Preface to the Second Edition

This is a revised and extended version of my 1995 elementary introduction to partial differential equations. The material is essentially the same except for three new chapters. The first (Chapter 8) is about non-linear equations of first order and in particular Hamilton–Jacobi equations. It builds on the continuing idea that PDEs, although a branch of mathematical analysis, are closely related to models of physical phenomena. Such underlying physics in turn provides ideas of solvability. The Hopf variational approach to the Cauchy problem for Hamilton–Jacobi equations is one of the clearest and most incisive examples of such an interplay. The method is a perfect blend of classical mechanics, through the role and properties of the Lagrangian and Hamiltonian, and calculus of variations. A delicate issue is that of identifying “uniqueness classes.” An effort has been made to extract the geometrical conditions on the graph of solutions, such as quasi-concavity, for uniqueness to hold.

Chapter 9 is an introduction to weak formulations, Sobolev spaces, and direct variational methods for linear and quasi-linear elliptic equations. While terse, the material on Sobolev spaces is reasonably complete, at least for a PDE user. It includes all the basic embedding theorems, including their proofs, and the theory of traces. Weak formulations of the Dirichlet and Neumann problems build on this material. Related variational and Galerkin methods, as well as eigenvalue problems, are presented within their weak framework. The Neumann problem is not as frequently treated in the literature as the Dirichlet problem; an effort has been made to present the underlying theory as completely as possible. Some attention has been paid to the local behavior of these weak solutions, both for the Dirichlet and Neumann problems. While efficient in terms of existence theory, weak solutions provide limited information on their local behavior. The starting point is a sup bound for the solutions and weak forms of the maximum principle. A further step is their local Hölder continuity.

An introduction to these local methods is in Chapter 10 in the framework of DeGiorgi classes. While originating from quasi-linear elliptic equations,
these classes have a life of their own. The investigation of the local and bound-
ary behavior of functions in these classes, involves a combination of methods
from PDEs, measure theory, and harmonic analysis. We start by tracing them
back to quasi-linear elliptic equations, and then present in detail some of
these methods. In particular, we establish that functions in these classes are
locally bounded and locally Hölder continuous, and we give conditions for the
regularity to extend up to the boundary. Finally, we prove that non-negative
functions on the DeGiorgi classes satisfy the Harnack inequality. This, on the
one hand, is a surprising fact, since these classes require only some sort of
Caccioppoli-type energy bounds. On the other hand, this raises the question
of understanding their structure, which to date is still not fully understood.
While some facts about these classes are scattered in the literature, this is per-
haps the first systematic presentation of DeGiorgi classes in their own right.
Some of the material is as recent as last year. In this respect, these last two
chapters provide a background on a spectrum of techniques in local behavior
of solutions of elliptic PDEs, and build toward research topics of current active
investigation.

The presentation is more terse and streamlined than in the first edi-
tion. Some elementary background material (Weierstrass Theorem, mollifiers,
Ascoli–Arzelá Theorem, Jensen’s inequality, etc..) has been removed.

I am indebted to many colleagues and students who, over the past fourteen
years, have offered critical suggestions and pointed out misprints, imprecise
statements, and points that were not clear on a first reading. Among these
Giovanni Caruso, Xu Guoyi, Hanna Callender, David Petersen, Mike O’Leary,
Changyong Zhong, Justin Fitzpatrick, Abey Lopez and Haichao Wang. Special
thanks go to Matt Calef for reading carefully a large portion of the manu-
script and providing suggestions and some simplifying arguments. The help
of U. Gianazza has been greatly appreciated. He has read the entire manu-
script with extreme care and dedication, picking up points that needed to be
clarified. I am very much indebted to Ugo.

I would like to thank Avner Friedman, James Serrin, Constantine
Dafermos, Bob Glassey, Giorgio Talenti, Luigi Ambrosio, Juan Manfredi,
John Lewis, Vincenzo Vespri, and Gui Qiang Chen for examining the manu-
script in detail and for providing valuable comments. Special thanks to David
Kinderlehrer for his suggestion to include material on weak formulations and
direct methods. Without his input and critical reading, the last two chapters
probably would not have been written. Finally, I would like to thank Ann
Kostant and the entire team at Birkhäuser for their patience in coping with
my delays.

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June 2009

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Partial Differential Equations
Second Edition
DiBenedetto, E.
2010, XX, 389 p. 19 illus., Hardcover
A product of Birkhäuser Basel