

Preface

Diophantus, the “father of algebra,” is best known for his book *Arithmetica*, a work on the solution of algebraic equations and the theory of numbers. However, essentially nothing is known of his life, and there has been much debate regarding precisely the years in which he lived.

Diophantus did his work in the great city of Alexandria. At this time, Alexandria was the center of mathematical learning. The period from 250 BCE to 350 CE in Alexandria is known as the Silver Age, also the Later Alexandrian Age. This was a time when mathematicians were discovering many ideas that led to our current conception of mathematics. The era is considered silver because it came after the Golden Age, a time of great development in the field of mathematics. This Golden Age encompasses the lifetime of Euclid.

The quality of mathematics from this period was an inspiration for the axiomatic methods of today's mathematics.

While it is known that Diophantus lived in the Silver Age, it is hard to pinpoint the exact years in which he lived. While many references to the work of Diophantus have been made, Diophantus himself made few references to other mathematicians' work, thus making the process of determining the time that he lived more difficult.

Diophantus did quote the definition of a polygonal number from the work of Hypsicles, who was active before 150 BCE, so we can conclude that Diophantus lived after that date. From the other end, Theon, a mathematician also from Alexandria, quoted the work of Diophantus in 350 CE. Most historians believe that Diophantus did most of his work around 250 CE. The greatest amount of information about Diophantus's life comes from the possibly fictitious collection of riddles written by Metrodorus around 500 CE. One of these is as follows:

His boyhood lasted $1/6$ of his life; he married after $1/7$ more; his beard grew after $1/12$ more, and his son was born five years later; the son lived to half his father's age, and the father died four years after the son.

Diophantus was the first to employ symbols in Greek algebra. He used a symbol (arithmos) for an unknown quantity, as well as symbols for algebraic operations and for powers. *Arithmetica* is also significant for its results in the theory of numbers, such as the fact that no integer of the form $8n + 7$ can be written as the sum of three squares.

Arithmetica is a collection of 150 problems that give approximate solutions to equations up to degree three. *Arithmetica* also contains equations that deal with indeterminate equations. These equations deal with the theory of numbers.

The original *Arithmetica* is believed to have comprised 13 books, but the surviving Greek manuscripts contain only six.

The others are considered lost works. It is possible that these books were lost in a fire that occurred not long after Diophantus finished *Arithmetica*.

In what follows, we call a *Diophantine equation* an equation of the form

$$f(x_1, x_2, \dots, x_n) = 0, \quad (1)$$

where f is an n -variable function with $n \geq 2$. If f is a polynomial with integral coefficients, then (1) is an *algebraic Diophantine equation*.

An n -uple $(x_1^0, x_2^0, \dots, x_n^0) \in \mathbb{Z}^n$ satisfying (1) is called a *solution* to equation (1). An equation having one or more solutions is called *solvable*.

Concerning a Diophantine equation three basic problems arise:

Problem 1. Is the equation solvable?

Problem 2. If it is solvable, is the number of its solutions finite or infinite?

Problem 3. If it is solvable, determine all of its solutions.

Diophantus's work on equations of type (1) was continued by Chinese mathematicians (third century), Arabs (eight through twelfth centuries) and taken to a deeper level by Fermat, Euler,

Lagrange, Gauss, and many others. This topic remains of great importance in contemporary mathematics.

This book is organized in two parts. The first contains three chapters. Chapter 1 introduces the reader to the main elementary methods in solving Diophantine equations, such as decomposition, modular arithmetic, mathematical induction, and Fermat's infinite descent. Chapter 2 presents classical Diophantine equations, including linear, Pythagorean, higher-degree, and exponential equations, such as Catalan's. Chapter 3 focuses on Pell-type equations, serving again as an introduction to this special class of quadratic Diophantine equations. Chapter 4 contains some advanced methods involving Gaussian integers, quadratic rings, divisors of certain forms, and quadratic reciprocity. Throughout Part I, each of the sections contains representative examples that illustrate the theory.

Part II contains complete solutions to all exercises in Part I. For several problems, multiple solutions are presented, along with useful comments and remarks. Many of the selected exercises and problems are original or have been given original solutions by the authors.

The book is intended for undergraduates, high school students and teachers, mathematical contest (including Olympiad and Putnam) participants, as well as any person interested in mathematics.

We would like to thank Richard Stong for his careful reading of the manuscript. His pertinent suggestions have been very useful in improving the text.

June 2010

Titu Andreescu
Dorin Andrica
Ion Cucuruzeanu



<http://www.springer.com/978-0-8176-4548-9>

An Introduction to Diophantine Equations

A Problem-Based Approach

Andreescu, T.; Andrica, D.; Cucurezeanu, I.

2010, XI, 345 p., Hardcover

ISBN: 978-0-8176-4548-9

A product of Birkhäuser Basel