phi_j = \phi \quad \text{in } C_0^\infty(X),
\]

if the following two conditions are both met:

(a) there exists a compact subset \( K \) of \( X \) such that \( \text{supp} \ \phi_j \subseteq K \) for all \( j \);
(b) for every multi-index \( \alpha \) the sequence \( (\partial^\alpha \phi_j)_{j \in \mathbb{N}} \) converges uniformly on \( X \) to \( \partial^\alpha \phi \).

Observe that the data above imply that \( \text{supp} \ \phi \subseteq K \). The notion of convergence introduced in the definition above is very strong. The stronger the convergence, the fewer convergent sequences there are, and the more readily a function defined on \( C_0^\infty(X) \) will be continuous.

Now we combine compactness and test functions in order to introduce the useful technical tool of a partition of unity over a compact set.

**Definition 2.14.** Let \( K \) be a compact subset of an open subset \( X \) of \( \mathbb{R}^n \) and \( \mathcal{U} \) an open cover of \( K \). A \( C_0^\infty(X) \) partition of unity over \( K \) subordinate to \( \mathcal{U} \) is a finite sequence \( \psi_1, \ldots, \psi_l \in C_0^\infty(X) \) with the following properties (see Fig. 2.4):

(i) \( \psi_j \geq 0 \), for every \( 1 \leq j \leq l \), and \( \sum_{j=1}^l \psi_j \leq 1 \) on \( X \);
(ii) there exists a neighborhood \( V \) of \( K \) in \( X \) with \( \sum_{j=1}^l \psi_j(x) = 1 \), for all \( x \in V \);
(iii) for every \( j \) there is a \( U = U(j) \in \mathcal{U} \) for which \( \text{supp} \ \psi_j \subseteq U \).
Given a function $f$ on $X$, write $f_j = \psi_j f$ in the notation above. Then we obtain functions $f_j$ with compact support contained in $U(j)$, while $f = \sum_{j=1}^{l} f_j$ on $V$. Furthermore, all $f_j \in C^k$ if $f \in C^k$. In the applications, the $U \in \mathcal{U}$ are small neighborhoods of points of $K$ with the property that we can reach certain desired conclusions for functions with support in $U$. For example, partitions of unity were used in this way in [7, Theorem 7.6.1] to prove the integral theorems for open sets $X \subset \mathbb{R}^n$ with $C^1$ boundary.

**Theorem 2.15.** For every compact set $K$ contained in an open subset $X$ of $\mathbb{R}^n$ and every open cover $\mathcal{U}$ of $K$ there exists a $C^\infty_0(X)$ partition of unity over $K$ subordinate to $\mathcal{U}$.

**Proof.** For every $a \in K$ there exists an open set $U_a \in \mathcal{U}$ such that $a \in U_a$. Select $r_a > 0$ such that $B(a; 2r_a) \subset U_a \cap X$. By criterion (c) in Theorem 2.2 for compactness, there exist finitely many $a(1), \ldots, a(l)$ such that $K$ is contained in the union $V$ of the $B(a(j), r_a(j))$, for $1 \leq j \leq l$. Now select the corresponding $\phi_j \in C^\infty_0(X)$ as in Lemma 2.11 and set

$$
\psi_1 = \phi_1; \quad \psi_{j+1} = \phi_{j+1} \prod_{i=1}^{j} (1 - \phi_i) \quad (1 \leq j < l).
$$

Then the conditions (i) and (iii) for a $C^\infty_0(X)$ partition of unity subordinate to $\mathcal{U}$ are satisfied by the $\psi_1, \ldots, \psi_l$. The relation

$$
\sum_{i=1}^{j} \psi_i = 1 - \prod_{i=1}^{j} (1 - \phi_i)
$$

is trivial for $j = 1$. If (2.11) is true for $j < l$, then summing (2.10) and (2.11) yields (2.11) for $j + 1$. Consequently (2.11) is valid for $j = l$, and this implies that the $\psi_1, \ldots, \psi_l$ satisfy condition (ii) for a partition of unity with $V$ as defined above. □

**Corollary 2.16.** Let $K$ be a compact subset in $\mathbb{R}^n$. For every open neighborhood $X$ of $K$ in $\mathbb{R}^n$ there exists a $\chi \in C^\infty_0(\mathbb{R}^n)$ with $0 \leq \chi \leq 1$, supp $\chi \subset X$ and $\chi = 1$ on an open neighborhood of $K$. In particular, for $\delta > 0$ sufficiently small, we can find such a function $\chi$ with $\chi = 1$ on $K_\delta$.

**Proof.** Consider the open cover $\{X\}$ of $K$ and let $\psi_1, \ldots, \psi_l$ be a subordinate partition of unity over $K$ as in the preceding theorem. Then $\chi = \sum_j \psi_j$ satisfies all requirements. For the second assertion, apply Corollary 2.4 and the preceding result with $K$ replaced by $C$ as in the corollary. □

The function $\chi$ is said to be a cut-off function for the compact subset $K$ of $\mathbb{R}^n$. Through multiplication by $\chi$ we can replace a function $f$ defined on $X$ by a function $g$ with compact support contained in $X$. Here $g = f$ on a neighborhood of $K$ and $g \in C^k$ if $f \in C^k$. 
We still have to verify the claim in (2.6); it follows from Lemma 2.18 below. In the case of $k$ equal to $\infty$, another proof will be given in Theorem 11.2. Later on, in demonstrating Theorem 11.22, we will need an analog of Corollary 2.16 in the case of not necessarily compact sets. To that end, we derive Lemma 2.19 below. In preparation, we introduce some concepts that are useful in their own right.

**Definition 2.17.** Let $X \subset \mathbb{R}^n$ be an open subset. A function $f : X \to \mathbb{C}$ is said to be **locally integrable** if for every $a \in X$, there exists an open rectangle $B \subset X$ with the properties that $a \in B$ and that $f$ is integrable on $B$.

The characteristic function or indicator function $1_U$ of a subset $U$ of $\mathbb{R}^n$ is defined by

$$1_U(x) = 1 \quad \text{if} \quad x \in U, \quad 1_U(x) = 0 \quad \text{if} \quad x \in \mathbb{R}^n \setminus U.$$ 

$U$ is said to be **measurable** if $1_U$ is locally integrable.

For the purposes of this book it will almost invariably be sufficient to interpret the concept of integrability, as we use it here, in the sense of Riemann. However, for distributions it is common to work with Lebesgue integration, which leads to a more comprehensive theory. Loosely speaking, Lebesgue’s theory is more powerful than Riemann’s, in the sense that it leads to a process of integration for more functions and to a simpler treatment of singular behavior of functions. On the other hand, a thorough treatment of Lebesgue integration is technically more demanding than that of Riemann integration. The distinction between the two concepts rarely arises in the case of the functions that will be encountered in this text. It is primarily in the description of spaces of all functions satisfying certain properties that the difference becomes important.

Readers who are not familiar with Lebesgue integration can find a way around this by restricting themselves to locally integrable functions with an absolute value whose improper Riemann integral exists, and otherwise taking our assertions about Lebesgue integration for granted. Some of these assertions do not apply to Riemann integration, but this need not be a reason for serious concern; we will discuss this issue when the need arises.

Nonetheless, for the benefit of readers who are interested in the relation between the theory of distributions and that of (Lebesgue) integration we concisely but fairly completely discuss integration in Chap. 20. In particular, local integrability is introduced in Definition 20.37.

**Lemma 2.18.** Let $f$ be locally integrable on $\mathbb{R}^n$ and $g \in C_0^k(\mathbb{R}^n)$. Then $f \ast g \in C^k(\mathbb{R}^n)$ and

$$\text{supp} \,(f \ast g) \subset \text{supp} \,f + \text{supp} \,g.$$ 

Here $\text{supp} \,f + \text{supp} \,g$ is a closed subset of $\mathbb{R}^n$, compact if $f$, too, has compact support; in that case $f \ast g \in C_0^k(\mathbb{R}^n)$.

**Proof.** We study $(f \ast g)(x)$ for $x \in U$, where $U \subset \mathbb{R}^n$ is bounded and open. Define $h(x, y) := f(y) \,g(x - y)$. Then the function $x \mapsto h(x, y)$ belongs to $C^k(U)$ for every $y \in \mathbb{R}^n$, because for every multi-index $\alpha \in (\mathbb{Z}_{\geq 0})^n$ with $|\alpha| \leq k$, 

$$f \ast g(x) = \int_{\mathbb{R}^n} f(x - y) \,g(y) \,dy$$

for every $x \in U$. Then the function $x \mapsto f \ast g(x)$ belongs to $C^k(U)$ for every $y \in \mathbb{R}^n$, because for every multi-index $\alpha \in (\mathbb{Z}_{\geq 0})^n$ with $|\alpha| \leq k$, \n
$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} (f \ast g)(x) = \int_{\mathbb{R}^n} \frac{\partial^{|\alpha|}}{\partial x^\alpha} f(x - y) \,g(y) \,dy.$$
Let $B(r)$ be a ball about 0 of radius $r > 0$ such that $\text{supp } g \subseteq B(r).$ Then there exists an $r' > 0$ with $B(r') + U \subseteq B(r);$ furthermore, the characteristic function $\chi$ of $B(r')$ is integrable on $\mathbb{R}^n.$ For every $x \in U$ the function $\frac{\partial^\alpha h}{\partial x^\alpha} (x, \cdot)$ vanishes outside $B(r);$ consequently, the latter function does not change upon multiplication by $\chi.$ In addition, we have

$$\left| \frac{\partial^\alpha h}{\partial x^\alpha} (x, y) \right| \leq \sup_{x \in \mathbb{R}^n} |\partial^\alpha g(x)| |f(y)| \chi(y) \quad ((x, y) \in U \times \mathbb{R}^n),$$

where $|f| \chi$ is an absolutely integrable function on $\mathbb{R}^n.$ In view of a well-known theorem on changing the order of differentiation and integration (in the context of Riemann integration, see [7, Theorem 6.12.4]) we then know that $\int_{\mathbb{R}^n} h(x, y) dy$ is a $C^k$ function of $x$ whose derivatives equal the integral with respect to $y$ of the corresponding derivatives according to $x$ of the integrand $h(x, y).$

Furthermore, $h(x, y) = 0$ if $x \in U$ and $y \notin K_U,$ where

$$K_U := (\text{supp } f) \cap (\overline{U} + (-\text{supp } g)).$$

Now suppose $u \notin \text{supp } f + \text{supp } g.$ Then there exists a neighborhood $U$ of $u$ in $\mathbb{R}^n$ such that $x \notin \text{supp } f + \text{supp } g$ for all $x \in \overline{U},$ because the complement of $\text{supp } f + \text{supp } g$ is open. But this means $K_U = \emptyset,$ which implies that $(f \ast g)(x) = 0$ for all $x \in U.$

**Lemma 2.19.** Let $\phi \in C_0^\infty(\mathbb{R}^n),$ $\phi \geq 0,$ $\int \phi(x) dx = 1,$ and $\|x\| \leq 1$ if $x \in \text{supp } \phi.$ Suppose that the subset $U$ of $\mathbb{R}^n$ is measurable; see Definition 2.17. Select $\delta > 0$ arbitrarily and define, for $0 < \epsilon < \delta,$

$$\phi_\epsilon(x) = \frac{1}{\epsilon^n} \phi\left(\frac{1}{\epsilon} x\right) \quad \text{and} \quad \chi_{U, \epsilon} := 1_U \ast \phi_\epsilon.$$

Then

$$\chi_{U, \epsilon} \in C^\infty(\mathbb{R}^n), \quad 0 \leq \chi_{U, \epsilon} \leq 1, \quad \text{supp } \chi_{U, \epsilon} \subseteq U_\delta.$$

Finally, $\chi_{U, \epsilon} = 1$ on a neighborhood of $U - \delta.$

**Proof.** We have $\chi_{U, \epsilon} \in C^\infty(\mathbb{R}^n)$ by Lemma 2.18. Because $\phi \geq 0,$ we obtain

$$0 = 0 \ast \phi_\epsilon \leq 1_U \ast \phi_\epsilon \leq 1 \ast \phi_\epsilon = 1(\phi_\epsilon) = 1.$$
Usually in applications of Lemma 2.19, the set $U$ is either open or closed, but even then its characteristic function $1_U$ will not always be locally integrable in the sense of Riemann (see [7, Exercise 6.1]), whereas it is in the sense of Lebesgue; see Proposition 20.36. The only occasion in the text where this issue might play a role is in the proof of Theorem 11.22.

Finally, there are many situations in which one prefers to use, instead of $\phi \in C_0^\infty(\mathbb{R}^n)$, functions like (see Fig. 2.5)

$$\gamma(x) = \gamma_n(x) = \pi^{-\frac{n}{2}} e^{-\|x\|^2}. \quad (2.12)$$

The numerical factor is chosen such that the integral of $\gamma$ over $\mathbb{R}^n$ equals 1; this $\gamma$

![Graph of $\gamma_2$, with different horizontal and vertical scales](image)

is the Gaussian density or the probability density of the normal distribution, with expectation 0 and variance

$$\int_{\mathbb{R}^n} \|x\|^2 \gamma_n(x) \, dx = \frac{n}{2}. \quad (2.13)$$

For larger values of $\|x\|$ the values $\gamma(x)$ are so extremely small that in many situations we may just as well consider $\gamma$ as having compact support. Naturally, this is only relative: if we were to use $\gamma$ to test a function that grows at least like $e^{\|x\|^2}$ as $\|x\| \to \infty$, this would utterly fail.

For the sake of completeness we recall the calculation of $I_n := \int \gamma_n(x) \, dx$. Because $\gamma_n(x) = \prod_{j=1}^n \gamma_1(x_j)$, we have $I_n = (I_1)^n$. The change of variables $x = r(\cos \alpha, \sin \alpha)$ in a dense open subset of $\mathbb{R}^2$ now yields
\[ I_2 = \frac{1}{\pi} \int_{\mathbb{R}^2} e^{-(x_1^2 + x_2^2)} \, dx_1 \, dx_2 = \frac{1}{\pi} \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} e^{-r^2} \, r \, d\alpha \, dr = 1, \]

or
\[ \int_{\mathbb{R}} e^{-x^2} \, dx = \sqrt{\pi}. \quad (2.14) \]

We refer to [7, Exercises 2.73 and 6.15, or 6.41] for other proofs of this identity. For the computation of (2.13) introduce spherical coordinates in \( \mathbb{R}^n \) by \( x = r \omega \) with \( r > 0 \) and \( \omega \) belonging to the unit sphere \( \{ x \in \mathbb{R}^n \mid \|x\| = 1 \} \) in \( \mathbb{R}^n \); see [7, Example 7.4.12] and (13.37). Next use the substitution \( r^2 = s \) and formulas (13.30) and (13.31) below.

## Problems

2.1. Let \( U \subset \mathbb{R}^n \) be a closed set. Prove that the corresponding distance function satisfies \( |d(x, U) - d(y, U)| \leq \|x - y\| \), for all \( x \) and \( y \in \mathbb{R}^n \).

2.2. Let \( \phi \in C_0^\infty(\mathbb{R}) \), \( \phi \neq 0 \), and \( 0 \notin \text{supp } \phi \). Decide whether the sequence \( (\phi_j)_{j \in \mathbb{N}} \) converges to 0 in \( C_0^\infty(\mathbb{R}) \) if:

(i) \( \phi_j(x) = j^{-1} \phi(x - j) \).
(ii) \( \phi_j(x) = j^{-p} \phi(j \, x) \). Here \( p \) is a given positive integer.
(iii) \( \phi_j(x) = e^{-j} \phi(j \, x) \).

In each of these cases verify that for every \( x \in \mathbb{R} \) and every \( k \in \mathbb{Z}_{\geq 0} \), the sequence \( (\phi_j^{(k)}(x))_{j \in \mathbb{N}} \) converges to 0, and in addition, that in case (i) the convergence is even uniform on \( \mathbb{R} \).

2.3. Let \( \phi \) and \( \phi_\epsilon \) be as in Lemma 2.19. Prove that for every \( \psi \in C_0^\infty(X) \), the function \( \psi \ast \phi_\epsilon \) converges to \( \psi \in C_0^\infty(X) \) as \( \epsilon \downarrow 0 \).

2.4. Consider \( \beta \in C_0^\infty(\mathbb{R}) \) with \( \beta \geq 0 \) and \( \beta(x) = 0 \) if and only if \( |x| \geq 1 \). Further assume that \( \int \beta(x) \, dx = 1 \). Let \( 0 < \epsilon < 1 \), \( \beta_\epsilon(x) = \frac{1}{\epsilon} \beta(\frac{x}{\epsilon}) \), \( I = [-1, 1] \subset \mathbb{R} \), and let \( \psi = 1_I \ast \beta_\epsilon \). Determine where one has \( \psi = 0 \), where \( 0 < \psi < 1 \), and where \( \psi = 1 \), and in addition, where \( \psi' = 0 \), \( \psi' > 0 \), and \( \psi' < 0 \), respectively.

Now let \( \phi(x) = \beta(x_1) \beta(x_2) \) for \( x \in \mathbb{R}^2 \) and let \( U = I \times I \), a square in the plane. Consider \( \chi = \chi_{U_j \epsilon} \) as in Lemma 2.19. Prove that \( \chi(x) = \psi(x_1) \psi(x_2) \). Determine where one has \( \chi = 0 \), or \( 0 < \chi < 1 \), or \( \chi = 1 \), and in addition, for \( j = 1 \) and 2, where \( \partial_j \chi = 0 \), \( \partial_j \chi > 0 \), \( \partial_j \chi < 0 \). Verify that if \( 0 < \chi < 1 \), there is a \( j \) such that \( \partial_j \chi \neq 0 \). Prove by the Submersion Theorem (see [7, Theorem 4.5.2]) that for every \( 0 < c < 1 \) the level set \( N(c) := \{ x \in \mathbb{R}^2 \mid \chi(x) = c \} \) is a \( C^\infty \) curve in the plane. Is this also true for the boundary of the support of \( \chi \) and of \( 1 - \chi \)? Give a description, as detailed as possible, of the level curves of \( \chi \), including a sketch.

2.5. For \( \epsilon > 0 \), define \( \gamma_\epsilon \in C^\infty(\mathbb{R}) \) by
\[ \gamma_\epsilon(x) = \frac{1}{\epsilon \sqrt{\pi}} e^{-x^2/\epsilon^2}. \]
Calculate $|\cdot| * \gamma_{\epsilon}$. Prove that this function is analytic on $\mathbb{R}$ and examine how closely it approximates the function $|\cdot|$. Also calculate its derivatives of first and second order. See Fig. 2.6.

![Fig. 2.6 Illustration for Problem 2.5. Graphs of $\gamma_{\epsilon}$ and $|\cdot| * \gamma_{\epsilon}$ with $\epsilon = 1/50$](image)

#### 2.6. Let $U$ be a proper open subset of $\mathbb{R}^n$. Let $\gamma(x)$ be as in (2.12) and $\gamma_{\epsilon}(x) = \frac{1}{\epsilon^n} \gamma \left( \frac{1}{\epsilon} x \right)$, for $\epsilon > 0$. Denote the “probability of distance to 0 larger than $r$” by

$$
\rho(r) = \int_{\|x\| > r} \gamma(x) \, dx.
$$

Give an estimate of the $r$ for which $\rho(r) < 10^{-6}$. Prove that $\chi_{\epsilon} := 1_U * \gamma_{\epsilon}$ is analytic and that $0 < \chi_{\epsilon} < 1$. Further prove that $\chi_{\epsilon}(x) \leq \rho(\delta/\epsilon)$ if $d(x, U) = \delta > 0$; finally, show that $\chi_{\epsilon}(x) \geq 1 - \rho(\delta/\epsilon)$ if $d(x, \mathbb{R}^n \setminus U) = \delta > 0$.

#### 2.7. Show, for $a > 0$, that

$$
\int_{\mathbb{R}^n} e^{-a\|x\|^2/2} \, dx = \left( \frac{2\pi}{a} \right)^{n/2} \quad \text{and} \quad \int_{\mathbb{R}^n} \|x\|^2 e^{-a\|x\|^2/2} \, dx = \frac{n}{a} \left( \frac{2\pi}{a} \right)^{n/2}.
$$
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