Behavior Near Time Infinity of Solutions of the Vorticity Equations

The Navier–Stokes equations are famous as fundamental equations of fluid mechanics and have been well studied as typical nonlinear partial differential equations in mathematics. It is not too much to say that various mathematical methods for analyzing nonlinear partial differential equations have been developed through studies of the Navier–Stokes equations. There have been many studies of the behavior of solutions of the Navier–Stokes equations near time infinity. In this chapter, as an application of the previous section, we study the behavior of the vorticity of a two dimensional flow near time infinity. In particular, we study whether or not the vorticity converges to a self-similar solution.

The main purpose of this chapter is to show that the vorticity of a two-dimensional flow asymptotically converges to a constant multiple of the Gauss kernel (called the Gaussian vortex, which is self-similar) if the total circulation is sufficiently small. This result is applicable (as mentioned in §2.6) to the problem of the formation of the Burgers vortex in a three-dimensional flow, which is a very interesting topic in fluid mechanics. (Very recently, the smallness assumption has been removed. We shall mention this improvement at the end of this chapter.) This type of asymptotic behavior of the vorticity (§2.2.2) is proved in papers cited in §2.7.1. To estimate a limit of rescaled solutions is an important step in the proof, and it has not been mentioned in the literature so far. In this chapter we will present a new result concerning this step and complete the whole proof. Moreover, we give a clearer proof of the asymptotic formula (§2.4 and §2.5) by introducing recent improvements of the fundamental $L^q$-$L^1$ estimate (§2.3) of the linear heat equation with a convective term. The estimates of several quantities, including the derivatives of vorticities and velocities, are established by applying the fundamental $L^q$-$L^1$ estimate, in which various fundamental inequalities in calculus (§2.4) play essential roles. These inequalities are proved in Chapter 6. In this chapter, we often rewrite differential equations as integral equations. Such an operation is justified in Chapter 4. The existence and the uniqueness of solutions to the vorticity equations are stated in §2.2.1 without proofs. We admit these results
here. When we consider the vorticity equations it is useful to study the heat equation with a convective term, for it is considered a linearized version of the original equations. The existence of solutions to this linearized equation is again admitted in this chapter. Several properties of the fundamental solution to the heat equation with a convective term are presented in Lemma 2.5.2 without proof. They are effectively used to obtain the estimates for the limit of rescaled solutions. Throughout this chapter we try to establish sharp results as elementarily as possible. For example, an elementary proof is presented for the estimates of derivatives of the vorticity (§2.4.2), which gives new results for the cases $p = 1$ and $p = \infty$. In §2.1, we derive the vorticity equations from the Navier–Stokes equations, and in §2.7, we mention the history of research on the vorticity equations and related topics. This chapter intends to give an elementary approach without Lebesgue integrals or distribution theory, so the only prerequisite to reading it is a basic knowledge of differential and integral calculus for functions of several variables. For this reason some assumptions of the results are not optimal.

2.1 Navier–Stokes Equations and Vorticity Equations

We consider the Navier–Stokes equations, which are used to model the motion of incompressible viscous flows and which are the fundamental equations of fluid mechanics:

$$
\rho_0 \left\{ \frac{\partial u^i}{\partial t}(x, t) + \sum_{j=1}^{n} u^j(x, t) \frac{\partial u^i}{\partial x_j}(x, t) \right\} - \nu \sum_{j=1}^{n} \frac{\partial^2 u^i}{\partial x_j^2}(x, t) + \frac{\partial p}{\partial x_i}(x, t) = 0
$$

for $x \in \mathbb{R}^n$, $0 < t < T$, and $i = 1, 2, \ldots, n$, and

$$
\sum_{j=1}^{n} \frac{\partial u^j}{\partial x_j}(x, t) = 0
$$

for $x \in \mathbb{R}^n$ and $0 < t < T$.

Here we assume $T > 0$ or $T = \infty$, and $n$ denotes an integer greater than or equal to 2. The vector

$$
u(x, t) = (u^1(x, t), u^2(x, t), \ldots, u^n(x, t))
$$

denotes the velocity vector of the fluid at a point $x \in \mathbb{R}^n$ and at time $t \in (0, T)$; $p(x, t)$ denotes the pressure of the fluid. Of course, $u^i(x, t)$ ($i = 1, \ldots, n$) and $p(x, t)$ are real-valued functions, and $\rho_0$ and $\nu$ are positive constants that describe the density and the kinematic viscosity of the fluid, respectively. We note that the above system is the Navier–Stokes equations with no external force term. For given $\rho_0$, $\nu$, and the initial velocity $u(x, 0)$, the problem to find $u$ and $p$ satisfying the above Navier–Stokes equations is called the initial
value problem for the Navier–Stokes equations. Here we have \( n + 1 \) equations and \( n + 1 \) unknown functions. Observe that by assuming an initial condition also for the pressure, the conditions are overdetermined and we cannot solve the initial value problem. Hence, we do not assign the initial value of the pressure. In physics one often adds the word “field” to describe physical quantities depending on \( x \). For example, \( u \) is called the velocity vector field and \( p \) is called the pressure field. However, in this book we do not use this terminology.

We often express the Navier–Stokes equations in a concise form using notation of vector analysis:

\[
\rho_0 \{ \partial_t u + (u, \nabla)u \} - \nu \Delta u + \nabla p = 0 \quad \text{in } \mathbb{R}^n \times (0, T),
\]

\[
\text{div } u = 0 \quad \text{in } \mathbb{R}^n \times (0, T).
\]

Here, div and \( \nabla \) denote the divergence and the gradient with respect to the spatial variable \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), respectively. Moreover, \( (u, \nabla) \) denotes the operator \( \sum_{j=1}^n u^j \frac{\partial}{\partial x_j} \), and we assume that it acts on each element \( u^i \) of \( u \). Namely, the \( i \)th component of \( (u, \nabla)u \) is \( \sum_{j=1}^n u^j \frac{\partial u^i}{\partial x_j} \); and the Laplacian \( \Delta u \) for the vector-valued function \( u \) is \( (\Delta u^1, \Delta u^2, \ldots, \Delta u^n) \). The first equation describes the momentum conservation law, and the second equation describes the mass conservation law, which expresses the incompressibility.

Using a suitable scaling transformation for the dependent variables \( u \) and \( p \), and independent variables \( x \) and \( t \), we may assume that \( \rho_0 = 1 \), and \( \nu = 1 \). In fact, for example, if we set \( \tilde{t} = (\rho_0 \nu)^{-1/3} t, \tilde{x} = (\rho_0/\nu^2)^{1/3} x, \tilde{u} = (\rho_0^2/\nu)^{1/3} u, \) and \( \tilde{p} = (\rho_0/\nu^2)^{1/3} p \), then we obtain (at least formally) the Navier–Stokes equations for \( \tilde{u}(\tilde{x}, \tilde{t}) \) and \( \tilde{p}(\tilde{x}, \tilde{t}) \) on \( \mathbb{R}^n \times (0, \tilde{T}) \) with \( \rho_0 = 1 \) and \( \nu = 1 \). (We may obtain the Navier–Stokes equations with \( \rho_0 = 1 \) and \( \nu = 1 \) also by another transformation.) Here we set \( \tilde{T} = (\rho_0 \nu)^{-1/3} T \). Thus we assume that the positive constants \( \rho_0 \) and \( \nu \) are 1, unless otherwise claimed. That is, we consider

\[
\partial_t u + (u, \nabla)u - \Delta u + \nabla p = 0 \quad \text{in } \mathbb{R}^n \times (0, T),
\]

\[
\text{div } u = 0 \quad \text{in } \mathbb{R}^n \times (0, T).
\]

### 2.1.1 Vorticity

Let a set \( v = (v^1, v^2, \ldots, v^n) \) of functions \( v^i \) \( (i = 1, 2, \ldots, n) \) be an \( n \)-dimensional (real) vector-valued function defined on \( \mathbb{R}^n \), namely, a vector field on \( \mathbb{R}^n \). (Here and hereinafter, we simply call \( v \) a function, or \( \mathbb{R}^n \)-valued function, if we need to emphasize that \( v \) is vector-valued. Let curl be the differential operator that represents the rotation. (It is also expressed as rot.) That is, for spatial variables \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), and for \( v \) whose component \( v^i \) is \( C^1 \) on \( \mathbb{R}^n \), we define
\[ \text{curl } v = \begin{cases} \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}, & \text{if } n = 2, \\ \left( \frac{\partial v^3}{\partial x_2} - \frac{\partial v^2}{\partial x_3}, \frac{\partial v^1}{\partial x_3}, \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2} \right), & \text{if } n = 3. \end{cases} \]

If \( v \) denotes the velocity, then \( \text{curl } v \) is called the \textit{vorticity}. In case of spatial dimension \( n = 3 \), if \( v^3 \equiv 0 \) and \((v^1, v^2)\) depends only on \((x_1, x_2)\), so that \( v = (v^1(x_1, x_2), v^2(x_1, x_2), 0) \), then
\[
\text{curl } v = \left( 0, 0, \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2} \right).
\]
Thus we may identify the third component with \( \text{curl } (v^1, v^2) \) for \( n = 2 \).

Next we consider \( n = 3 \) and \( v = (0, 0, \varphi) \). Here we assume that \( \varphi = \varphi(x_1, x_2) \) depends only on \( x_1 \) and \( x_2 \) (is independent of \( x_3 \)) and that \( \varphi \) is continuously differentiable. In this case
\[
\text{curl } v = \left( \frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1}, 0 \right),
\]
and we may identify this by \( \nabla^\perp \varphi \). Here, we define the differential operator \( \nabla^\perp \) by \( \nabla^\perp \varphi = \left( \frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1} \right) \). By definition, \( \langle \nabla^\perp \varphi, \nabla \varphi \rangle = 0 \), namely, \( \nabla^\perp \varphi \) is perpendicular to \( \nabla \varphi \), so we use the notation \( \nabla^\perp \).

In the following, we explain a convenient formula for deriving the vorticity equations. The proofs are left to the reader as Exercise 2.1.

### 2.1.2 Vorticity and Velocity

**Proposition.** Assume that \( n = 2 \) or \( n = 3 \). Let \( v = (v^1, v^2, \ldots, v^n) \) be a vector field on \( \mathbb{R}^n \). Assume that its components \( v^j \) (\( j = 1, 2, \ldots, n \)) are continuous up to second order derivatives (i.e., \( v^j \in C^2(\mathbb{R}^n) \)). Then for \( n = 3 \) we have
\[
-\Delta v = \text{curl curl } v - \nabla \text{ div } v \quad \text{in } \mathbb{R}^3; \tag{2.3.1}
\]
for \( n = 2 \) we have
\[
-\Delta v = \nabla^\perp \text{curl } v - \nabla \text{ div } v \quad \text{in } \mathbb{R}^2. \tag{2.3.2}
\]

We assume that the velocity \( v \) and the pressure \( p \) (in the Navier–Stokes equations) are sufficiently smooth, and we write the vorticity as \( \omega(x, t) = \text{curl } u(x, t) \). For \( n = 3 \) vorticity \( \omega \) is an \( \mathbb{R}^3 \)-valued function; for \( n = 2 \) vorticity \( \omega \) is a scalar real-valued function. By the above proposition and (2.2) we see that \(-\Delta u(x, t) = \text{curl } \omega(x, t)\) when \( n = 3 \); \(-\Delta u(x, t) = \nabla^\perp \omega(x, t)\) when \( n = 2 \).

Applying \text{curl} to (2.1), for \( n = 3 \) we obtain
\[
\partial_t \omega + (u, \nabla) \omega - (\omega, \nabla) u - \Delta \omega = 0 \quad \text{in } \mathbb{R}^3 \times (0, T).
\]
Here, we have used \( \text{curl} \left( (u, \nabla)u \right) = (u, \nabla)\omega - (\omega, \nabla)u + \omega(\text{div} \ u) \), \( \text{curl} \ (\Delta u) = \Delta \omega \), and \( \text{curl} \ (\nabla p) = 0 \). In the case \( n = 2 \), using \( \text{curl} \left( (u, \nabla)u \right) = (u, \nabla)\omega \), we obtain
\[
\partial_t \omega + (u, \nabla)\omega - \Delta \omega = 0 \quad \text{in} \ \mathbb{R}^2 \times (0, T) \tag{2.4}
\]
for \( u \) satisfying \( \text{div} \ u = 0 \).

Hence from the Navier–Stokes equations, we obtain the following equation for the vorticity \( \omega \) and the velocity \( u \); in the case \( n = 3 \),
\[
\partial_t \omega + (u, \nabla)\omega - (\omega, \nabla)u - \Delta \omega = 0 \quad \text{in} \ \mathbb{R}^3 \times (0, T),
\]
\[-\Delta u = \text{curl} \ \omega \quad \text{in} \ \mathbb{R}^3 \times (0, T). \tag{2.5}\]

2.1.3 Biot–Savart Law

In the sequel we assume that the dimension of the space is two. We set \( E(x) = -\frac{1}{2\pi} \log |x|, \ x \in \mathbb{R}^2, x \neq 0 \). This is called the fundamental solution of the Laplace operator. The next proposition is proved in §6.3.5.

**Proposition.** For \( f \in C_0^\infty(\mathbb{R}^2) \), \( -\Delta(E \ast f) = f \) in \( \mathbb{R}^2 \).

Here \( \ast \) denotes convolution, i.e., \( (E \ast f)(x) = \int_{\mathbb{R}^2} E(x - y)f(y)dy \). We use the same notation for the convolution \( a \ast b = (a \ast b^1, a \ast b^2) \) of a scalar function \( a = a(x) \) and an \( \mathbb{R}^2 \)-valued function \( b = (b^1(x), b^2(x)) \) that are defined on \( \mathbb{R}^2 \).

For a smooth real-valued function \( w \) defined on \( \mathbb{R}^2 \) we set \( v = E \ast (\nabla^\perp w) \). If the support of \( \nabla^\perp w \) is compact, then by the above proposition, \( w \) satisfies
\[-\Delta v = \nabla^\perp w \quad \text{in} \ \mathbb{R}^2.\]

Conversely, \( v \) satisfying this equation is expressed by \( v = E \ast (\nabla^\perp w) \), under a suitable decay condition on \( v \) at space infinity.\(^1\) Thus \( v = E \ast (\nabla^\perp w) \) is formally equivalent to \( -\Delta v = \nabla^\perp w \), in this sense.

We define a vector field \( K \) (which is defined on the domain \( \mathbb{R}^2 \) excluding the origin) as
\[
K(x) = \frac{1}{2\pi} \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right), \quad x \in \mathbb{R}^2, \ x \neq 0. \tag{2.6}
\]

\(^1\) To show this, use the fact that a bounded harmonic function on the whole plane is a constant. This statement is called Liouville’s theorem.
Since
\[ \nabla^\perp E(x) = \left( \frac{\partial E}{\partial x_2}(x), -\frac{\partial E}{\partial x_1}(x) \right) = K(x), \quad x \in \mathbb{R}^2, \quad x \neq 0, \]
we obtain
\[ E \ast (\nabla^\perp w) = (\nabla^\perp E) \ast w = K \ast w \quad \text{in} \quad \mathbb{R}^2 \]
(at least for \( w \in C_0^\infty \mathbb{R}^2 \)). (For the justification of the commutation of convolution and differential, see §4.1.4 and §6.3.5. See also §6.3.6.) The formula \( v = K \ast w \) determining \( v \) from \( w \) is called the Biot–Savart law. The function \( v \) obtained by this relation satisfies \( \text{div} \, v = \text{div} \, (K \ast w) = \text{div} \, \nabla^\perp (E \ast w) = 0 \) in \( \mathbb{R}^2 \).

Here and hereinafter we use the symbol \( K \) as defined in (2.6).

2.1.4 Derivation of the Vorticity Equations

We have obtained (2.4) and (2.5) from the two-dimensional Navier–Stokes equations of §2.1.2. Instead of (2.5) we consider the Biot–Savart law, which is formally equivalent to (2.5) (We note that (2.5) has been derived from the mass conservation law (2.2).) So we consider
\[
\begin{align*}
\partial_t \omega + (u, \nabla) \omega - \Delta \omega &= 0 \quad \text{in} \quad \mathbb{R}^2 \times (0, T), \\
u &= K \ast \omega \quad \text{in} \quad \mathbb{R}^2 \times (0, T).
\end{align*}
\]
This system is called the two-dimensional vorticity equations. For a given function \( \omega_0 \) on \( \mathbb{R}^2 \), the problem of finding a real-valued function \( \omega = \omega(x, t) \) and an \( \mathbb{R}^2 \)-valued function \( u = (u^1(x, t), u^2(x, t)) \) satisfying
\[ \omega(x, 0) = \omega_0(x), \quad x \in \mathbb{R}^2, \quad (2.9) \]
and the vorticity equations is called the initial value problem for the vorticity equations. In this chapter, we analyze the asymptotic behavior of the vorticity near time infinity.

As stated above, the vorticity equations are derived from the Navier–Stokes equations. Conversely, we can also derive the Navier–Stokes equations from the vorticity equations by determining the pressure \( p \) suitably. (For example, see [Giga Miyakawa Osada 1988].) Hence the analysis for solutions of the vorticity equations is equivalent to the analysis for solutions of the Navier–Stokes equations.

2.2 Asymptotic Behavior Near Time Infinity

Consider the initial value problem of the vorticity equations (2.7), (2.8), and (2.9) in the plane. Similarly to the heat equation, if we assume that \( \omega_0 \) does
not grow at space infinity, it is known that problem (2.7), (2.8), and (2.9) has a unique global-in-time smooth solution. The existence and uniqueness problem has been well studied in various situations. In this chapter, we consider the problem in the case that the initial vorticity ω₀ is a continuous function with compact support. The existence and the uniqueness problems will be commented on in §2.7.2 together with the research history, but we will not give their proofs. In this chapter we focus on the asymptotic behavior of ω as \( t \to \infty \), admitting the following unique existence theorem.

2.2.1 Unique Existence Theorem

**Theorem.** For the initial vorticity \( ω₀ \in C₀(\mathbb{R}^2) \) there exists a unique pair of smooth functions \((ω, u)\) satisfying (2.7), (2.8), and (2.9) in \( \mathbb{R}^2 \times (0, \infty) \), and having the following properties:

(i) We have \( ω \in C(\mathbb{R}^2 \times [0, \infty)) \) and ω satisfies the initial condition (2.9). Moreover, \( \lim_{t \to 0} ||ω - ω₀||p(t) = 0 \) for any \( p \) with \( 1 \leq p \leq \infty \).

(ii) For any \( t₀ \) and \( t₁ \) with \( 0 < t₀ < t₁ \), \( \sup_{t₀ \leq t \leq t₁} ||∂ᵢᵣ^α ω||p(t) < \infty \), where \( 1 \leq p \leq \infty \), \( α \) is an arbitrary multi-index, and \( ℓ = 0, 1, 2, \ldots \).

(iii) For any \( t₀ \) and \( t₁ \) with \( 0 < t₀ < t₁ \), \( \sup_{t₀ \leq t \leq t₁} ||∂ᵢᵣ^α u||r(t) < \infty \), where \( 2 < r \leq \infty \), \( α \) is an arbitrary multi-index, and \( ℓ = 0, 1, 2, \ldots \). (For a vector-valued function \( v \), \( ||v||p \) denotes \( ||v||p \) and \( ∂ᵢᵣ^α v \) denotes the vector with \( i \)th component \( ∂ᵢᵣ^α vᵢ \), where \( vᵢ \) denotes the \( i \)th component of \( v \).)

The conditions (ii) and (iii) imply that for each \( t > 0 \), \( ω(x,t) \) and \( u(x,t) \) decay at space infinity as functions of \( x \). Moreover, \( ||ω - ω₀||p(t) \to 0 \) (as \( t \to 0 \)) in (i) means the \( L^p \)-continuity of the map in \( t \) with values \( ω(x,t) \) (which is a function of \( x \)). This property is important and is also valid for solutions of the heat equation as mentioned in Exercise 7.3 and Theorem 4.2.1.

Remark. By (2.8) we have \( u = K * ω \) (in \( \mathbb{R}^2 \times (0, \infty) \)), but for each \( t > 0 \), \( ω(x,t) \) is not compactly supported as a function of \( x \). Thus there is a problem as to whether \( K * ω \) is well defined. Fortunately, as remarked in §6.3.5, the property (ii) of the solution ω is sufficient to define (the components of ) \( K * ω \) as a smooth function on \( \mathbb{R}^2 \times (0, \infty) \) satisfying

\[
∂ᵢᵣ^α (K * ω) = K * (∂ᵢᵣ^α ω)
\]

in \( \mathbb{R}^2 \times (0, \infty) \), where \( α \) is an arbitrary multi-index and \( ℓ = 0,1,2,\ldots \).

Hereinafter, we simply define a *solution* of (2.7), (2.8), and (2.9) to be a pair of smooth solutions \((ω, u)\) that satisfies (2.7), (2.8), and (2.9) on \( \mathbb{R}^2 \times (0, \infty) \) and that satisfies properties (i), (ii), and (iii) of the unique existence theorem. The main purpose of this chapter is to establish the asymptotic behavior of ω as \( t \to \infty \) for the vorticity equations, which is similar to Theorem 1.1.4. As we will see later, ω decays as \( t \to \infty \). Our aim is to obtain the leading part of the decay.
2.2.2 Theorem for Asymptotic Behavior of the Vorticity

**Theorem.** Let the pair of functions $(\omega, u)$ denote the solution of (2.7), (2.8), and (2.9) with initial vorticity $\omega_0(\in C_0(\mathbb{R}^2))$. Furthermore, we set $m = \int_{\mathbb{R}^2} \omega_0(y)dy$. Then there exists a (small) constant $m_0$ such that for any $m$ with $|m| < m_0$, 

$$\lim_{t \to \infty} t\|\omega - mg\|_{\infty}(t) = 0$$

holds. Here $g(x,t) = G_t(x)$ is the Gauss kernel.

**Remark.** According to very recent results of Th. Gallay and C. E. Wayne [Gallay Wayne 2005], the smallness assumption $|m| < m_0$ can be removed. We shall discuss this topic in §2.8. Thus, the result exactly corresponds to Theorem 1.1.4 with $n = 2$ for the heat equation.

We can prove (2.10) by regarding the term $(u, \nabla)\omega$ of equation (2.7) as a perturbation of the heat equation and using the expression of the solution of the heat equation. However, to carry out this strategy we need the stronger assumption that 

$$\|\omega_0\|_1 = \int_{\mathbb{R}^2} |\omega_0(y)|dy$$

is sufficiently small. We give an example to show that the latter assumption is actually stronger. Consider $\omega_0$ with $\omega_0(x_1, x_2) = A \cos x_2 \sin x_1$, $|x_1| < \pi$, $|x_2| < \pi/2$, and $\omega_0(x_1, x_2) = 0$ otherwise. Although $m = 0$, $\|\omega_0\|_1$ can be chosen as large as one likes by choosing the constant $A$ large. In this book we introduce the method of the scaling transformation to prove the above theorem. Just as for the heat equation we begin by studying the scaling invariance of the vorticity equations.

2.2.3 Scaling Invariance

**Proposition.** Assume that the pair of functions $(\omega, u)$ satisfies (2.7) and (2.8) in $\mathbb{R}^2 \times (0, \infty)$. For $k > 0$ define $(\omega_k, \overline{u}_k)$ by 

$$\omega_k(x, t) = k^2\omega(kx, k^2t), \quad \overline{u}_k(x, t) = ku(kx, k^2t).$$

Then $(\omega_k, \overline{u}_k)$ satisfies (2.7) and (2.8) in $\mathbb{R}^2 \times (0, \infty)$, by replacing $\omega$ by $\omega_k$ and $u$ by $\overline{u}_k$.

**Proof.** It is easy to show that $(\omega_k, \overline{u}_k)$ satisfies (2.7) by an argument similar to the heat equation (§1.2.1). We shall check how the Biot–Savart law (2.8) varies under the scaling transformation. For $(x, t) \in \mathbb{R}^2 \times (0, \infty)$ we calculate $(K * \omega_k)(x, t)$ to get 

$$(K * \omega_k)(x, t) = \int_{\mathbb{R}^2} K(x-y)\omega_k(y, t)dy = k^2 \int_{\mathbb{R}^2} K(x-y)\omega(ky, k^2t)dy$$

$$= \int_{\mathbb{R}^2} K\left(\frac{kx-z}{k}\right)\omega(z, k^2t)dz.$$
Using the property $K(\lambda x) = \lambda^{-1} K(x)$ for $\lambda > 0$, we obtain
\[(K \ast \omega_k)(x, t) = ku(kx, k^2 t) = \overline{u}_k(x, t),\]
which implies that $(\omega_k, \overline{u}_k)$ satisfies (2.8).

Thus we obtain an invariance of the vorticity equations under the scaling transformation of (2.11). As mentioned in §1.2.1 for the heat equation, there are some other scaling transformations under which the heat equation is invariant. But observe that equations (2.7) and (2.8) are not invariant under such scaling transformations, since (2.7) includes the term $(u, \nabla)\omega$.

If a pair of functions $(\omega, u)$ satisfies the vorticity equations (2.7) and (2.8) on $\mathbb{R}^2 \times (0, \infty)$, and for any $k > 0$, $\omega = \omega_k$ and $u = \overline{u}_k$ hold on $\mathbb{R}^2 \times (0, \infty)$, then $(\omega, u)$ is called a forward self-similar solution of the vorticity equations (or simply a self-similar solution).

We next observe that (2.7), (2.8), and (2.9) have a conserved quantity similar to that of the heat equation.

### 2.2.4 Conservation of the Total Circulation

**Proposition.** Assume that a pair of functions $(\omega, u)$ is the solution of (2.7), (2.8), and (2.9) with the initial vorticity $\omega_0 (\in C_0(\mathbb{R}^2))$. Then
\[
\int_{\mathbb{R}^2} \omega(x, t) \, dx = \int_{\mathbb{R}^2} \omega_0(x) \, dx
\]
for all $t > 0$. In particular, for any $\omega_k$ ($k > 0$) defined in (2.11), and for any $t > 0$,
\[
\int_{\mathbb{R}^2} \omega_k(x, t) \, dx = \int_{\mathbb{R}^2} \omega_0(x) \, dx.
\]

**Proof.** Formally, calculating similarly to the case of the heat equation in §1.2.2, we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \omega(x, t) \, dx = \int_{\mathbb{R}^2} \Delta \omega \, dx - \int_{\mathbb{R}^2} (u, \nabla)\omega \, dx = -\int_{\mathbb{R}^2} (u, \nabla)\omega \, dx.
\]
Here we recall
\[
\text{div } u = \text{div } (K \ast \omega) = \text{div } \nabla^\perp (E \ast \omega) = 0
\]
to get $(u, \nabla)\omega = \text{div } (u\omega)$. By integration by parts (§4.5.2) we now obtain
\[
\int_{\mathbb{R}^2} (u, \nabla)\omega \, dx = \int_{\mathbb{R}^2} \text{div } (u\omega) \, dx = 0.
\]
Thus we have shown that $\int_{\mathbb{R}^2} \omega(x, t) \, dx$ is independent of $t$, and we formally obtain the first identity. Using (ii) and (iii) of the unique existence theorem, one can justify this calculation by Theorem 7.2.1. Since
\[ \int_{\mathbb{R}^2} \omega_k(x,t)dx = k^2 \int_{\mathbb{R}^2} \omega(kx,k^2t)dx = \int_{\mathbb{R}^2} \omega(y,k^2t)dy, \]

for the rescaled \( \omega_k \) and for \( t > 0 \), we get the latter identity. \( \square \)

In fluid mechanics \( \int_{\mathbb{R}^2} \omega(x,t)dx \) is called the total circulation. For this reason we used this word in the title of this section.

The Gauss kernel is a self-similar solution not only for the heat equation, but also for the vorticity equations, as we will see in \( \S 2.2.5 \).

### 2.2.5 Rotationally Symmetric Self-Similar Solutions

**Lemma.** Assume that a smooth real-valued function \( \rho(x) \) on \( \mathbb{R}^2 \) depends only on the length \( |x| = \sqrt{x_1^2 + x_2^2} \) of \( x = (x_1, x_2) \). (That is, it is invariant under rotations centered at the origin.) Assume that \( E \ast \rho \) is defined as a \( C^1 \)-function on \( \mathbb{R}^2 \) and that the vector field \( v \) is expressed by \( v = K \ast \rho = \nabla \perp (E \ast \rho) \). Then

\[ (v, \nabla)\rho \equiv 0 \text{ in } \mathbb{R}^2. \]

In particular, for \( m \in \mathbb{R} \), set \( \omega = mg (= mG_1) \) and \( u = K \ast \omega \). Then \( (\omega, u) \) satisfies the vorticity equations (2.7) and (2.8) on \( \mathbb{R}^2 \times (0, \infty) \). Hence \((mg, mK \ast g)\) is a self-similar solution.

As we will mention in Proposition 6.3.5, if \( \rho \in C_0^\infty(\mathbb{R}^2) \), then \( E \ast \rho \in C^\infty(\mathbb{R}^2) \) and \( K \ast \rho = \nabla \perp (E \ast \rho) \) in \( \mathbb{R}^2 \). The same properties hold for \( E \ast \rho \), even if the support of \( \rho \) is not compact and its decay rate as \( |x| \to \infty \) is fast like the Gauss kernel \( G_t(x) \).

**Proof.** Since \( \rho(x) \) is rotationally symmetric, \( \rho(Qx) = \rho(x) \) for any \( 2 \times 2 \) rotation matrix \( Q \) (i.e., \( Q = (\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} ) \) with \( \theta \in \mathbb{R} \)). On the other hand, by a coordinate transformation of the integral, we obtain

\[
(E \ast \rho)(Qx) = \int_{\mathbb{R}^2} E(Qx - y)\rho(y)dy = \int_{\mathbb{R}^2} E(Qx - Qz)\rho(Qz)dz = \int_{\mathbb{R}^2} E(Q(x - z))\rho(Qz)dz.
\]

Since \( E \) and \( \rho \) are rotationally symmetric, \( E \ast \rho \) is also rotationally symmetric, i.e., \( (E \ast \rho)(Qx) = (E \ast \rho)(x) \). That is to say, \( E \ast \rho \) is a function depending only on \( |x| \).

Hence \( \nabla \rho(x) \) and \( \nabla (E \ast \rho)(x) \) are parallel to \( x/|x| \) for \( x \neq 0 \). In particular, \( \nabla \rho(x) \) is parallel to \( \nabla (E \ast \rho)(x) \) for \( x \neq 0 \). On the other hand, for \( h \in C^1(\mathbb{R}^2) \), \( \text{curl } h = \nabla \perp h \) is orthogonal to the gradient vector \( \nabla h \) of \( h \). Hence \( v = K \ast \rho = \nabla \perp (E \ast \rho) \) is orthogonal to \( \nabla (E \ast \rho) \). Therefore, \( v \) is orthogonal to \( \nabla \rho \), i.e.,

\[ (v, \nabla)\rho = \langle v, \nabla \rho \rangle = \langle \nabla \perp (E \ast \rho), \nabla \rho \rangle \equiv 0. \]
(By the assumption of the smoothness of $\rho$, we have $\nabla \rho(0) = 0$. Hence the above equality is still valid on all of $\mathbb{R}^2$ including $x = 0$.) For fixed $t > 0$, the Gauss kernel $g(x, t) = G_t(x)$ is a function depending only on $|x|$. So the first result implies that $(\omega, u) = (mg, mK \ast g)$ satisfies (2.7) and (2.8), since $mg$ is a solution of the heat equation.

As in the case of the heat equation ($\S$1.2.6), to prove the asymptotic formula (2.10), it suffices to prove that the rescaled functions $\{\omega_k\}$ uniformly converge to $mg$ as $k \to \infty$ at $t = 1$, i.e., $\lim_{k \to \infty} \|\omega_k - mg\|_{\infty}(1) = 0$. The strategy of the proof is also similar to the case of the heat equation ($\S$1.2.7), but each step, i.e., to show the “compactness” or the “characterization of the limit function,” becomes more complicated. In $\S$2.3 and $\S$2.4, we shall prove estimates that play an important role in the proof of “compactness,” and we will prove the “compactness” in the first part of $\S$2.5. In $\S$2.5.1 to $\S$2.5.4, we give the “characterization of the limit function,” and complete the proof of (2.10) in $\S$2.5.5.

To prove the “compactness” we begin by deriving decay estimates for solutions of (2.7) and (2.8). Observe that a decay estimate derived from (2.7) will in general depend on $u$. Here, by the fact that the function $u$ in $(u, \nabla)\omega$ in (2.7) depends on $\omega$, it is necessary to provide a suitable estimate of $u$ and $\omega$. This is different from the case of the heat equation. If possible, we obtain a decay estimate of $\omega$ that is independent of $u$. As we prove in the next section, we fortunately obtain such an estimate from (2.7).

2.3 Global $L^q$-$L^1$ Estimates for Solutions of the Heat Equation with a Transport Term

First, since $u$ satisfying (2.8) always satisfies $\text{div} \ u = 0$, we consider (2.7) for a given $u$ satisfying $\text{div} \ u = 0$ in this section. We regard (2.7) as a linear equation with respect to $\omega$. For an unknown function $\omega$ and a given coefficient $v$ with $\text{div} \ v = 0$, consider a heat equation with terms of first derivatives (which are also called transport terms) as

$$\partial_t \omega - \Delta \omega + (v, \nabla)\omega = 0.$$  \hspace{1cm} (H_v)

Here, $v$ is an $\mathbb{R}^2$-valued function $v(x, t) = (v^1(x, t), v^2(x, t))$ defined on $\mathbb{R}^2 \times (0, \infty)$. In this section ($\S$2.3), we establish an $L^q$-$L^1$ estimate (independent of $v$) for the solution $\omega$ of this linear equation.

2.3.1 Fundamental $L^q$-$L^r$ Estimates

Theorem. Assume that the functions $v^1, v^2 \in C^\infty(\mathbb{R}^2 \times (0, \infty))$ satisfy $\text{div} \ v = 0$ in $\mathbb{R}^2 \times (0, \infty)$, where $v = (v^1, v^2)$. Assume that $\omega \in C^\infty(\mathbb{R}^2 \times (0, \infty))$ satisfies $(H_\omega)$ in $\mathbb{R}^2 \times (0, \infty)$. Moreover, they satisfy the following initial condition (I) and conditions (at space infinity) (a) and (b):
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(1) **Assume that the function** $\omega_0 \in C(\mathbb{R}^2)$ **satisfies** $\|\omega_0\|_1 < \infty$. **Assume that** $\omega \in C(\mathbb{R}^2 \times [0, \infty))$, $\omega(x, 0) = \omega_0(x), x \in \mathbb{R}^2$, **and that** $\|\omega\|_1(t)$ **is continuous at** $t = 0$.

(a) **For any** $t_0, t_1 (0 < t_0 < t_1)$, **sup$_{t_0 \leq t \leq t_1}$** $\|\partial_t^\alpha \partial_x^\beta \omega\|_p(t) < \infty$, **where** $\alpha$ **is a multi-index satisfying** $|\alpha| + 2\ell \leq 2$, **and** $\ell = 0, 1, 1 \leq p \leq \infty$.

(b) $\|v\|_\infty(t) < \infty$ **for each** $t > 0$.

Then there exists a universal constant $\kappa > 0$ that is independent of $v, \omega, \omega_0, t, q$, **such that**

$$\|\omega\|_q(t) \leq \frac{1}{(kt)^{1-1/q}} \|\omega_0\|_1$$

holds for all $t > 0$ and $q$ with $1 \leq q \leq \infty$.

In the case of $v \equiv 0$, this estimate corresponds to the $L^p$-$L^q$ estimate (1.5) for the heat equation with $q = 1$ and $\kappa = 4\pi$ in two-dimensional space. The important aspect of this estimate lies in the fact that we may take $\kappa$ independent of the special profile of $v$, provided that $\text{div} v = 0$ even if $v$ diverges to infinity as $t \to 0$. To prove this estimate, we first establish a quantitative estimate that implies that the $L^r$-norm of $\omega$ is nonincreasing as a function of $t$.

### 2.3.2 Change Ratio of $L^r$-Norm per Time: Integral Identities

**Lemma.** **Assume that the functions** $v$ **and** $\omega$ **satisfy the assumptions of the theorem in §2.3.1 except condition (I).** Then for $r = 2^m$ ($m = 0, 1, 2, \ldots$), $\|\omega\|_r(t)$ **is differentiable for** $t > 0$ and

$$\frac{d}{dt} \int_{\mathbb{R}^2} (\omega(x, t))^r dx = -4 \left(1 - \frac{1}{r}\right) \int_{\mathbb{R}^2} |\nabla((\omega(x, t))^{r/2})|^2 dx$$

for $t > 0$. In particular, for $m \geq 1$, $\|\omega\|_r(t)$ **is nonincreasing with respect to** $t$ **for** $t > 0$. Hence, for $m \geq 1$, if $\|\omega\|_r(t)$ **is continuous at** $t = 0$, **then** $\|\omega\|_r(t) \leq \|\omega\|_r(0)$ **holds for** $t \geq 0$.

**Proof.** When $r = 1$, this integral identity describes the conservation of the total circulation, and the proof is the same as in §2.2.4. We shall prove the identity for $r = 2^m, m \geq 1$. By assumption (a) in §2.3.1 we may differentiate under the integral sign (§7.2.1) to get

$$\frac{d}{dt} \int_{\mathbb{R}^2} \omega^r dx = \int_{\mathbb{R}^2} r\omega^{r-1} \partial_t \omega \, dx = \int_{\mathbb{R}^2} r\omega^{r-1} \Delta \omega \, dx - \int_{\mathbb{R}^2} r\omega^{r-1} (v, \nabla) \omega \, dx$$

for $t > 0$. Integrating by parts (§4.5.2) for the first term of the right hand side, we obtain

$$\int_{\mathbb{R}^2} r\omega^{r-1} \Delta \omega \, dx = \int_{\mathbb{R}^2} \text{div} (r\omega^{r-1} \nabla \omega) \, dx - \int_{\mathbb{R}^2} \langle r\nabla (\omega^{r-1}), \nabla \omega \rangle \, dx$$

$$= -\int_{\mathbb{R}^2} r(r-1)\omega^{r-2} |\nabla \omega|^2 \, dx;$$
moreover, using the chain rule for the composition of functions, we have
\[ = -4 \left(1 - \frac{1}{r}\right) \int_{\mathbb{R}^2} |\nabla (\omega^{r/2})|^2 dx. \]
Since \( \text{div} \, v = 0 \), integrating by parts (§4.5.2), we obtain for the second term
\[ \int_{\mathbb{R}^2} r\omega^{r-1}(v, \nabla) \omega dx = \int_{\mathbb{R}^2} (v, \nabla)\omega^r dx = \int_{\mathbb{R}^2} \text{div} (v\omega^r) dx = 0. \]
We thus obtain the integral identity at least formally. In the above calculation, we impose the decay conditions (assumptions (a), (b) in §2.3.1) for \( v, \omega \), and \( \nabla \omega \) at space infinity to justify the integration by parts. For details the reader is referred to the divergence theorem in the whole space in §4.5.2.

Using this idea, we can prove the estimate corresponding to \( q = 1 \) in §2.3.1.

2.3.3 Nonincrease of \( L^1 \)-Norm

**Lemma.** Assume that the functions \( v \) and \( \omega \) satisfy the assumption in §2.3.1. Then \( \|\omega\|_{1}(t) \leq \|\omega_{0}\|_{1} \) for all \( t \geq 0 \).

**Proof.** Since \( |\omega| \) is not differentiable by the operation \( |\cdot| \), we cannot calculate \( \frac{d}{dt} \int |\omega| \, dx \) directly. To overcome this difficulty we approximate \( |\omega| \) by smooth functions as follows. Using the function
\[ \psi_{\varepsilon}(z) = (z^2 + \varepsilon)^{1/2} - \varepsilon^{1/2}, \quad z \in \mathbb{R}, \quad \varepsilon > 0, \]
we calculate
\[ \frac{d}{dt} \int_{\mathbb{R}^2} \psi_{\varepsilon}(\omega) dx \quad \text{for} \quad t > 0. \]
Since \( |\psi_{\varepsilon}(z)| \leq |z|, \quad z \in \mathbb{R}, \quad \int_{\mathbb{R}^2} \psi_{\varepsilon}(\omega) dx \) is finite for any \( t > 0 \), provided \( \|\omega\|_{1}(t) < \infty \). Similarly to §2.3.2, by integration by parts (§4.5.2), we obtain
\[ \frac{d}{dt} \int_{\mathbb{R}^2} \psi_{\varepsilon}(\omega) dx = \int_{\mathbb{R}^2} \psi'_{\varepsilon}(\omega) \Delta \omega dx - \int_{\mathbb{R}^2} \psi'_{\varepsilon}(\omega)(v, \nabla) \omega dx \]
\[ = -\int_{\mathbb{R}^2} \langle \nabla (\psi'_{\varepsilon}(\omega)), \nabla \omega \rangle dx + \int_{\mathbb{R}^2} \text{div} (\psi'_{\varepsilon}(\omega) \nabla \omega) dx \]
\[ - \int_{\mathbb{R}^2} \text{div} (\psi_{\varepsilon}(\omega) v) dx \]
\[ = -\int_{\mathbb{R}^2} \psi''_{\varepsilon}(\omega) |\nabla \omega|^2 dx. \]
Since \( \psi_{\varepsilon} \) is convex, so that \( \psi''_{\varepsilon} > 0 \), we obtain
\[ \frac{d}{dt} \int_{\mathbb{R}^2} \psi_{\varepsilon}(\omega) dx \leq 0. \]
For $0 < \delta < t$, integrating both sides on the interval $(\delta, t)$ and letting $\varepsilon \to 0$, we obtain $\|\omega\|_1(t) \leq \|\omega\|_1(\delta)$. (The fact that limit and integration commute easily follows from $\|\omega\|_1(t) < \infty$. In fact, we may apply the dominated convergence theorem in §7.1.1.) By the assumption of continuity of $\|\omega\|_1(t)$ at $t = 0$, sending $\delta \to 0$ yields the desired result. □

In the next section we shall derive a system of differential inequalities for

$$y_r(t) = \|\omega\|_r^2(t), \quad r = 2^m, \quad m = 0, 1, 2, \ldots,$$

from the integral identity of the change ratio of the $L^r$-norm. Since the non-increasing property of $y_r(t)$ is not enough to yield the fundamental $L^q-L^1$ estimate in §2.3.1, we have to estimate the right hand side of the integral identity in Lemma 2.3.2 in a more precise way.

For this purpose, we use the Nash inequality in $\mathbb{R}^2$:

$$\|\varphi\|_2^2 \leq C \|\varphi\|_1 \|\nabla \varphi\|_2,$$

Here $\varphi$ is a continuously differentiable function defined on $\mathbb{R}^2$ (namely, $\varphi \in C^1(\mathbb{R}^2)$), $\|\varphi\|_1 < \infty$, and $C$ is a constant that is independent of $\varphi$. (The proof of this inequality is given in §6.1.2.) The Nash inequality is one of the important inequalities frequently used in the analysis of partial differential equations. Defining $\kappa$ by $C = 1/\kappa^{1/2}$, we obtain

$$\|\nabla \varphi\|_2^2 \geq \kappa \|\varphi\|_2 \|\varphi\|_1^2.$$

(Take the constant $C$ in the Nash inequality as the best possible constant obtained in [Carlen Loss 1993]. Then we may take $\kappa$ as $\kappa = \pi(j_{1,1}/2)^2 \approx 3.670 \cdot \pi$, which is still smaller than $4\pi$. Hence quantitatively, the fundamental $L^q-L^1$ estimate in §2.3.1 is still weaker than the $L^q-L^1$ estimate of the heat equation in the case $v = 0$ (§1.1.2). Here $j_{1,1}$ denotes the smallest positive zero of the Bessel function $J_1$.) Applying the Nash inequality to $\varphi = \omega^{r/2}$ ($r = 2^m, m = 1, 2, \ldots$), we obtain

$$\|\nabla (\omega^{r/2})\|_2^2(t) \geq \kappa \|\omega^{r/2}\|_2^2(t)/\|\omega^{r/2}\|_2^2(t) = \kappa \|\omega\|_{r/2}^2(t) \|\omega\|_{r/2}^{-r/2}(t)$$

for $t > 0$ (provided that $\|\omega\|_{r/2}(t) \neq 0$). By the integral identity in Lemma 2.3.2, we now obtain

$$\frac{d}{dt}\|\omega\|_r^2(t) \leq -4\kappa \left(1 - \frac{1}{r}\right) \|\omega\|_{r/2}^2(t) \|\omega\|_{r/2}^{-r/2}(t), \quad t > 0.$$

We thus obtain the following system of differential inequalities.

2.3.4 Application of the Nash Inequality

**Proposition.** Assume that $v$ and $\omega$ satisfy the assumptions in Theorem 2.3.1 except for condition (I). For $r = 2^m$ ($m = 0, 1, 2, \ldots$) and $y_r(t) = \|\omega\|_r^2(t)$ the following differential inequalities hold:
\[
\frac{y_r^2}{2} \frac{dy_r}{dt} \leq -4\kappa \left(1 - \frac{1}{r}\right) y_r^2, \quad t > 0, \quad (r = 2^m, m = 1, 2, 3, \ldots),
\]

where \(\kappa\) is a universal constant independent of \(v\) and \(\omega\).

Remark. Both the differential inequalities in the above proposition and the integral identities in Lemma 2.3.2 hold for any strictly positive \(t\). So we need not assume condition (I) of Theorem 2.3.1.

This system of differential inequalities leads to a successive estimate for \(y_{2^m}(t)\) by the following lemma. Note that this lemma itself is independent of the above proposition.

Lemma. Let \(r = 2^m (m = 0, 1, 2, \ldots)\) and \(a > 0\). Assume that \(y_r = y_r(t)\) is a positive function defined in \((0, \infty)\), belonging to \(C^1(0, \infty)\) for \(m \geq 1\), and satisfying\(^1\)

\[
\frac{dy_r}{dt} \leq -a \left(1 - \frac{1}{r}\right) \frac{y_r^2}{y_{r/2}^2}, \quad t > 0, \quad (r = 2^m, m = 1, 2, \ldots).
\]

Moreover, assume that \(y_1\) is bounded in \((0, \infty)\), namely, there exists a constant \(M_1 > 0\) such that \(y_1(t) \leq M_1 (t > 0)\). Then the following two statements are valid:

(i) The inequality

\[
y_r(t) \leq M_r t^{1-r}, \quad t > 0, \quad r = 2^m, \quad m = 0, 1, 2, \ldots,
\]

holds, where \(M_r\) is defined by \(M_r = a^{-1} r M_{r/2}^2\) successively.

(ii) If the inequality in (i) holds for \(r = 2^m (m = 0, 1, 2, \ldots)\), then for sufficiently large \(m\),

\[
(y_r(t))^{1/r} \leq \frac{4}{a} M_1 t^{-1+1/r}, \quad t > 0.
\]

Proof. (i) Let us prove the claim by an induction argument with respect to \(m\). It is obvious in the case \(m = 0\). We shall prove \(y_{2r}(t) \leq M_{2r} t^{1-2r}\) with \(r = 2^m\) under the assumption that the claim is valid for any positive integer less than or equal to \(m\). Applying the assumption of the induction to the differential inequality, we obtain

\[
\frac{dy_{2r}}{dt} \leq -a \left(1 - \frac{1}{2r}\right) \frac{y_{2r}^2(t)}{M_{r}^2} t^{2r-2}
\]

for \(t > 0\). Dividing both sides by \(-y_{2r}^2\), we get

\[
-\frac{dy_{2r}}{dt} / y_{2r}^2(t) \geq a \left(1 - \frac{1}{2r}\right) M_{r}^{-2} t^{2r-2} \quad (> 0).
\]

\(^1\) \(y_r^2(t)\) denotes \((y_r(t))^2\).
Integrating both sides over the interval \([s, t] \subset (0, \infty)\), we obtain

\[
\frac{1}{y_{2r}(t)} - \frac{1}{y_{2r}(s)} \geq a \left( 1 - \frac{1}{2r} \right) M_r^{-2} \int_s^t \tau^{2r-2} d\tau = \frac{a}{2r M_r^2} (t^{2r-1} - s^{2r-1}).
\]

Recalling that \(y_{2r}(s) \geq 0\), we obtain

\[
\frac{1}{y_{2r}(t)} \geq \frac{a}{2r M_r^2} (t^{2r-1} - s^{2r-1}),
\]

and letting \(s \to 0\) results in

\[
\frac{1}{y_{2r}(t)} \geq \frac{a}{2r M_r^2} t^{2r-1}, \quad t > 0.
\]

Hence \(y_{2r}(t) \leq M_{2r} t^{1-2r}\), and the proof is now complete.

(ii) Setting \(\mu_r = M_r^{1/r}\), we obtain

\[
(y_r(t))^{1/r} \leq \mu_r t^{-1+1/r}, \quad t > 0, \ m = 0, 1, 2, \ldots,
\]

by (i). So we shall estimate \(\mu_r\) by successive equalities to prove \(\mu_r \leq 4\mu_1/a\) for large \(r\). To obtain this estimate, for \(r = 2^m\), we set \(b_m = \log \mu_r\). Since \(M_r^{1/r} = (r/a)^{1/r} M_{r/2}^{2/r}\), for \(b_m\) we obtain the following successive equalities:

\[
b_0 = \log \mu_1,
\]

\[
b_m = b_{m-1} + \frac{1}{2^m} \log \left( \frac{2^m}{a} \right), \quad m = 1, 2, \ldots.
\]

We thus deduce

\[
b_m = b_0 + \sum_{j=1}^{m} \left( \frac{j}{2^j} \log 2 - \frac{1}{2^j} \log a \right), \quad m = 1, 2, \ldots.
\]

If \(j\) is sufficiently large, say \(2^j/a > 1\), each summand is positive. Hence, for sufficiently large \(m\), we obtain (say \(2^m > a\))

\[
b_m \leq b_0 + \left( \sum_{j=1}^{\infty} \frac{j}{2^j} \right) \log 2 - \left( \sum_{j=1}^{\infty} \frac{1}{2^j} \right) \log a
\]

\[
= b_0 + 2 \log 2 - \log a = b_0 + \log(4/a).
\]

(Since the series or its derivative is a geometric series, it is easy to determine their values (Exercise 2.2).) Applying \(\exp\) to both sides, for a sufficiently large \(r\), we obtain \(\mu_r \leq 4\mu_1/a\). The proof of (ii) is now complete. \(\square\)
2.3.5 Proof of Fundamental $L^q-L^1$ Estimates

We are now in position to prove Theorem 2.3.1. By an application of the Nash inequality, we obtain a system of differential inequalities for $y_r$ as in Proposition 2.3.4 with $y_r(t) = \|\omega\|_{r}(t)$ for $r = 2^m(m = 0, 1, 2, \ldots)$. By the nonincrease of the $L^1$-norm, which is obtained in Lemma 2.3.3, we have $y_1(t) \leq M_1$, $t \geq 0$ with $M_1 = \|\omega_0\|_1$. Note that $y_r(t)$ is positive for $t \geq 0$ unless $\omega_0 \equiv 0$. (Indeed, if $y_r(t_0) = 0$ for some $t = t_0 > 0$, then by the strong maximum principle ($\S$2.3.8 and [Protter Weinberger 1967]) $\omega$ must be zero for $t \in [0, t_0]$.) The result for $\omega \equiv 0$ is trivial, so we may assume that $y_r(t) > 0$ for all $t \geq 0$. Using Lemma 2.3.4 with $a = 4\kappa$, for sufficiently large $m$ we obtain $\|\omega\|_{r}(t) \leq \frac{1}{\kappa t^{1-1/r}}\|\omega_0\|_1$, $t > 0$, with $r = 2^m$. Since $\|\omega\|_{\infty}(t) = \lim_{r \to \infty} \|\omega\|_{r}(t)$ for $t > 0$ (Exercise 2.3), we get $\|\omega\|_{\infty}(t) \leq \frac{1}{\kappa t}\|\omega_0\|_1$, $t > 0$.

By the H"older inequality ($\S$4.1.1), for $q$ with $1 \leq q \leq \infty$, $\|\omega\|_{q}(t) \leq \|\omega\|_{1}(t)^{1/q}\|\omega\|_{\infty}^{1-1/q}(t)$, $t > 0$.

(This may also be derived by a direct calculation (Exercise 2.4).) Thus the nonincrease of the $L^1$-norm $\|\omega\|_{1}(t) \leq \|\omega_0\|_1$ for $t \geq 0$ and the estimate of $\|\omega\|_{\infty}(t)$ imply $\|\omega\|_{q}(t) \leq \frac{1}{(\kappa t)^{1-1/q}}\|\omega_0\|_1$, $t > 0$,

which yields the assertion. 

One may prove the fundamental $L^q-L^1$ estimate by the system of differential inequalities, nonincrease of the $L^1$-norm, and by a duality argument without using Lemma 2.3.4. We shall give the idea of this method of proof. Setting $r = 2$ in the system of differential inequalities in $\S$2.3.4, and recalling that $\|\omega\|_{1}(t) \leq \|\omega_0\|_1$ in $\S$2.3.2, we obtain $\frac{dy_2}{dt} \leq -2\kappa y_2^2 \|\omega_0\|^{-2}_1$.

Since $y_2 \geq 0$, this inequality implies that $y_2(t)$ is nonincreasing with respect to $t$. Thus, if $y_2(t_1) = 0$ for some $t_1 \geq 0$, then $y_2(t) = 0$ for all $t \geq t_1$. If there exists no such $t_1$, then $y_2(t) > 0$ for $t > 0$ and $\lim_{t \to \infty} y_2(t) = 0$. Let $t_*$ be the minimum for such $t_1$ (admit $t_* = \infty$). If $t_* = 0$, by the continuity of $\|\omega\|_{1}(t)$ at $t = 0$, we obtain $\omega_0 \equiv 0$, so that $\omega \equiv 0$ by $\|\omega\|_{1}(t) \leq \|\omega_0\|_1$. Thus, we may assume $t_* > 0$. Dividing both sides of the above differential inequality by $y_2$ for $0 < t < t_*$, and integrating over $(0, t)$, we obtain
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\[ y_2(t) \leq (2\kappa t)^{-1}\|\omega_0\|_1^2, \]

or the \(L^2-L^1\) estimate

\[ \|\omega\|_2(t) \leq (2\kappa t)^{-1/2}\|\omega_0\|_1, \quad t > 0. \]

Next we consider the “duality problem” for \((H_v)\). Fix \(t_0 > 0\) and consider

\[ \partial_t \psi + (v, \nabla) \psi + \Delta \psi = 0 \quad \text{in } \mathbb{R}^2 \times (0, t_0), \]

\[ \psi(x, t_0) = \psi_0(x), \quad x \in \mathbb{R}^2. \]

Here we assume \(\psi_0 \in C_0(\mathbb{R}^2)\) and \(\|\psi_0\|_1 \leq 1\). Admitting the existence of a solution \(\psi\) (§4.4.4), we multiply both sides of the evolution equation by \(\omega\) and integrate them over \(\mathbb{R}^2 \times (0, t_0)\). Since \(\text{div } v = 0\) by assumption, integrating by parts yields

\[ \int_{\mathbb{R}^2} \omega(x, t_0)\psi_0(x)dx = \int_{\mathbb{R}^2} \omega_0(x)\psi(x, 0)dx. \]

By the \(L^2-L^1\) estimate for solutions of \((H_v)\) obtained at the beginning of the proof, we observe that \(\|\psi(x, 0)\|_2 \leq (2\kappa t_0)^{-1/2}\|\psi_0\|_1 \leq (2\kappa t_0)^{-1/2}\). We thus obtain

\[ \sup_{\|\psi_0\|_1 \leq 1} \left| \int_{\mathbb{R}^2} \omega(x, t_0)\psi_0(x)dx \right| \leq \|\omega_0\|_2 \|\psi(x, 0)\|_2 \leq (2\kappa t_0)^{-1/2}\|\omega_0\|_2. \]

Here we used the \textit{Schwarz inequality} (§4.1.1). The latter term is equal to \(\|\omega\|_\infty(t_0)\) in view of the characterization of the norm by duality (Chapter 6 (6.8)). We thus obtain the \(L^\infty-L^2\) estimate

\[ \|\omega\|_\infty(t_0) \leq (2\kappa t_0)^{-1/2}\|\omega_0\|_2. \]

For general \(t > 0\) we set \(t_0 = t/2\) and use the \(L^2-L^1\) estimate and the \(L^\infty-L^2\) estimate to obtain

\[ \|\omega\|_\infty(t) \leq (\kappa t)^{-1/2}\|\omega\|_2(t/2) \leq (\kappa t)^{-1}\|\omega_0\|_1. \]

As in the proof given in the first paragraph of this section, we obtain the fundamental \(L^q-L^1\) estimate from this estimate by interpolating with the \(L^1-L^1\) estimate.

By similar arguments to establish the fundamental \(L^q-L^1\) estimates it is also possible to derive the following \(L^q-L^r\) estimate \((r = 2^m)\) for \(\omega\), as in the case of the heat equation.
2.3.6 Extension of Fundamental $L^q$-$L^1$ Estimates

**Theorem.** Assume that $v$ and $\omega$ satisfy the same assumptions as in §2.3.1. Let $\kappa$ be the universal constant in §2.3.1. Then for $\rho = 2^k$ ($k = 0, 1, 2, \ldots$) and $q$ with $\rho \leq q \leq \infty$, we have

$$\|\omega\|_{q}(t) \leq \frac{1}{(\kappa t)^{1/\rho - 1/q}} \|\omega_0\|_{\rho}, \quad t > 0.$$  

Here $\|\omega\|_{\rho}(t)$ is assumed to be continuous at $t = 0$.

**Proof.** Replacing $y_1 \leq M_1$ by $y_\rho \leq N_\rho$ with a constant $N_\rho$ in Lemma 2.3.4, arguing as in the proof of the lemma for the differential inequality with $m \geq k + 1$ we obtain

$$y_s(t) \leq N_s t^{1-s/\rho}, \quad s = 2^m \geq \rho = 2^k, \quad t > 0,$$

instead of (i). We thus define $N_s = a^{-1} \rho^{-1} s N_s^{2/s}$ inductively. Then for sufficiently large $s = 2^m$, instead of (ii), we obtain

$$(y_s(t))^{1/s} \leq \left(\frac{4}{a}\right)^{1/\rho} N_\rho^{1/\rho} t^{-1/\rho + 1/s}, \quad t > 0$$

(Exercise 2.5).

Since we assume that $\|\omega\|_{\rho}(t)$ is continuous at $t = 0$, by §2.3.2 and §2.3.3 we obtain $\|\omega\|_{\rho}(t) \leq \|\omega_0\|_{\rho}$, $t > 0$. We set

$$y_s(t) = \|\omega\|_{s}(t), \quad s = 2^m, \quad m = k, k + 1, k + 2, \ldots, \quad t > 0,$$

$N_\rho = \|\omega_0\|_{\rho}^s$, and $a = 4\kappa$. Then we observe that similar inequalities as in Lemma 2.3.4 are valid for the above $t_s$. By similar arguments as in §2.3.5, we obtain $\|\omega\|_{\infty}(t) \leq \frac{1}{(\kappa t)^{1/\rho}} \|\omega_0\|_{\rho}$ for $t > 0$. By the Hölder inequality (§4.1.1) for general $q$ with $\rho \leq q \leq \infty$ we then conclude that

$$\|\omega\|_{q}(t) \leq \|\omega\|_{\infty}^{1-p/q}(t) \|\omega\|_{\rho}^{p/q}(t)$$

$$\leq \left(\frac{1}{(\kappa t)^{1/\rho}} \|\omega_0\|_{\rho}\right)^{1-p/q} \|\omega_0\|_{\rho}^{p/q}$$

$$= \frac{1}{(\kappa t)^{1/\rho - 1/q}} \|\omega_0\|_{\rho}, \quad t > 0.$$  

2.3.7 Maximum Principle

**Proposition.** Assume that $v$ and $\omega$ satisfy the same assumptions as in §2.3.1. Then

$$\|\omega\|_{\infty}(t) \leq \|\omega_0\|_{\infty}, \quad t \geq 0.$$  

Here $\|\omega\|_{\rho}(t)$ is assumed to be continuous at $t = 0$ for sufficiently large $\rho$ ($< \infty$).
This is easy to prove. Indeed, if we set $q = \infty$ in §2.3.6 and send $\rho \to \infty$, then we get the desired result, since $\lim_{\rho \to \infty} \|\omega_\rho\|_\rho = \|\omega_0\|_\infty$ (Exercise 2.3).

This proposition is called the maximum principle, since it estimates the upper bound of $|\omega(x,t)|$ as a function of $x$ on $t > 0$. If one assumes the boundedness of $\|\omega\|_\infty(t)$ and $\|v\|_\infty(t)$, we may prove the proposition without assuming $\text{div } v = 0$. We shall discuss this property in the next section.

2.3.8 Preservation of Nonnegativity

**Theorem.** Let $T$ be a given positive number. Assume that the functions $v^i$ and $w$ are bounded on $\mathbb{R}^n \times (0,T)$, for $i = 1, \ldots, n$. Moreover, assume that $w \in C(\mathbb{R}^n \times [0,T)) \cap C^2(\mathbb{R}^n \times (0,T))$ satisfies

$$\partial_t w - \Delta w + (v, \nabla)w = 0 \quad \text{in } \mathbb{R}^n \times (0,T)$$

for $v = (v^1, \ldots, v^n)$. Then we have the following properties:

(i) If $w(\cdot, 0)$ is nonnegative on $\mathbb{R}^n$, then $w$ is also nonnegative on $\mathbb{R}^n \times [0,T)$. Namely, if $w(x,0) \geq 0$, $x \in \mathbb{R}^n$, then $w(x,t) \geq 0$, $x \in \mathbb{R}^n$, $t \in [0,T)$.

(ii) $\sup_{x \in \mathbb{R}^n} w(x,t) \leq \sup_{x \in \mathbb{R}^n} w(x,0)$ for $t \in (0,T)$, and $\inf_{x \in \mathbb{R}^n} w(x,t) \geq \inf_{x \in \mathbb{R}^n} w(x,0)$ for $t \in [0,T)$.

(iii) $\|w\|_\infty(t) \leq \|w\|_\infty(0)$, for $t \in [0,T)$.

**Proof.** Property (iii) immediately follows from (ii). Property (ii) follows from (i). Indeed, we set $\sup_{x \in \mathbb{R}^n} w(x,0) = M_0$ and $\tilde{w} = -(w - M_0)$ and observe that $\tilde{w}$ satisfies $\partial_t \tilde{w} - \Delta \tilde{w} + (v, \nabla)\tilde{w} = 0$ in $\mathbb{R}^n \times (0,T)$. Thus by (i), we obtain $\sup_{x \in \mathbb{R}^n} w(x,t) \leq M_0$. The claim for the infimum is proved by similar arguments.

It remains to prove (i).

We transform the dependent variable $w$ to $u = e^{-t}w$. Then $u$ satisfies

$$\partial_t u + u - \Delta u + (v, \nabla)u = 0 \quad \text{in } \mathbb{R}^n \times (0,T).$$

Let $L$ be the operator acting on $u$ which is defined by the left hand side of this equation. That is to say, the left hand side is denoted by $Lu$. We assume that $w$ has a negative value at $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$, and we shall derive a contradiction. By the definition of $u$, we get $(-\alpha =) u(x_0, t_0) < 0$. If there exists a point $(\hat{x}, \hat{t})$ in $\mathbb{R}^n \times (0,T)$ at which $\inf_{\mathbb{R}^n \times [0,T]} u$ is attained, then we have

$$\partial_t u(\hat{x}, \hat{t}) \leq 0, \quad \nabla u(\hat{x}, \hat{t}) = 0, \quad \Delta u(\hat{x}, \hat{t}) \geq 0;$$

hence we obtain

$$Lu(\hat{x}, \hat{t}) \leq u(\hat{x}, \hat{t}) \leq u(x_0, t_0) = -\alpha < 0.$$ 

This contradicts $Lu = 0$ (in $\mathbb{R}^n \times (0,T)$). However, unfortunately, since $\mathbb{R}^n$ is unbounded, we do not know whether there exists a point at which $\inf_{\mathbb{R}^n \times [0,T]} u$
is attained. For this reason, we use the following trick. Let $A > 0$ and $\varepsilon > 0$ be constants to be determined later, and set

$$u_{\varepsilon} = u + \varepsilon(At + |x|^2).$$

Then we have

$$Lu_{\varepsilon} = Lu + \varepsilon(A + At + |x|^2 - 2n + 2 \langle v, x \rangle).$$

Since $Lu = 0$ in $\mathbb{R}^n \times (0, T)$, we choose $A > 0$ such that

$$A \geq \sup \{2n + 2|x| \|v\|_\infty(t) - |x|^2 : x \in \mathbb{R}^n, t \in [0, t_0]\},$$

and conclude that $Lu_{\varepsilon} \geq 0$ in $\mathbb{R}^n \times (0, t_0)$. (By the assumption of the boundedness of $v$, the above supremum is finite.) We fix such an $A$ and take $\varepsilon > 0$ so small that $u_{\varepsilon}(x_0, t_0) = u(x_0, t_0) + \varepsilon(At_0 + |x_0|^2) \leq -\alpha/2 < 0$.

Since $w$ is bounded on $\mathbb{R}^n \times [0, T)$, $u$ is also bounded on $\mathbb{R}^n \times [0, T)$. Moreover, if

$$|x| > \varepsilon^{-1/2} \left( - \inf_{\mathbb{R}^n \times (0, T)} u \right)^{1/2} =: R, \ x \in \mathbb{R}^n,$$

then $u_{\varepsilon}(x, t) > 0$ ($t \in (0, T)$). Since the function $u_{\varepsilon}$ is continuous in $\mathbb{R}^n \times [0, t_0]$, $u_{\varepsilon}$ has a minimum value on $\overline{B}_R \times [0, t_0]$ (the Weierstrass theorem). This means that there exists a point $(x_1, t_1) \in \overline{B}_R \times [0, t_0]$ such that

$$u_{\varepsilon}(x_1, t_1) = \inf \{u_{\varepsilon}(x, t) : x \in \overline{B}_R, t \in [0, t_0]\}.$$

Note that $u_{\varepsilon}(x_1, t_1) \leq u_{\varepsilon}(x_0, t_0) < 0$. Since $u_{\varepsilon}(x, 0) \geq 0$ ($x \in \mathbb{R}^n$) by assumption and since $u_{\varepsilon}(x, t) \geq 0$ ($|x| = R, t \in [0, t_0]$) by the choice of $R$, we conclude that $|x_1| < R$ and $t_1 \in (0, t_0]$. Since $u_{\varepsilon}$ attains its minimum in $\overline{B}_R \times [0, t_0]$ at $(x_1, t_1)$, we obtain

$$\partial_t u_{\varepsilon}(x_1, t_1) \leq 0, \quad \nabla u_{\varepsilon}(x_1, t_1) = 0, \quad \Delta u_{\varepsilon}(x_1, t_1) \geq 0.$$

Thus

$$(Lu_{\varepsilon})(x_1, t_1) \leq u_{\varepsilon}(x_1, t_1) \leq u_{\varepsilon}(x_0, t_0) = -\alpha/2 < 0,$$

which contradicts $Lu_{\varepsilon} \geq 0$ in $\mathbb{R}^n \times (0, T)$. We thus conclude that $w$ is nonnegative on $\mathbb{R}^n \times (0, T)$. \(\square\)

Following the lines of the proof, we see that in order to prove (i) it suffices to assume that $Lw \geq 0$ (in $\mathbb{R}^n \times (0, T)$) instead of $Lw = 0$ (in $\mathbb{R}^n \times (0, T)$). It is known that $w$ is positive on $t > 0$ unless $w$ is identically zero under the situation of (i). This is called the strong maximum principle, which, however, is not discussed in this book. For details about the maximum principle and the strong maximum principle readers are referred to [Protter Weinberger 1967], [Kumanogo 1978]. Also in [Ito 1979] the strong maximum principle is discussed in detail, although the main theme is the construction of fundamental solutions of diffusion equations.
2.4 Estimates for Solutions of Vorticity Equations

Using the results of the previous section, we shall derive estimates for solutions of the vorticity equations (2.7) and (2.8). As in the case of the heat equation, it is important to derive estimates depending only on the $L^1$-norm of the initial value $\omega_0$. Using the fundamental $L^q-L^1$ estimate in §2.3.1, we shall first derive estimates for the vorticity and the velocity of (2.7) and (2.8).

2.4.1 Estimates for Vorticity and Velocity

**Theorem.** Let the initial vorticity $\omega_0$ be in $C_0(\mathbb{R}^2)$ and let $\kappa$ be the universal constant of §2.3.1. Then there exist positive constants $L_j(p)$ ($j = 1, 2$) depending only on $p$ and satisfying the following properties for all solutions $(\omega, u)$ of (2.7), (2.8), and (2.9):

(i) We have
$$ \|\omega\|_q(t) \leq \frac{1}{(\kappa t)^{\frac{1}{q}}} \|\omega_0\|_1 $$
for all $t > 0$ and $q$ with $1 \leq q \leq \infty$.

(ii) For each $p$ satisfying $2 < p < \infty$ define $q$ by $\frac{1}{p} = \frac{1}{q} - \frac{1}{2}$. Then
$$ \|u\|_p(t) \leq L_1(p) \|\omega\|_q $$
for all $t > 0$. Moreover, for each $p$ satisfying $2 < p \leq \infty$ the estimate
$$ \|u\|_p(t) \leq \frac{L_1(p)}{(\kappa t)^{\frac{1}{2} - \frac{1}{p}}} \|\omega_0\|_1 $$
is valid for all $t > 0$.

(iii) For each $q$ satisfying $1 < q < \infty$ the estimate
$$ \|\nabla u\|_q(t) \leq L_2(q) \|\omega\|_q(t) \leq \frac{L_2(q)}{(\kappa t)^{\frac{1}{q}}} \|\omega_0\|_1 $$
is valid for all $t > 0$.

(iv) For each $p$ satisfying $2 < p \leq \infty$ the convergence
$$ \lim_{t \to 0} \|u - u_0\|_p(t) = 0 $$
is valid for $u_0 = K * \omega_0$.

For an $\mathbb{R}^2$-valued function $v = (v^1, v^2)$ defined on $\mathbb{R}^2$, $\nabla v$ denotes the matrix $(\partial_{x_i} v^j)_{1 \leq i, j \leq 2}$, whereas the expression $|\nabla v|$ denotes the Hilbert–Schmidt norm $(\sum_{j=1}^{2} |\nabla v^j|^2)^{1/2}$. For $p$ with $1 \leq p \leq \infty$, we define $\|\nabla v\|_p = ||\nabla v||_p$.

**Proof.** The first estimate (i) is obvious by applying the fundamental $L^q-L^1$ estimate in §2.3.1 to (2.7). Estimates (ii) and (iii) are derived from (i) together with various fundamental estimates in differential and integral calculus discussed in Section 6. One will see the importance of such fundamental inequalities through the proof of the theorem.
We first note that the velocity is expressed by the Biot–Savart law (2.8) using the vorticity $\omega$. If we write $x = (x_1, x_2)$ and $K = (K_1, K_2)$, then we obtain

$$|K_i(x)| \leq \frac{1}{2\pi} \frac{1}{|x|}, \quad i = 1, 2,$$

since $|x_1| \leq |x|$, $|x_2| \leq |x|$. Thus $u = (u^1, u^2)$ is also estimated by

$$|u^i(x)| \leq \int_{\mathbb{R}^2} \frac{1}{2\pi |x-y|} |\omega(y)| dy, \quad i = 1, 2.$$

In the proof below we suppress the dependence of $u$ and $\omega$ with respect to $t$ unless it is necessary to clarify. In other words, we simply write $u(x)$ and $\omega(x)$ instead of $u(x, t)$ and $\omega(x, t)$, respectively. For a function $f$ defined on $\mathbb{R}^2$ we define the operator $I_1$ by

$$(I_1(f))(x) = \frac{1}{|x|} * f = \int_{\mathbb{R}^2} \frac{1}{|x-y|} f(y) dy, \quad x \in \mathbb{R}^2.$$

Then we obtain

$$|u^i(x)| \leq \frac{1}{2\pi} (I_1(|\omega|))(x), \quad x \in \mathbb{R}^2.$$

For this operator $I_1$ it is known that

$$\|I_1(f)\|_p \leq C \|f\|_q, \quad 1/p = 1/q - 1/2, \quad 1 < q < 2,$$

which is a special case of the Hardy–Littlewood–Sobolev inequality proved in §6.2.1. Here $C$ is a constant depending only on $p$. The above inequality is at least valid for a continuous function $f$ with $\|f\|_q < \infty$. For more details see §6.2 (especially the theorem and the remark in §6.2.1). We apply this inequality to $I_1(|\omega|)$ and observe that there exists a constant $L_1$ depending only on $p$ such that

$$\|u\|_p \leq L_1 \|\omega\|_q, \quad 1/p = 1/q - 1/2, \quad 1 < q < 2.$$

(By the unique existence theorem in §2.2.1, $\omega$ is continuous and satisfies $\|\omega\|_q < \infty$. Thus we may apply Theorem 6.2.1 to $\omega$.) This inequality is considered as an estimate of the velocity by the vorticity. (Thus the Hardy–Littlewood–Sobolev inequality is considered as a generalization of the estimate for the velocity by the vorticity in a two-dimensional fluid.)

Combining this estimate and (i), we obtain (ii) for $2 < p < \infty$. Since we cannot remove the restriction of the index $1 < q < 2$ in the Hardy–Littlewood–Sobolev inequality, we have to prove the case $p = \infty$ separately.

Postponing the proof of (ii) in the case of $p = \infty$, we consider (iii). The operator that maps $\omega$ to $\partial_{x_j} u^i$ is a typical example of singular integral operator (§6.4.2). In this case, the $L^p$ estimate of the singular integral operator is well known as the Calderon–Zygmund inequality; this will be discussed with a proof.
in §6.4. By the Calderón–Zygmund inequality (§6.4.2) the derivative of the velocity is estimated by the vorticity as

\[ \| \nabla u \|_q \leq C \| \omega \|_q, \quad 1 < q < \infty, \]

where \( C \) is a constant depending only on \( q \). We remark that the above inequality is not valid for \( q = 1 \) and \( q = \infty \). (By the unique existence theorem (§2.2.1), for a fixed \( t > 0 \), \( \omega \) is \( C^1 \) on \( \mathbb{R}^2 \) as a function of \( x \), and \( \omega \) and \( |\nabla \omega| \) are bounded and integrable on \( \mathbb{R}^2 \). Hence we obtain the last inequality by Theorem 6.4.2 and remark (i) afterward.) (Note that in the case of \( q = 2 \) it is easy to prove that

\[ \| \nabla u \|_2^2 = \| \omega \|_2^2, \]

provided that the following integration by parts on \( \mathbb{R}^2 \) is justified:

\[
\begin{align*}
\| \nabla u \|_2^2 &= \sum_{j=1}^{2} \int_{\mathbb{R}^2} \langle \nabla u^j, \nabla u^j \rangle dx = - \int_{\mathbb{R}^2} \langle u, \Delta u \rangle dx \\
&= \int_{\mathbb{R}^2} \langle u, (\nabla^\perp \text{curl } - \nabla \text{div}) u \rangle dx = \int_{\mathbb{R}^2} \langle u, \nabla^\perp \text{curl } u \rangle dx \\
&= \int_{\mathbb{R}^2} (\text{curl } u)(\text{curl } u) dx = \| \omega \|_2^2.
\end{align*}
\]

Here we have used the property \( \Delta = -\nabla^\perp \text{curl } + \nabla \text{div} \) (§2.1.2), \( \text{div } u = 0 \), and \( \text{curl } u = \omega \). Physically, up to constant multiples, \( \| \nabla u \|_2^2 \) and \( \| \omega \|_2^2 \) correspond to the enstrophy and the energy of vorticity, respectively.) In the case of \( 1 < q < \infty \), combining the Calderón–Zygmund inequality and (i), we obtain (iii) with \( L_2 = C \).

We now consider (ii) in the case of \( p = \infty \). Sometimes the modulus of a function is estimated by the modulus of its derivatives. There are several types of such inequalities including the Sobolev inequality. Here, we use a special case of the Gagliardo–Nirenberg inequality (§6.1.1) of the form

\[ \| u \|_\infty \leq \tilde{C} \| u \|_r^{1-2/r} \| \nabla u \|_r^{2/r}, \quad 2 < r < \infty. \]

(By Theorem 2.2.1, each component of \( u \) is \( C^1 \) and satisfies \( \| u \|_r < \infty \); hence we may apply the above inequality to \( u \).) In this inequality, it is always assumed that \( \| u \|_r \) is finite. Here \( \tilde{C} \) is a constant depending on \( r \) and independent of \( u \). We shall discuss the Gagliardo–Nirenberg inequality in detail in §6.1. (Without the finiteness of \( \| u \|_r \) this inequality fails in the case of a nonzero constant function. If \( \| u \|_r \) is finite, the inequality is valid, since \( u \) vanishes identically when \( |\nabla u| = 0 \) on \( \mathbb{R}^2 \).) We choose an \( r \in (2, \infty) \) in the Gagliardo–Nirenberg inequality and apply (ii) and (iii) to the right-hand side to get
\[ \|u\|_\infty(t) \leq \tilde{C} \left( \frac{L_1(r)}{(kt)^{2-\frac{1}{r}}} \|\omega_0\|_1 \right)^{1-\frac{2}{p}} \left( \frac{L_2(r)}{(kt)^{1-\frac{1}{r}}} \|\omega_0\|_1 \right)^{\frac{2}{p}} \]

\[ = \tilde{C}(L_1(r))^{1-2/r}(L_2(r))^{2/r} \frac{(kt)^{1/2}}{(kt)^{1/2}} \|\omega_0\|_1 \]

for \( t > 0 \). The exponent of \( 1/t \) is calculated as

\[ \left( \frac{1}{2} - \frac{1}{r} \right) \left( 1 - \frac{2}{r} \right) + \left( 1 - \frac{1}{r} \right) \frac{2}{r} = \frac{1}{2}. \]

We thus obtain (ii) in the case of \( p = \infty \) with

\[ L_1(\infty) = \tilde{C}(L_1(r))^{1-2/r}(L_2(r))^{2/r}. \]

Finally, we shall prove (iv). Since \( u - u_0 = K * (\omega - \omega_0) \), for \( p \) with \( 2 < p < \infty \), using the Hardy–Littlewood–Sobolev inequality, we obtain

\[ \|u - u_0\|_p(t) \leq L_1(p)\|\omega - \omega_0\|_q(t), \quad 1/p = 1/q - 1/2, \quad 1 < q < 2. \]

On the other hand, by the unique existence theorem (§2.2.1 (i)), we have \( \|\omega - \omega_0\|_q(t) \to 0 (t \to 0) \); hence (iv) follows for \( p \) with \( 2 < p < \infty \). In the case of \( p = \infty \) (each component of) \( u_0 \) is \( C^1 \) at least for \( \omega_0 \in C_0(\mathbb{R}^2) \cap C^1(\mathbb{R}^2) \) (Remark (ii) in §6.3.5), so we may argue similarly as in the proof of (ii). The Gagliardo–Nirenberg inequality yields

\[ \|u - u_0\|_\infty(t) \leq \tilde{C}\|u - u_0\|_r^{1-2/r}(t)\|\nabla (u - u_0)\|_p^{2/r}(t), \quad 2 < r < \infty, \]

for all \( t > 0 \). By the Calderón–Zygmund inequality (theorem and remark (i) in §6.4.2) we have

\[ \|\nabla (u - u_0)\|_r(t) \leq C\|\omega - \omega_0\|_r(t). \]

We also have

\[ \|u - u_0\|_r(t) \leq L_1(r)\|\omega - \omega_0\|_q(t), \quad 1/r = 1/q - 1/2, \quad 1 < q < 2, \]

which is obtained from the Hardy–Littlewood–Sobolev inequality, and now observe that

\[ \|u - u_0\|_\infty(t) \leq \tilde{C}(L_1(r))^{1-2/r} \tilde{C}^{2/r}\|\omega - \omega_0\|_q^{1-2/r}(t)\|\omega - \omega_0\|_r^{2/r}(t). \]

Using \( \|\omega - \omega_0\|_s(t) \to 0 (t \to 0) (1 \leq s \leq \infty) \) again (see the unique existence theorem (§2.2.1(i))), we conclude (iv) in the case of \( p = \infty \) for \( \omega_0 \in C_0(\mathbb{R}^2) \cap C^1(\mathbb{R}^2) \). For general \( \omega_0 \in C_0(\mathbb{R}^2) \), \( u_0 \) may not be \( C^1 \), so additional work is necessary. However, by Remark (ii) in §6.4.2, the last inequality is still valid, so we can prove (iv) in the case of \( p = \infty \). \( \square \)
The Gagliardo–Nirenberg inequality, the Hardy–Littlewood–Sobolev inequality, and the Calderón–Zygmund inequality are valid in higher-dimensional spaces, under suitable corrections for the relation of the exponents. These general inequalities are discussed in Chapter 6.

For the second half of the proof of (ii) including the case $p = \infty$, it is possible to prove it by applying only the fundamental $L^q$-$L^1$ estimate §2.3.1. We shall give its proof only for the case of $p = \infty$. Using the operator $I_1$, we have

$$|u^i(x,t)| \leq \frac{1}{2\pi} I_1(|\omega(\cdot,t)|)(x), \quad i = 1, 2, \quad t > 0.$$}

For $A > 0$ we write

$$I_1(|\omega(\cdot,t)|)(x) = \int_{|x-y| \leq A} \frac{|\omega(y,t)|}{|x-y|} dy + \int_{|x-y| > A} \frac{|\omega(y,t)|}{|x-y|} dy.$$  

We use the fundamental $L^\infty$-$L^1$ estimate (§2.3.1) to obtain

$$I_1(|\omega(\cdot,t)|)(x) \leq \|\omega\|_\infty(t) \int_{|x-y| \leq A} \frac{dy}{|x-y|} + \frac{1}{A} \|\omega\|_1(t) \leq \{2\pi A(\kappa t)^{-1} + A^{-1}\}\|\omega_0\|_1.$$  

We set $A = (\kappa t/2\pi)^{1/2}$ (so that $2\pi A(\kappa t)^{-1} = A^{-1}$). Thus we obtain

$$I_1(|\omega(\cdot,t)|)(x) \leq 2(2\pi/\kappa t)^{1/2}\|\omega_0\|_1, \quad t > 0.$$  

Therefore, for $t > 0$, we get $|u^i(x,t)| \leq 2(2\pi \kappa t)^{-1/2}\|\omega_0\|_1$. In the case of $2 < p < \infty$, to estimate $I_1(|\omega(\cdot,t)|)(x)$ it is sufficient to use the Young inequality (§4.1.1).

The estimates of derivatives of the vorticity for $p = 1$ and $p = \infty$ in the corollary below are new. The key step for the proof is to establish a new Gronwall-type lemma.

### 2.4.2 Estimates for Derivatives of the Vorticity

**Theorem.** Let the initial vorticity $\omega_0$ be in $C_0(\mathbb{R}^2)$. Then there exists a positive constant $W$ depending only on $\|\omega_0\|_1$ such that any solution $(\omega, u)$ of (2.7), (2.8), and (2.9) satisfies

(i) $\|\nabla \omega\|_p(t) \leq \frac{W}{t^{2-p}}\|\omega_0\|_1$ for $t > 0, 1 \leq p \leq \infty$,

(ii) $\|\partial_x, \partial_y \omega\|_p(t) \leq \frac{W}{t^{2-p}}\|\omega_0\|_1$ for $t > 0, 1 \leq p \leq \infty, 1 \leq i, j \leq 2$,

(iii) $\|\partial_t \omega\|_p(t) \leq \frac{W}{t^{2-p}}\|\omega_0\|_1$ for $t > 0, 1 \leq p \leq \infty$.

Moreover, the constant $W = W(\|\omega_0\|_1)$ may be chosen such that it is non-decreasing with respect to $\|\omega_0\|_1$.  

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Corollary. Let the initial vorticity $\omega_0$ be in $C_0(\mathbb{R}^2)$. Then there exist positive constants $W_1, W_2$ depending only on a multi-index $\beta$, a nonnegative integer $b$, and $\|\omega_0\|_1$ such that any solution $(\omega, u)$ of (2.7), (2.8), and (2.9) satisfies

$$
\|\partial_t^b \partial_x^\beta \omega\|_p(t) \leq \frac{W_1}{t^{b+|\beta|+1-\frac{1}{p}}} \|\omega_0\|_1 \quad \text{for } 1 \leq p \leq \infty,
$$

$$
\|\partial_t^b \partial_x^\beta u\|_p(t) \leq \frac{W_2}{t^{b+|\beta|+1-\frac{1}{p}}} \|\omega_0\|_1 \quad \text{for } 2 < p \leq \infty.
$$

Moreover, the constant $W_j$ may be chosen such that it is nondecreasing with respect to $\|\omega_0\|_1$.

Note that in this corollary the estimates of derivatives for all orders are obtained. Especially, the estimates in the above theorem are special cases of this corollary for $2b + |\beta| \leq 2$. These estimates are obtained using estimates for vorticities and velocities §§2.4.1 and the $L^p$-$L^q$ estimate for derivatives of solutions of the heat equation §1.1.3.

In the following we regard the nonlinear term $-(u, \nabla)\omega$ as a given function, and apply the results of the linear heat equations with inhomogeneous terms. This argument is called the perturbation argument, which is one of the standard methods for analyzing nonlinear partial differential equations.

Proof of Theorem. The basic idea of the proof is as follows: Since $\text{div } u = 0$, we may rewrite the nonlinear term $(u, \nabla)\omega$ of (2.7) as $\text{div } (u\omega)$. Here we consider equation (2.7) on $\mathbb{R}^2 \times (0, \infty)$, and write it as

$$
\partial_t \omega - \Delta \omega = \text{div } h_1, \quad h_1 = -u\omega.
$$

We often suppress the $x$-dependence of functions of $t$ and $x$ for simplicity. For example, $f(t)$ denotes the function $f(\cdot, t)$ of $x$ on $\mathbb{R}^2$. By the estimates for the vorticity and the velocity obtained in §2.3.3 and §2.4.1, $h_1$ satisfies

$$
\|h_1\|_1(t) \leq \|u\|_{\infty}(t) \|\omega\|_1(t) \leq \frac{L_1(\infty)}{(\kappa t)^{1/2}} \|\omega_0\|_1^2, \quad t > 0,
$$

where $\kappa$ is the universal constant of §2.3.1. The function $\|h_1\|_1(t)$ is not always bounded near $t = 0$, but regarding $h_1$ as a known function and using Theorem 4.4.3, by the “variation-of-constant formula” as it is known for ordinary differential equations of first order, we obtain

$$
\omega(t) = e^{t\Delta} \omega_0 + \int_0^t \text{div } (e^{(t-s)\Delta} h_1(s)) ds \quad \text{in } \mathbb{R}^2, \quad t > 0.
$$

Hence $\omega$ is expressed by integrals. This is an equality of functions on $\mathbb{R}^2$ with parameter $t > 0$. As explained at the beginning of §4.3, $e^{t\Delta} f$ denotes the function of $x \in \mathbb{R}^2$ at $t > 0$ and it is the solution of the heat equation with initial value $f$. That is to say, $e^{t\Delta}$ is an operator with a parameter $t$ that
operates on functions on $\mathbb{R}^2$. As in §1.1, let $G_t$ be the Gauss kernel. Then $(e^{t\Delta}f)(x) = (G_t * f)(x)$. (For a solution $(\omega, u)$ of (2.7), (2.8), and (2.9) it is easy to see that $\omega$ is a weak solution (its definition will be given in §4.3.4) of $\partial_t \omega - \Delta \omega = \text{div} \, h_1$ with initial value $\omega_0$, where the term $\text{div} \, h_1$ is regarded as a known inhomogeneous term. Hence we can apply Theorem 4.4.3 to this equation.)

Using a positive constant $0 < \varepsilon < 1$, we rewrite this representation\(^1\) for $\omega$ by

$$
\omega(t) = e^{t\Delta} \omega_0 + \int_{t(1-\varepsilon)}^{t} e^{(t-s)\Delta} (\text{div} \, h_1(s)) ds
+ \int_{0}^{t(1-\varepsilon)} \text{div} \, (e^{(t-s)\Delta} \, h_1(s)) ds.
$$

Here we use the commutativity of $e^{(t-s)\Delta}$ and $\text{div}$ given in Proposition 4.1.6. Differentiating both sides with respect to the spatial variables, and taking the $L^p$-norm ($1 \leq p \leq \infty$), we obtain\(^2\)

$$
\| \nabla \omega \|_p(t) \leq \| \nabla (e^{t\Delta} \omega_0) \|_p + \int_{t(1-\varepsilon)}^{t} \| \nabla (e^{(t-s)\Delta} (\text{div} \, h_1(s))) \|_p ds
+ \int_{0}^{t(1-\varepsilon)} \| \nabla (\text{div} \, (e^{(t-s)\Delta} \, h_1(s))) \|_p ds.
$$

Here we write each term of the right-hand side as $J_1(t)$, $J_2(t)$, and $J_3(t)$, respectively. (We divide the integral over the interval $(0, t)$ into integrals over $(0, t(1-\varepsilon))$ and $(t(1-\varepsilon), t)$, since this integral may be unbounded near $s = 0$ and $s = t$. For this reason, for the term with the interval of integration $(0, t(1-\varepsilon))$, we first take the convolution with the Gauss kernel and then differentiate with respect to spatial variables. For the term with the interval of integration $(t(1-\varepsilon), t)$ we differentiate $h_1$ with respect to the spatial variables. We later take $\varepsilon$ small.) We shall estimate $J_1$, $J_2$, and $J_3$. First, by the $L^p$-$L^1$ estimate for derivatives of solutions of the heat equation §1.1.3, there exists a constant $C_1$ that is independent of $\omega_0$ and $t$ (by analyzing the constant that appears in the proof of the estimate in §1.1.3, we can take $C_1$ even independent of $p$) such that

$$
J_1(t) \leq C_1 \| \omega_0 \|_1 t^{\frac{1}{p} - \frac{3}{2}}, \quad t > 0.
$$

Similarly, by §1.1.3, for the integrand of $J_2$ we have

$$
\| \nabla (e^{(t-s)\Delta} (\text{div} \, h_1(s))) \|_p \leq \frac{C_1}{(t-s)^{1/2}} \| \text{div} \, h_1 \|_p(s), \quad 0 < s < t.
$$

\(^1\) The expression $e^{t\Delta} h$ for an $\mathbb{R}^n$-valued function $h = (h^1, \ldots, h^n)$ stands for the $\mathbb{R}^n$-valued function with $e^{t\Delta} h^i$ as the $i$th component.

\(^2\) In these calculations we always use the property $\| \int f dt \|_p \leq \int \| f \|_p dt$ for a function $f$ of $x$ and $t$. For the proof we refer to Exercise 6.5.
Moreover, using (ii) in §2.4.1 for \( \text{div} \ h_1 = -(u, \nabla)\omega \) we observe that
\[
\|\text{div} \ h_1\|_p(s) \leq \|u\|_\infty(s)\|\nabla \omega\|_p(s)
\leq L_1(\infty)(\kappa s)^{-1/2}\|\omega_0\|_1\|\nabla \omega\|_p(s), \quad 0 < s < t,
\]
for \( t > 0 \). We obtain
\[
J_2(t) \leq C_1 L_1(\infty)\kappa^{-1/2}\|\omega_0\|_1 \int_{t(1-\varepsilon)}^t \frac{1}{(t-s)^{1/2}} \frac{1}{s^{1/2}} \|\nabla \omega\|_p(s)ds.
\]
For the integrand of \( J_3 \), using §1.1.3 as \( q = 1 \), and §2.4.1 we obtain
\[
\|\nabla (\text{div} (e^{(t-s)\Delta} h_1(s)))\|_p \leq \frac{C_2}{(t-s)^{2-\frac{3}{p}}} \|u\|_1(s) \leq \frac{C_2 L_1(\infty)}{(t-s)^{2-\frac{3}{p}}(\kappa s)^{1/2}} \|\omega_0\|_1
\]
for \( 0 < s < t \). Here \( C_j (j = 2, 3) \) denotes a constant independent of \( p, \omega_0, t, \) and \( s \). If \( 1 \leq p < \infty \), then the integral
\[
\int_0^{t(1-\varepsilon)} (t-s)^{-2+\frac{1}{p}} s^{-\frac{1}{2}} ds = A_\varepsilon t^{-\alpha}, \quad A_\varepsilon = \int_0^{1-\varepsilon} (1-\tau)^{-2+\frac{1}{p}} \tau^{-\frac{1}{2}} d\tau
\]
converges. Hence, for \( 1 \leq p \leq \infty \) we obtain
\[
J_3(t) \leq C_2 L_1(\infty)\|\omega_0\|_1^2 A_\varepsilon \kappa^{-\frac{1}{2}} t^{-\alpha}, \quad t > 0.
\]
By these estimates for \( J_1, J_2, \) and \( J_3 \), we obtain
\[
\|\nabla \omega\|_p(t)
\leq \|\omega_0\|_1 \left\{ (C_1 + W_1 A_\varepsilon)t^{-\alpha} + C_3 \int_{t(1-\varepsilon)}^t \frac{1}{(t-s)^{1/2}} \frac{1}{s^{1/2}} \|\nabla \omega\|_p(s)ds \right\}.
\]
Here, we set
\[
W_1 = C_2 \kappa^{-1/2} L_1(\infty)\|\omega_0\|_1, \quad C_3 = C_1 L_1(\infty)\kappa^{-1/2}.
\]
These constants are not only independent of \( \omega \), but also independent of \( \varepsilon \) except for \( A_\varepsilon \). (\( A_\varepsilon \) diverges to infinity as \( \varepsilon \to 0 \).) We apply the following Gronwall-type lemma to the above estimate for \( \|\nabla \omega\|_p(t) \) with \( \psi(t) = \|\nabla \omega\|_p(t) \) to get (i). Here, by (ii) in §2.2.1, \( \psi(t) \) is continuous in \( t > 0 \). However, the boundedness of \( t^\alpha \psi(t) \) near \( t = 0 \) is not clear, so we cannot apply the next lemma directly. We regard \( t = \eta > 0 \) as an initial time and argue in the same way as above. We then apply the next lemma for \( \psi(t) = \|\nabla \omega\|_p(t + \eta) \) to get
\[
\|\nabla \omega\|_p(t + \eta) \leq W t^{1/p-3/2}\|\omega_0\|_1.
\]
Note that \( t^\alpha \psi(t) \) is bounded near \( t = 0 \) by Theorem 2.1.1. Since \( W \) is independent of \( \eta \) we obtain (i).
Lemma. Assume that $\psi$ is a continuous function defined on $(0,T)$, where $0 < T \leq \infty$ (it suffices to assume that $\psi$ is locally integrable to prove the claim). Let $\alpha$ be a real number, $\gamma$, $\delta$ positive numbers, and $\gamma + \delta = 1$, $0 < \gamma < 1$. Assume that $t^\alpha \psi(t)$ is bounded near $t = 0$. Moreover, let $b_\varepsilon$ be a positive number determined by a positive number $\varepsilon < 1$. (For the sake of simplicity, we assume that $b_\varepsilon$ is nonincreasing with respect to $\varepsilon$.) Assume that there exists a constant $\sigma$ such that

$$0 \leq \psi(t) \leq \sigma \left( b_\varepsilon t^{-\alpha} + \int_{t(1-\varepsilon)}^t \frac{\psi(s)}{(t-s)^{\gamma s^\delta}} \, ds \right),$$

for any $0 < t < T$, $0 < \varepsilon < 1$, and that is independent of $\varepsilon$ and $t$. Then there exists a constant $C$ depending only on $\sigma$, $\alpha$, $\delta$, $\gamma$ (and $b_\varepsilon$, which is a function of $\varepsilon$) such that

$$\psi(t) \leq C \sigma t^{-\alpha},$$

for any $0 < t < T$. Moreover, we may take $C$ nondecreasing with respect to $\sigma$.

Proof of Lemma. By the assumption, for $0 < t < T$, we have

$$\psi(t)t^\alpha \leq \sigma \left( b_\varepsilon + t^\alpha \int_{t(1-\varepsilon)}^t \frac{\psi(s)s^\alpha}{(t-s)^{\gamma s^\delta}} \, ds \right).$$

We consider

$$\varphi(t) = \sup_{0 \leq \tau \leq t} \tau^\alpha \psi(\tau).$$

Then, for $t > 0$, we have

$$\varphi(t) \leq \sigma \left( b_\varepsilon + t^\alpha \int_{t(1-\varepsilon)}^t \frac{ds}{(t-s)^{\gamma s^\delta + \alpha}} \right)$$

$$= \sigma \left( b_\varepsilon + \varphi(t) \int_{1-\varepsilon}^1 \frac{d\tau}{(1-\tau)^{\gamma \tau^{\delta + \alpha}}} \right).$$

The last equality is obtained by the coordinate transformation $s = t \tau$ and $\gamma + \delta = 1$. Since $0 < \gamma < 1$,

$$I(\varepsilon) = \int_{1-\varepsilon}^1 \frac{1}{(1-\tau)^{\gamma \tau^{\delta + \alpha}}} \, d\tau$$

converges for $0 < \varepsilon < 1$. Since $I(\varepsilon)$ is an increasing function with respect to $\varepsilon$, for $\sigma > 0$, there exists a unique $\varepsilon > 0$ such that $I(\varepsilon) = \min(\frac{1}{2\sigma}, I(1))$. ($I(1)$ can be $\infty$.) For such an $\varepsilon = \varepsilon(\sigma)$, we have

$$\varphi(t) \leq \sigma b_{\varepsilon(\sigma)} + \frac{1}{2} \varphi(t), \quad \eta/(1-\varepsilon(\sigma)) < t < T.$$
Hence we obtain
\[ \varphi(t) \leq 2\sigma b_\varepsilon(s), \quad 0 < t < T. \]
Since \( \varepsilon(s) \) is nonincreasing with respect to \( s \), \( C = 2b_\varepsilon(s) \) is nonincreasing with respect to \( s \). Therefore, the lemma is proved. \( \square \)

Now we return to the proof of the theorem. First, as mentioned in Remark 2.2.1, since \( \partial_x u = K * \partial_x \omega \) (\( j = 1, 2 \)), by the estimate of \( \| \nabla \omega \|_p \) in (i), similarly as in the proof of (ii) and (iii) of Theorem 2.4.1, we obtain
\[ \| \nabla u \|_p(t) \leq L_1(p)W t^{\frac{1}{p}-1} \| \omega_0 \|_1 \quad (2 < p \leq \infty), \quad t > 0. \]
To obtain the estimates for second derivatives of \( \omega \), by similar arguments as in (i), we differentiate twice the integral equation satisfied by \( \omega \), and then estimate it. This yields
\[
\| \nabla \nabla \omega \|_p(t) \leq \| \nabla \nabla (e^{t\Delta} \omega_0) \|_p + \int_{t(1-\varepsilon)}^t \frac{C_1}{(t-s)^{1/2}} \| \nabla (u, \nabla) \omega \|_p(s) ds \\
+ \int_0^{t(1-\varepsilon)} \frac{C_2'}{(t-s)^{3/2-1/p}} \| h_1 \|_1(s) ds
\]
\( (C_2' \) is a constant independent of \( p, \omega_0, t, s. \)

Since \( \| \nabla (u, \nabla) \omega \|_p \leq \| \nabla u \|_\infty \| \nabla \omega \|_p + \| u \|_\infty \| \nabla \nabla \omega \|_p \) holds, by substituting the estimates of \( \| \nabla u \|_\infty(t), \| u \|_\infty(t), \| \nabla \omega \|_p(t), \) and \( \| h_1 \|_1(t), \) we obtain an inequality for \( \| \nabla \nabla \omega \|_p(t), \) to which the Gronwall-type lemma is applicable. Hence we obtain (ii) from the lemma. Using (2.7), if we apply (i), (ii), and the estimate for \( \| u \|_\infty \) in \( \S 2.4.1 \), then we obtain (iii) from \( \partial_t \omega = \Delta \omega - (u, \nabla) \omega. \) Hence Theorem 2.4.2 is proved. \( \square \)

Next let us state the outline of the proof of the corollary. First we consider the case \( b = 0 \). Recalling (ii) of Theorem 2.4.2 and \( \partial_x^2 u = K * (\partial_x^2 \omega) \), we can obtain estimates not only for \( \| u \|_p(t), \| \nabla u \|_p(t), \) but also for \( \| \partial_x^2 u \|_p(t) \) \( (2 < p \leq \infty, \| \beta \| = 2) \) (which are mentioned in the corollary), analogously to the calculation in the proof of (iii). By similar calculations as in the proof of the theorem, if we differentiate the integral equation of \( \omega \) three times and use the estimates of \( \| \partial_x^2 u \|_\infty(t) \) for \( \| \beta \| \leq 2 \), then we obtain an estimate for
\[
\psi(t) = \sum_{|\beta|=3} \| \partial_x^2 \omega \|_p(t).
\]
By applying the Gronwall-type lemma, we obtain the estimate \( \| \partial_x^2 \omega \|_p(t) \) for \( \| \beta \| = 3 \), which is claimed in the corollary.

In general, once the claim in the corollary for \( \| \partial_x^2 \omega \|_p(t) \) \( (1 \leq p \leq \infty, \| \beta \| = k \geq 1) \) is established, then by estimating \( K * (\partial_x^2 \omega) \), we obtain the claim for \( \| \partial_x^2 u \|_p(t) \) \( (2 < p \leq \infty, \| \beta \| = k) \). Next, by differentiating the integral equation of \( \omega \) \( k+1 \) times and using the estimate for \( \| \partial_x^2 u \|_\infty(t) \) for \( \| \gamma \| \leq k \), we get the estimate in the corollary for \( \| \partial_x^k \omega \|_p(t) \) \( (1 \leq p \leq \infty, \| \mu \| = k+1) \) by the Gronwall-type lemma. Hence by induction with respect to \( k \), we obtain the estimates in the corollary for \( b = 0 \).
For the estimate of \( b > 0 \), using
\[
\frac{\partial_t \omega}{\Delta \omega} - (u, \nabla)\omega, \quad \frac{\partial_t u}{K \ast \partial_t \omega}
\]
repeatedly, we can replace the time derivative by spatial derivatives. Then
from the estimates of \( \omega \) and \( u \) for \( b = 0 \), we obtain the desired estimates for
the time derivative.

Similarly to the case of the heat equation, we will obtain an estimate for
the vorticity \( \omega \) at space infinity. However, the proof is not so simple as in the
case of the heat equation.

### 2.4.3 Decay Estimates for the Vorticity in Spatial Variables

**Proposition.** Let the initial value \( \omega_0 \) be in \( C_0(\mathbb{R}^2) \). Then there exists a constant \( W' \) satisfying the following property: Assume that \( (\omega, u) \) is a solution of (2.7), (2.8), and (2.9), and that the support of \( \omega_0 \) \( \text{supp} \omega_0 \) is contained in the open ball \( B_{R_0} \) with radius \( R_0 \). Then we have
\[
\sup_{|x| \geq R} |\omega(x, t)| \leq \frac{W'}{R} \|\omega_0\|_1 \left( \frac{1}{t^{1/2}} + 1 \right),
\]
for all \( R \geq \max(2R_0, 1) \), \( t > 0 \), where \( W' \) depends only on \( \|\omega_0\|_1 \) and is nondecreasing with respect to \( \|\omega_0\|_1 \).

**Proof.** The outline of the proof is as follows: First we multiply a suitable function to \( \omega \) to construct a modified function \( \omega_R \) that vanishes in \( B_{R/2} \) and coincides with \( \omega \) outside \( B_R \). We calculate the equation that \( \omega_R \) satisfies, and then establish estimates for \( \|\omega_R\|_\infty \).

### The First Step (Construction of \( \omega_R \))

First we choose a function \( \theta \in C^\infty([0, \infty)) \) satisfying \( 0 \leq \theta \leq 1 \),
\[
\theta(\rho) = \begin{cases} 
0, & \rho \leq 1/2, \\
1, & \rho \geq 1,
\end{cases}
\]
and \( \theta' \geq 0 \). Then we set
\[
\varphi_R(x) = \theta \left( \frac{|x|}{R} \right), \quad x \in \mathbb{R}^2,
\]
and define the function \( \omega_R \) as
\[
\omega_R(x, t) = \omega(x, t) \varphi_R(x), \quad x \in \mathbb{R}^2, \ t \geq 0.
\]
(By definition, \( \omega_R(x, t) = \omega(x, t) \) if \( |x| \geq R, \ t > 0 \). We may construct such a \( \theta \) as in the first step of the proof in §4.4.2.) Since \( \omega \) satisfies (2.7), \( \omega_R \) satisfies
\[
\frac{\partial_t \omega_R}{\Delta \omega_R} - (u, \nabla)\omega_R = h_2,
\]
\[
h_2 = ((u, \nabla)\varphi_R)\omega - 2(\nabla \varphi_R, \nabla \omega) - (\Delta \varphi_R)\omega
\]
on $\mathbb{R}^2 \times (0, \infty)$. In the proof below we use the results on inhomogeneous heat equations with a transport term, in which $h_2$ is regarded as a given inhomogeneous term, while $(u, \nabla)\omega_R$ is not. This is because it seems difficult to derive the desired estimate if we apply the results on inhomogeneous heat equations without a transport term by regarding $(u, \nabla)\omega_R$ as a given function.

By the unique existence theorem ($\S 2.2.1$), $u \in C^\infty(\mathbb{R}^2 \times (0, \infty))$, and for any $t_1 > t_0 > 0$, multi-indices $\alpha$, and $\ell = 0, 1, 2, \ldots$, we have

$$\sup_{t_0 \leq t \leq t_1} \| \partial_x^\alpha \partial_t^\ell u \|_\infty(t) < \infty.$$  

Then by Theorem 4.4.4 we may define an evolution system $U(t, s)$, $t > s$, that corresponds to $\partial_t - \Delta + (u, \nabla)$, for $s > 0$. Namely, when $f \in C(\mathbb{R}^2)$ is bounded and integrable, ($\|f\|_\infty < \infty$ and $\|f\|_1 < \infty$), then we may express a solution $V$ of

$$\begin{cases} 
\partial_t V - \Delta V + (u, \nabla)V = 0 & \text{in } \mathbb{R}^2 \times (s, \infty), \\
V|_{t=s} = f & \text{in } \mathbb{R}^2,
\end{cases}$$

by $V(x, t) = (U(t, s)f)(x)$. Moreover, $h_2$ is a $C^\infty$ function on $\mathbb{R}^2 \times (0, \infty)$, and is zero on $|x| \geq R$. Hence if $1 \leq p \leq \infty$, $\|h_2\|_p(t)$ is finite for $t > 0$. Therefore $U(t, s)h_2(s)$ for $s > 0$ is well defined. Here $U(t, s)h_2(s)$ is a function of $t > s$ and the spatial variable $x$. Here and in the sequel, we suppress the $x$-dependence for simplicity. Similarly as above, for $\omega_R \in C(\mathbb{R}^2 \times (0, \infty))$, $\omega_R(t)$ denotes the function $\omega_R(x, t)$ of $x$ on $\mathbb{R}^2$. Here, $\omega_R$ is also $C^\infty$ on $\mathbb{R}^2 \times (0, \infty)$, and by the unique existence theorem ($\S 2.2.1$), for $0 < t_0 < t_1$, we have

$$\sup_{t_0 \leq t \leq t_1} \| \omega_R \|_p(t) < \infty, \quad 1 \leq p \leq \infty.$$  

Since we have $L^\infty$-estimates for higher derivatives, we may use the evolution system $U(t, s)$. By Theorem 4.4.4 and (ii) of the remark afterward, $\omega_R$ is given by

$$\omega_R(t) = U(t, \varepsilon)\omega_R(\varepsilon) + \int_{\varepsilon}^{t} U(t, s)h_2(s)ds, \quad 0 < \varepsilon < t,$$

in $\mathbb{R}^2$. This is an equality as functions in $\mathbb{R}^2$ with the parameter $t$ ($> \varepsilon$). (Since $\nabla u$ may diverge to infinity at $t = 0$, in order to use Theorem 4.4.4 we introduced $\varepsilon > 0$.)

**The Second Step (Estimates for $\int_{\varepsilon}^{t} U(t, s)h_2(s)ds$)**

First, by the Hölder inequality, for $1 \leq q \leq \infty$, $h_2$ is bounded by

$$\|h_2\|_q(t) \leq \|u\|_\infty(t)\|\nabla \varphi_R\|_\infty \|\omega\|_q(t) + 2\|\nabla \varphi_R\|_\infty \|\nabla \omega\|_q(t) + \|\Delta \varphi_R\|_\infty \|\omega\|_q(t).$$

By the chain rule and using the constants $C_\theta$ and $C'_\theta$, which depend only on $\theta$, we have
\[ \| \nabla \varphi_R \|_{\infty} \leq \frac{C_{\theta}}{R}, \quad \| \Delta \varphi_R \|_{\infty} \leq \frac{C_{\theta}'}{R^2} \leq \frac{C_{\theta}'}{R}, \quad R \geq 1. \]

Therefore, using the estimate for derivatives in §2.4.2 and the estimates for \( u \) and \( \omega \) in §2.4.1, for \( 1 \leq q \leq \infty \), we obtain

\[ \| h_2 \|_q(t) \leq \frac{W_1}{R} \left( t^{-\frac{1}{2}} + 1 \right) t^{\frac{1}{q} - 1} \| \omega_0 \|_1, \quad t > 0, \quad R \geq 1. \]

Here \( W_j \) \(( j = 1, 2, 3)\) are constants, which are independent of \( R \) and have the same property as \( W \) in §2.4.2.

On the other hand, for \( p \) with \( 1 \leq p \leq \infty \), by recalling that \( \| U(t, s)h_2(s) \|_p \) is continuous at \( t = s \) as a function of \( t \) \(( t \geq s)\) (§4.4.4), and using the fundamental \( L^q-L^1 \) estimate (§2.3.1) and its generalization (§2.3.6) we obtain

\[ \| U(t, s)h_2(s) \|_{\infty} \leq \frac{1}{\kappa(t-s)} \| h_2 \|_1(s), \quad t > s, \]
\[ \| U(t, s)h_2(s) \|_{\infty} \leq \frac{1}{[\kappa(t-s)]^{1/2}} \| h_2 \|_2(s), \quad t > s. \]

Using the above estimates, we will estimate \( \int_{\varepsilon}^t U(t, s)h_2(s)ds \). Since the integrand may be infinite at \( s = 0 \) and \( s = t \), we divide the interval of integration. If \( 0 < \varepsilon < t/2 \), we obtain

\[ \left\| \int_{\varepsilon}^t U(t, s)h_2(s)ds \right\|_{\infty} \]
\[ \leq \int_{\varepsilon}^{t/2} \| U(t, s)h_2(s) \|_{\infty} ds + \int_{t/2}^t \| U(t, s)h_2(s) \|_{\infty} ds \]
\[ \leq \int_0^{t/2} \frac{1}{\kappa(t-s)} \| h_2 \|_1(s) ds + \int_{t/2}^t \frac{1}{[\kappa(t-s)]^{1/2}} \| h_2 \|_2(s) ds \]
\[ \leq \frac{W_2}{R} \| \omega_0 \|_1 \left\{ \int_0^{t/2} \frac{1}{(t-s)} \left( \frac{1}{s^{1/2}} + 1 \right) ds \right. \]
\[ \left. + \int_{t/2}^t \frac{1}{(t-s)^{1/2}} \left( \frac{1}{s^{1/2}} + 1 \right) \frac{1}{s^{1/2}} ds \right\}. \]

Setting \( s = t\tau \) and calculating the integral, we obtain

\[ \int_0^{t/2} \frac{1}{(t-s)} \left( \frac{1}{s^{1/2}} + 1 \right) ds \]
\[ = \frac{1}{t^{1/2}} \int_0^{1/2} \frac{1}{(1-\tau)^{1/2}} d\tau + \int_0^{1/2} \frac{1}{1-\tau} d\tau = A_0 t^{-1/2} + A_1, \]
\[
\int_{t/2}^{t} \frac{1}{(t-s)^{1/2}} \left( \frac{1}{s^{1/2}} + 1 \right) \frac{1}{s^{1/2}} ds \\
= \frac{1}{t^{1/2}} \int_{1/2}^{1} \frac{1}{(1-\tau)^{1/2}\tau} d\tau + \int_{1/2}^{1} \frac{1}{(1-\tau)^{1/2}\tau} d\tau = A_2 t^{-1/2} + A_3,
\]

where \(A_0, A_1, A_2, A_3\) are real numbers independent of \(t\). Hence we obtain

\[
\left\| \int_{\varepsilon}^{t} U(t, s) h_2(s) ds \right\|_{\infty} \leq \frac{W_3}{R} \|\omega_0\|_1 \left( \frac{1}{t^{1/2}} + 1 \right),
\]

for \(t \geq 2\varepsilon > 0, R \geq 1\).

**The Third Step (Estimate for \(U(t, \varepsilon)\omega_R(\varepsilon)\))**

By the maximum principle (§2.3.7), for \(\varepsilon > 0\), we have

\[
\|U(t, \varepsilon)\omega_R(\varepsilon)\|_{\infty} \leq \|\omega_R(\varepsilon)\|_{\infty}, \quad t \geq \varepsilon.
\]

(By property (§2.2.1) for \(\omega, \omega_R(\varepsilon)\) belongs to \(C(\mathbb{R}^2)\) and \(\|\omega_R(\varepsilon)\|_p < \infty\) \((1 \leq p \leq \infty)\). Moreover, by Theorem 4.4.4, \(\|U(t, \varepsilon)\omega_R(\varepsilon)\|_p\) is continuous at \(t = \varepsilon\) as a function of \(t\) \((t \geq \varepsilon)\). Hence we may apply §2.3.7.) On the other hand, if \(R > 2R_0 > 0, \varphi_R\) is zero on the ball \(B_{R_0}\); hence from the assumption on the support of \(\omega_0\), we obtain \(\omega_R(0) = 0\). By the continuity of \(\omega(t)\) at \(t = 0\) (§2.2.1 (i)),

\[
\lim_{\varepsilon \to 0} \|\omega_R(\varepsilon)\|_{\infty} = \lim_{\varepsilon \to 0} \|\omega_R(\varepsilon) - \omega_R(0)\|_{\infty} \leq \lim_{\varepsilon \to 0} \|\varphi_R\|_{\infty} \|\omega(\varepsilon) - \omega_0\|_{\infty} = 0
\]

is valid for \(R > 2R_0 > 0\). Hence we obtain

\[
\lim_{\varepsilon \to 0} \|U(t, \varepsilon)\omega_R(\varepsilon)\|_{\infty} = 0, \quad t > 0,
\]

for \(R > 2R_0 > 0\).

**The Final Step (Completion of the proof)**

Taking \(R \geq \max(2R_0, 1)\) and estimating the \(L^\infty\)-norm of the formula for \(\omega_R\) at the end of the first step, we obtain

\[
\|\omega_R\|_{\infty}(t) \leq \|U(t, \varepsilon)\omega_R(\varepsilon)\|_{\infty} + \frac{W_3}{R} \|\omega_0\|_1 \left( \frac{1}{t^{1/2}} + 1 \right), \quad t > 2\varepsilon,
\]

by the second step. Using the result of the third step, as \(\varepsilon \to 0\), we obtain

\[
\|\omega_R\|_{\infty}(t) \leq \frac{W_3}{R} \|\omega_0\|_1 \left( \frac{1}{t^{1/2}} + 1 \right), \quad t > 0.
\]

Hence, recalling that \(\sup_{|x| \geq R} |\omega(x, t)| \leq \|\omega_R\|_{\infty}(t)\), we obtain the desired inequality. \(\Box\)
2.5 Proof of the Asymptotic Formula

Now we prove the asymptotic formula (2.10) in §2.2.2. Assume that the initial vorticity $\omega_0$ is in $C_0(\mathbb{R}^2)$, and that $(\omega, u)$ is a solution of (2.7), (2.8), and (2.9). We consider the family $\{(\omega_k, \overline{u}_k)\}_{k \geq 1}$, which is rescaled as in (2.11). First we consider the “compactness” that is announced in §2.2.5. By Proposition 2.2.3, $(\omega_k, \overline{u}_k)$ satisfies (2.7) and (2.8), and its initial values are $\omega_k|_{t=0} = \omega_0k$, where we define $\omega_0k(x) = k^2\omega(kx), x \in \mathbb{R}^2$. In the estimates in §2.4.1, §2.4.2, and §2.4.3, set $\omega = \omega_k$ and $u = \overline{u}_k$. Since $\|\omega_0k\|_1 = \|\omega_0\|_1$, we may take coefficients $W$ and $W'$ independent of $k$. Hence using the Ascoli–Arzelà-type compactness theorem (§1.3.2), as in the case of the heat equation (§1.3.5), for any subsequence $\{(\omega_{k(\ell)})\}_{\ell=1}^\infty$ of $\{\omega_k\}_{k \geq 1}$, there exists a subsequence $\{(\omega_{k(\ell(i))})\}_{i=1}^\infty$ of $\{\omega_{k(\ell)}\}$ such that $\omega_{k(\ell(i))}$ converges pointwise to some function $\overline{\omega} \in C(\mathbb{R}^2 \times (0, \infty))$ on $\mathbb{R}^2 \times (0, \infty)$ as $i \to \infty$. Moreover, its convergence is uniform on $\mathbb{R}^2 \times [\eta, 1/\eta]$ for any $\eta \in (0, 1)$.

For the limit function $\overline{\omega}$, we define $\overline{u} = K \ast \overline{\omega}$. Then $\overline{\omega}$ is a “weak solution” of

$$\begin{align*}
\partial_t \overline{\omega} - \Delta \overline{\omega} + (\overline{u}, \nabla) \overline{\omega} &= 0, \\
\overline{u} &= K \ast \overline{\omega}, \\
\overline{\omega}|_{t=0} &= m\delta, \quad m = \int_{\mathbb{R}^2} \omega_0 \, dx,
\end{align*}$$

in $\mathbb{R}^2 \times (0, \infty)$, where $\delta$ denotes the Dirac $\delta$ distribution. We will state this fact in §2.5.1 in a precise form. In §2.5.4 we will prove the uniqueness of this limit function and we will characterize the limit function that is mentioned in the end of §2.2.5.

Before discussing the uniqueness, we will show that if $\overline{\omega}(x, t)$ is smooth on $x \in \mathbb{R}^2$, $t > 0$, and if $1 \leq p \leq \infty$, then for any multi-index $\beta$ and $b = 0, 1, 2, \ldots,$

$$\sup_{t>0} t^{\frac{|\beta|}{2} + b + 1 - \frac{1}{p}} \|\partial_t^b \partial_x^\beta \overline{\omega}(t)\|_p < \infty. \quad (2.12a)$$

By Corollary 2.4.2, there exists a positive constant $W$ such that for $\omega_k$, we have

$$\sup_{t>0} t^{\frac{|\beta|}{2} + b + 1 - \frac{1}{p}} \|\partial_t^b \partial_x^\beta \omega_k(t)\|_p \leq W(\|\omega_0k\|_1, \beta, b)\|\omega_0k\|_1$$

$$= W(\|\omega_0\|_1, \beta, b)\|\omega_0\|_1,$$

where the right-hand side is independent of $k$. In particular, for any $b = 0, 1, 2, \ldots$, any multi-index $\beta$, and any $\eta > 0$, we have

$$\sup_{k \geq 1} \sup_{t \geq \eta} \|\partial_t^b \partial_x^\beta \omega_k\|_\infty(t) < \infty.$$  

By these estimates, using the theorem on the convergence of higher derivatives (§5.2.5) that is obtained as an application of the Ascoli–Arzelà theorem, we see that $\overline{\omega} \in C^\infty(\mathbb{R}^2 \times (0, \infty))$ and $\partial_t^b \partial_x^\beta \omega_{k(\ell(i))}$ converges uniformly to $\partial_t^b \partial_x^\beta \overline{\omega}$.
on any compact subset of $\mathbb{R}^2 \times (0, \infty)$ as $i \to \infty$. On the other hand, for any $t > 0$,
\[
\|\partial_t^i \partial_x^j \omega\|_p(t) \leq \lim_{i \to \infty} \|\partial_t^i \partial_x^j \omega_{k(\ell(i))}\|_p(t)
\]
(Exercise 2.6); hence by the estimates
\[
t^{\frac{|\alpha|}{2} + b + 1 - \frac{1}{p}} \|\partial_t^i \partial_x^j \omega_k\|_p(t) \leq W(\|\omega_0\|_1, \beta, b) \|\omega_0\|_1 < \infty,
\]
(2.12a) follows.

We will rigorously show that $(\omega, u)$ is a weak solution of the vortex equation with initial value $m\delta$. We note that by (2.12a) and Remark (iv) in §6.3.5, $u = K \ast \omega$ is defined as a smooth function on $\mathbb{R}^2 \times (0, \infty)$.

2.5.1 Characterization of the Limit Function as a Weak Solution

**Theorem.** The function $\omega$, which is defined by the limit of $\omega_{k(\ell(i))}$ as $i \to \infty$, satisfies
\[
0 = m\varphi(0, 0) + \int_0^\infty \int_{\mathbb{R}^2} \{\left(\varphi_t + \Delta \varphi\right) \omega + \langle \nabla \varphi, u \omega \rangle\} dx dt
\]
for any $\varphi \in C_0^\infty(\mathbb{R}^2 \times [0, \infty))$, where we set $\omega = K \ast \omega$, and $m = \int_{\mathbb{R}^2} \omega_0 dx$.

Here the pair $(\omega, \omega)$ with $\omega = K \ast \omega$ is called a weak solution of (2.7) and (2.8) with initial value $m\delta$ if $(\omega, \omega)$ satisfies the above integral equality for any $\varphi \in C_0^\infty(\mathbb{R}^2 \times [0, \infty))$. For $\omega$ and $\varphi = (\bar{\omega}, \bar{\omega})$, we assume that every term in the above integral equality makes sense and that $\omega = K \ast \omega$ is well defined. For example, it suffices to assume that $\bar{\omega}$, $|\bar{\omega}|$, and $|\bar{\omega}|$ are locally integrable (§1.4.3) on $\mathbb{R}^2 \times [0, \infty)$, and that $\|\omega\|_q(t) < \infty$, $t > 0$, $1 < q < \infty$. Let $g = G_t$. Then $(mg, K \ast (mg))$ is a weak solution of (2.7) and (2.8) with initial value $m\delta$ by Lemma 2.2.5 and Exercise 1.9.

The basic idea of the proof is the same as in the case of the heat equation. Since $(\omega_k, \bar{\omega}_k)$ is a solution of the vortex equation (2.7), (2.8), and (2.9) with initial value $\omega_{0k}$, for $\varphi \in C_0^\infty(\mathbb{R}^2 \times [0, \infty))$, by integration by parts it satisfies
\[
0 = \int_{\mathbb{R}^2} \varphi(x, 0)\omega_{0k}(x) dx + \int_0^\infty \int_{\mathbb{R}^2} (\varphi_t + \Delta \varphi) \omega_k dx dt + \int_0^\infty \int_{\mathbb{R}^2} \langle \nabla \varphi, u_k \omega_k \rangle dx dt.
\]
By the estimate $\sup_{t > 0} \|\omega_k\|_1(t) \leq \|\omega_0\|_1$ in §2.3.3, as $k \to \infty$, the first and second terms on the right-hand side converge to
\[
m\varphi(0, 0) + \int_0^\infty \int_{\mathbb{R}^2} (\varphi_t + \Delta \varphi) \omega dx dt,
\]
by similar arguments as in Proposition 1.4.1 and in the proof of §1.4.4. In this proof, $\omega_k$ simply denotes a subsequence $\omega_{k(\ell(i))}$ of $\omega$. 
Moreover, for each fixed \( t > 0 \), \( \bar{u}_k \) converges uniformly to \( \bar{u} \) for \( x \in \mathbb{R}^2 \). This fact will be established at the end of the proof. Set

\[
F_k(t) = \int_{\mathbb{R}^2} \langle \nabla \phi, \bar{u}_k \omega_k \rangle dx.
\]

If \( \bar{u}_k \) converges uniformly to \( \bar{u} \), since the support of \( \phi(\cdot, t) \) is compact for each \( t > 0 \) and so the region of integration is actually bounded (hence we can interchange integrals and limits as in Proposition 7.1), we see that \( F_k(t) \) converges to

\[
F(t) = \int_{\mathbb{R}^2} \langle \nabla \phi, \bar{u} \omega \rangle dx
\]

for each \( t > 0 \). On the other hand, by Theorem 2.4.1, we get

\[
|F_k(t)| \leq C_\phi \| \pi_k \|_{L^\infty(t)} \| \omega_k \|_{L^1(t)} \leq C_\phi \| \omega_0 \|_{L^2}^2 \| L_1(\infty)(\kappa t)^{-1/2},
\]

where \( \kappa \) is the universal constant of §2.3.1 and \( C_\phi = \sup \{ |\nabla \phi(x, t)| : x \in \mathbb{R}^2, t \geq 0 \} < \infty \). Since \( t^{-1/2} \) is integrable on neighborhoods of \( t = 0 \), and \( \phi \) is zero for sufficiently large \( t \), by the dominated convergence theorem (§7.1.1), we obtain

\[
\lim_{k \to \infty} \int_0^\infty F_k(t) dt = \int_0^\infty F(t) dx.
\]

Hence, we have proved

\[
\lim_{k \to \infty} \int_0^\infty \int_{\mathbb{R}^2} \langle \nabla \phi, \bar{u}_k \omega_k \rangle dx \ dt = \int_0^\infty \int_{\mathbb{R}^2} \langle \nabla \phi, \bar{u} \omega \rangle dx \ dt,
\]

and \((\omega, \bar{u})\) is a weak solution of (2.7) and (2.8) with initial value \( m\delta \).

In the following, we show that \( \pi_k \) converges uniformly to \( \pi \) with respect to \( x \in \mathbb{R}^2 \) for each fixed \( t > 0 \). First set \( v_k = \bar{u}_k - \bar{u}, w_k = \omega_k - \bar{\omega} \). We note that \( v_k = K * w_k \). By the Gagliardo–Nirenberg inequality (§6.1.1), the Calderón–Zygmund inequality (§6.4.2), and the Hardy–Littlewood–Sobolev inequality (§6.2.1) (by fixing \( t > 0 \)) for \( 2 < p < \infty \) we have

\[
\| v_k \|_{L^\infty(t)} \leq C \| \nabla v_k \|_{L^p(t)}^{2/p} \| v_k \|_{L^{1-2/p}(t)}^{1-2/p} \leq C' \| w_k \|_{L^p(t)}^{2/p} \| w_k \|_{L^{1-2/p}(t)}^{1-2/p}, \quad \frac{1}{p} = \frac{1}{q} - \frac{1}{2}.
\]

(See (i) of the remark below.) Here the constants \( C \) and \( C' \) depend only on \( p \) and are independent of \( t \). Using the Hölder inequality, we obtain

\[
\| v_k \|_{L^\infty(t)} \leq C' \| w_k \|_{L^q(t)}^{1-2/p} \{ \| w_k \|_{L^q(t)}^{q/p(t)} \| w_k \|_{L^{1-q/p(t)}}^{1-q/p(t)} \}^{2/p}.
\]

Since \( q/p = 2/(p + 2) \), the right-hand side of the last inequality is

\[
C' \| w_k \|_{L^q(t)}^{2/2(\frac{p}{p+2})} \| w_k \|_{L^{1-q/p(t)}}^{\frac{2}{2(\frac{p}{p+2})}}(t).
\]
By (i) of Theorem 2.4.1, for $t > 0$, $1 \leq r \leq \infty$, we have $\| \omega_k \|_r(t) \leq (\kappa t)^{-1+1/r} \| \omega_0 \|_1$, which leads to $\| \omega \|_r(t) \leq (\kappa t)^{-1+1/r} \| \omega_0 \|_1$ as in (2.12a). Hence $\| w_k \|_r(t) \leq 2(\kappa t)^{-1} \| \omega_0 \|_1$ and we obtain

$$
\| v_k \|_\infty(t) \leq 2 \pi^{\frac{1}{p}} C'(\kappa t)^{\left(\frac{2}{2+1} - \frac{1}{p}\right)} \| \omega_0 \|_1^{\frac{p}{2+1}} \| w_k \|_\infty^{\frac{2}{2+1}}(t).
$$

By the definition of the subsequence $\{\omega_{k(t(i))}\}_{i=1}^\infty$ (§2.5), for $\eta \in (0, 1)$, $w_k$ converges uniformly to 0 on $\mathbb{R}^2 \times [\eta, 1/\eta]$. Hence $v_k$ converges uniformly to 0 on the same region. 

\[ \square \]

**Remark.**

(i) Since we do not use distribution theory or Lebesgue integrals, we need to check that $v_k(x,t)$ is $C^1$ on $\mathbb{R}^2$ as a function of $x$ in the step using the Gagliardo–Nirenberg inequality. By Theorem 2.2.1, $\overline{u}_k$ is smooth on $\mathbb{R}^2 \times (0, \infty)$. Moreover, similarly to Remark 2.5.1, $\overline{\omega} = K \ast \overline{\omega}$ is smooth on $\mathbb{R}^2 \times (0, \infty)$. Hence for each fixed $t > 0$, $v_k(x,t)$ is $C^\infty$ on $\mathbb{R}^2$ as a function of $x$. On the other hand, the continuity of $w_k$ and $\| w_k \|_q(t) < \infty$ ($1 < q < 2$), which is assumed when we apply the Hardy–Littlewood–Sobolev inequality (see also Remark 6.2.1), is also obtained by the continuity of $\overline{\omega}$ and (2.12a) for $\overline{\omega}$ in the case $b = 0$ and $|\beta| = 0$. We can also check that the Calderón–Zygmund inequality is available in our case, since $\overline{\omega}$ is smooth and $\| \overline{\omega} \|_p(t)$ is bounded for any $t > 0$, $1 \leq p \leq \infty$ by (2.12a) with $b = 0$ and $|\beta| = 0$. Note that these justifications are not needed if we use distributions and Lebesgue integrals. The key fact in this proof is Theorem 2.4.1, in particular, the $L^q$-$L^1$ estimate for $\omega$. Hence we need not assume that $\overline{\omega}$ and $\overline{\omega}$ are $C^\infty$ on $t > 0$.

(ii) The function $\overline{u}$ is $C^\infty$ on $\mathbb{R}^2 \times (0, \infty)$ and for any multi-index $\beta$, $b = 0$, 1, 2, ..., and $2 < p \leq \infty$ it satisfies

$$
\sup_{t > 0} t^{\frac{|\beta|}{2} + b + \frac{1}{2} - \frac{1}{p}} \| \partial_x^\beta \partial_t^b \overline{\omega} \|_p(t) < \infty.
$$

(2.12b)

To prove (2.12b) we first note that $\overline{\omega} \in C^\infty(\mathbb{R}^2 \times (0, \infty))$, and that $\partial_x^\beta \partial_t^b \overline{u}_{k(t(i))}$ converges uniformly to $\partial_x^\beta \partial_t^b \overline{\omega}$ on any compact subset of $\mathbb{R}^2 \times (0, \infty)$ as $i \to \infty$. This can be verified by §5.2.5 and the fact that $\overline{u}_k$ converges pointwise to $\overline{\omega}$ on $\mathbb{R}^2 \times (0, \infty)$. We can check that the assumption required in §5.2.5 is satisfied if we use Corollary 2.4.2 and the estimates

$$
\sup_{k \geq 1} \sup_{t > 0} t^{\frac{|\beta|}{2} + b + \frac{1}{2} - \frac{1}{p}} \| \partial_x^\beta \partial_t^b \overline{u}_k \|_p(t) < \infty, \quad 2 < p \leq \infty,
$$

which can be obtained as the estimates for $\omega_k$. Therefore, as in the proof of the estimates for $\overline{\omega}$ in (2.12a), we obtain (2.12b) from the estimates for $\overline{u}_k$. 

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Next, we will prove the estimate of $\bar{\omega}$ by $|m|$. As mentioned in the proof of the theorem, we have

$$\sup_{t>0}(\kappa t)^{1-1/p}\|\bar{\omega}\|_p(t) \leq \|\omega_0\|_1.$$  

If we try to prove the uniqueness of the limit $\bar{\omega}$ under the assumption of the smallness for $|m|$ but not for $\|\omega_0\|_1$, it is needed to prove the better estimate

$$\sup_{t>0}(\kappa t)^{1-1/p}\|\bar{\omega}\|_p(t) \leq |m|.$$  

(The proof of uniqueness without this estimate is not known so far.) This estimate by $|m|$ is stronger than the estimate by $\|\omega_0\|_1$. Indeed, it claims that if $m = \int \omega_0 dx = 0$ then $\bar{\omega} \equiv 0$. The next section is devoted to the proof of the above estimate by $|m|$. This estimate was first established in the Japanese edition of this book, but recently was also obtained by [Gallay Wayne 2005] in an implicit way (see \S 2.8). In the proof below we use some results on fundamental solutions of parabolic operators that are generalizations of the heat operator $\partial_t - \Delta$.

2.5.2 Estimates for the Limit Function

**Theorem.** Assume that the pair of functions $(\bar{\omega}, \bar{u})$ and $m$ are given as in Theorem 2.5.1. Then we have

$$\sup_{t>0}(\kappa t)^{1-1/p}\|\bar{\omega}\|_p(t) \leq |m|, \quad 1 \leq p \leq \infty,$$

where $\kappa$ is the universal constant in the fundamental $L^q$-$L^1$ estimate in \S 2.3.1.

**Proof.** The First Step

First we show that it is sufficient to prove the above theorem in the case $p = 1$ only. Since $\omega_k$ satisfies $(H_{\pi_k})$ on $\mathbb{R}^2 \times (0, \infty)$ for $k \geq 1$, $\bar{\omega}$ satisfies $(H_\bar{\pi})$ on $\mathbb{R}^2 \times (0, \infty)$ for the limit $(\bar{\omega}, \bar{u})$ of any subsequence of $(\omega_k, \bar{u}_k)$. This is because, as mentioned in the paragraph containing (2.12a) of \S 2.5 and in (ii) of Remark 2.5.1, subsequences $\{\omega_k(\ell(i))\}$ and $\{\bar{u}_k(\ell(i))\}$ of $\{\omega_k\}$ and $\{\bar{u}_k\}$ converge to $\bar{\omega}$ and $\bar{u}$ respectively together with their higher derivatives uniformly on each compact set in $\mathbb{R}^2 \times (0, \infty)$. Moreover, by (2.12a) and (2.12b), $\bar{\omega}$ and $\bar{u}$ satisfy the assumption of the fundamental $L^q$-$L^1$ estimates in \S 2.3.1, except for condition (I). Hence the system of differential inequalities in Proposition 2.3.4 holds. So if we show that

$$\|\omega\|_1(t) \leq |m|, \quad t > 0,$$

then we can prove the estimate in the theorem for general $p$, similarly to \S 2.3.5 using Lemma 2.3.4.
The Second Step

For an $\mathbb{R}^2$-valued function $v$ defined on $\mathbb{R}^2 \times (0, \infty)$, $\Gamma_v(x, t, y, s)$ $(x, y \in \mathbb{R}^2, t > s \geq 0)$ denotes the fundamental solution of the operator $\partial_t - \Delta + (v, \nabla)$. (The definition and basic properties of the fundamental solution will be given in §4.4.5.) As in the proof of Theorem 2.4.2, for $t > 0$, $(\omega_k, \overline{u}_k)$ satisfies

$$\omega_k(t) = e^{t\Delta} \omega_{0k} - \int_0^t \text{div} (e^{(t-s)\Delta}(\overline{u}_k \omega_k(s)))ds \quad \text{in} \quad \mathbb{R}^2.$$ 

By (iv) of Theorem 2.4.1, $\overline{u}_k$ is bounded near $t = 0$. Moreover, by (ii) of Theorem 2.4.1 we have $\sup_{0 < t < T} \|\overline{u}_k\|_{\infty}(t) \leq \infty$ for any $T > 0$. Since $\text{div} \overline{u}_k = 0$ in $\mathbb{R}^2 \times (0, \infty)$, there exists a unique fundamental solution $\Gamma_{\overline{u}_k}(x, t, y, s)$ on $t > s \geq 0$ (by the unique existence theorem, Theorem 2 in §4.4.5). On the other hand, by (i) of Theorem 2.4.1, we have $\sup_{t > 0} \|\omega_k\|_1(t) \leq \|\omega_{0k}\|_1 < \infty$; hence as in §4.4.5, by the lemma for the uniqueness in §4.4.4,

$$\omega_k(x, t) = \int_{\mathbb{R}^2} \Gamma_{\overline{u}_k}(x, t, y, 0) \omega_{0k}(y)dy, \quad t > 0, \quad x \in \mathbb{R}^2.$$ 

Since $\|\omega_k\|_1(t) \leq \|\omega_0\|_1$ for $t > 0$, by the following lemma the family of functions $\{\Gamma_{\overline{u}_k}(x, t, y, 0)\}_{k \geq 1}$ is uniformly bounded and equicontinuous as functions of $y \in \mathbb{R}^2$ for each $t > 0$, $x \in \mathbb{R}^2$. That is, for each $x \in \mathbb{R}^2$ and $t > 0$, we have

$$\sup_{k \geq 1} \sup_{y \in \mathbb{R}^2} |\Gamma_{\overline{u}_k}(x, t, y, 0)| < \infty,$$

$$\lim_{y' \to y} \sup_{k \geq 1} |\Gamma_{\overline{u}_k}(x, t, y', 0) - \Gamma_{\overline{u}_k}(x, t, y, 0)| = 0, \quad y \in \mathbb{R}^2.$$ 

Hence for each $R > 0$, by applying the Ascoli–Arzelà theorem (§5.1.1) to this family on a closed ball $\overline{B}_R$, $\{\Gamma_{\overline{u}_k}(x, t, y, 0)\}_{k \geq 1}$ contains a uniformly convergent subsequence on $\overline{B}_R$ as functions of $y$. In other words, there exist a subsequence $\{k_j\}$ of $\{k(\ell(i))\}_{i=1}^\infty$ and a continuous function $A_{t, x}(y)$ on $\overline{B}_R$ such that

$$\lim_{j \to \infty} \sup_{y \in \overline{B}_R} |\Gamma_{\overline{u}_{k_j}}(x, t, y, 0) - A_{t, x}(y)| = 0, \quad t > 0, \quad x \in \mathbb{R}^2.$$ 

(More precisely, we should write $\{k_j\}_{j=1}^\infty$ as $\{k(\ell(i(j)))\}_{j=1}^\infty$. For simplicity we abbreviate such a notation. We assume $k_j \to \infty$ as $j \to \infty$.)

Lemma. Assume that $v = (v^1, v^2)$ with $v^1, v^2 \in C^\infty(\mathbb{R}^2 \times (0, \infty))$ satisfies $\text{div} v = 0$ on $\mathbb{R}^2 \times (0, \infty)$, and that for each $S > 0$, $\sup_{0 < t < S} \|v\|_{\infty}(t) < \infty$. Then there exists a unique fundamental solution $\Gamma_v(x, t, y, s)$ of $\partial_t - \Delta + (v, \nabla)$ (see the unique existence theorem, Theorem 2 in §4.4.5) that is continuous on $\{(x, t, y, s) : x, y \in \mathbb{R}^2, 0 \leq s < t < \infty\}$. Moreover, assume that $\sup_{t_1 \leq t \leq t_2} \|\partial_x^\alpha \partial_t^\ell v\|_{\infty}(t) < \infty$ for each $t_2 > t_1 > 0$ and for any multi-index $\alpha$ and $\ell = 0, 1, 2, \ldots$. Then the following are valid:
We have \(0 \leq \Gamma_v(x, t, y, s) \leq (\kappa(t - s))^{-1}, \ x, y \in \mathbb{R}^2, \ 0 \leq s < t\), where \(\kappa\) is the universal constant in §2.3.1.

Assume that \(0 < t < T\), and that the function \(v\) is given as \(v = K \ast \omega\) with a function \(\omega \in C(\mathbb{R}^2 \times (0, T))\). Moreover, let \(\sup_{0 < t < T} \|\omega\|_1(t) \leq M_1\) and let \(t_0 > 0\). Then there exist a positive constant \(C\) depending only on \(t_0\) and \(M_1\) and a constant \(\mu \in (0, 1)\) such that

\[
|\Gamma_v(x, t, y, 0) - \Gamma_v(x', t, y', 0)| \leq C(|x - x'|^2 + |y - y'|^2)^{\mu/2},
\]

\(T > t_0, \ x, x', y, y' \in \mathbb{R}^2\).

### The Third Step

If we replace \(k\) by \(k_j\) and take \(j \to \infty\) in the expression of \(\omega_k\) by the fundamental solution in the second step, then we obtain

\[
\omega(x, t) = mA_{t, x}(0), \quad t > 0, \ x \in \mathbb{R}^2.
\]

In what follows we assume \(k_j \geq 1\). Choosing the radius \(R\) such that \(\text{supp} \omega_0 \subset B_R\), we calculate

\[
\omega_{k_j}(x, t) - mA_{t, x}(0) = \int_{B_R} \{\Gamma_{\omega_{k_j}}(x, t, y, 0) - A_{t, x}(y)\} \omega_{0k_j}(y) dy
\]

\[
+ \int_{B_R} A_{t, x}(y) \omega_{0k_j}(y) dy - mA_{t, x}(0).
\]

Since \(\text{supp} \omega_{0k_j} \subset B_R\), from the properties of the limit of the initial value (see Proposition 1.4.1, Remark 1.4.1, and §4.2.5), the equality

\[
\lim_{j \to \infty} \int_{B_R} A_{t, x}(y) \omega_{0k_j}(y) dy = mA_{t, x}(0), \quad t > 0, \ x \in \mathbb{R}^2,
\]

follows. On the other hand, by the uniform convergence of \(\Gamma_{\omega_{k_j}}\) and by \(\|\omega_{0k_j}\|_1 = \|\omega_0\|_1\), which are obtained in the Second Step, for \(t > 0\) and \(x \in \mathbb{R}^2\), we obtain

\[
\left| \int_{B_R} \{\Gamma_{\omega_{k_j}}(x, t, y, 0) - A_{t, x}(y)\} \omega_{0k_j}(y) dy \right|
\]

\[
\leq \sup_{y \in B_R} |\Gamma_{\omega_{k_j}}(x, t, y, 0) - A_{t, x}(y)| \|\omega_0\|_1 \to 0 \quad (j \to \infty).
\]

Hence for \(t > 0\) and \(x \in \mathbb{R}^2\), we have shown that \(\lim_{j \to \infty} \omega_{k_j}(x, t) = mA_{t, x}(0)\), and then \(\omega(x, t) = mA_{t, x}(0)\) follows, since \(\omega\) is the limit of \(\omega_{k_j}\).
The Fourth Step

Since \( \text{div} \, \mathbf{u}_{k_j} = 0 \), as in §4.4.5, it follows that

\[
\int_{\mathbb{R}^2} \Gamma_{\mathbf{u}_{k_j}}(x, t, y, 0) dx = 1, \quad y \in \mathbb{R}^2, \quad t > 0.
\]

Moreover, since \( \Gamma_{\mathbf{u}_{k_j}} \geq 0 \), we have \( A_{t, x}(0) \geq 0 \). Hence, by Fatou’s lemma (§7.1.2), we obtain

\[
\int_{\mathbb{R}^2} A_{t, x}(0) dx \leq \lim_{j \to \infty} \int_{\mathbb{R}^2} \Gamma_{\mathbf{u}_{k_j}}(x, t, 0, 0) dx = 1.
\]

Hence we obtain \( \|w\|_1(t) \leq |m| \) for \( t > 0 \). This completes the proof except for the proof of the lemma.

Proof of Lemma. (i) As stated in §4.4.5, \( \Gamma_v \geq 0 \) is an important property of the fundamental solution, which follows from the nonnegativity-preserving principle in §2.3.8. By the definition of the fundamental solution and the assumption on \( v \), the function \( w \) given by

\[
w(x, t) = \int_{\mathbb{R}^2} \Gamma_v(x, t, y, s) f(y) dy, \quad t > s > 0, \quad x \in \mathbb{R}^2,
\]

for \( f \in C_0(\mathbb{R}^2) \) satisfies \( (H_v) \) on \( \mathbb{R}^2 \times (s, T) \). By the assumption on higher derivatives of \( v \), if \( s > 0 \), then \( w \) coincides with the solution constructed in §4.4.4 (§4.4.5). Since we assumed \( s > 0 \), assumptions (I) and (a) in §2.3.1 are satisfied (\( \omega \) has to be replaced by \( w \)) by Theorem 4.4.4. By the fundamental \( L^q-L^1 \) estimate (§2.3.1), we obtain

\[
\|w\|_\infty(t) \leq (\kappa(t-s))^{-1} \|f_0\|_1, \quad t > s > 0.
\]

Now let \( w_0 \in C_0(\mathbb{R}^2) \) be a given function satisfying \( w_0 \geq 0 \) and \( w_0 \not\equiv 0 \), and set \( m = \int_{\mathbb{R}^2} w_0(y) dy \). For a given \( y_0 \in \mathbb{R}^2 \) and \( k \geq 1 \), set \( w_{0k}(y) = k^2 w_0(k(y - y_0) + y_0) \). Then, since \( \Gamma_v(x, t, y, s) \) is continuous with respect to \( y \) (Definition 4.4.5), we obtain

\[
m \Gamma_v(x, t, y_0, s) = \lim_{k \to \infty} w_k(x, t)
\]

and

\[
w_k(x, t) = \int_{\mathbb{R}^2} \Gamma_v(x, t, y, s) w_{0k}(y) dy, \quad s > 0,
\]

by Remark 1.4.1, Proposition 1.4.1, and §4.2.5. On the other hand, since \( w_k \geq 0 \) by \( \Gamma_v \geq 0 \) and \( w_{0k} \geq 0 \), using the above fundamental \( L^q-L^1 \) estimate, we obtain

\[
\int_{\mathbb{R}^2} \Gamma_v(x, t, y, s) w_{0k}(y) dy \leq \|w_k\|_\infty(t) \leq (\kappa(t-s))^{-1} m, \quad t > s > 0, \quad x \in \mathbb{R}^2.
\]
Here we used \( m = \int_{\mathbb{R}^2} w_0k(y)dy \). Therefore, by letting \( k \to \infty \), for \( t > 0 \) and \( x \in \mathbb{R}^2 \), we obtain
\[
0 \leq \Gamma_v(x, t, y_0, s) \leq (\kappa(t - s))^{-1}, \quad t > s > 0.
\]
By the continuity of \( \Gamma_v \) in \( s \in [0, t) \), for \( s \geq 0 \) we obtain inequality (i).

In order to prove (ii), we appeal to general results on elliptic and parabolic equations with discontinuous coefficients. This, however, exceeds the range of this book. For a proof the reader is referred to [Osada 1987] (see also [Giga Miyakawa Osada 1988]). There the structure of the Biot–Savart law and the Nash–Moser methods are effectively used to prove (ii).

To continue the proof of Theorem 2.5.2, instead of (ii) of the lemma, it is sufficient to prove that if
\[
\sup_{0 < t < T} t^{\frac{1}{2} + \frac{|\alpha|}{2}} \| \partial_x^\alpha v \|_\infty(t) \leq M_1, \quad |\alpha| \leq 1,
\]
then
\[
|\partial_y^\alpha \Gamma_v(x, t, y, 0)| \leq Ct^{-3/2}, \quad T > t > 0, \quad |\alpha| = 1, \quad x, y \in \mathbb{R}^2.
\]
(Here \( C \) is a constant depending only on \( M_1 \).) However, it is not known whether such an estimate is valid. On the other hand, one can prove the estimate
\[
|\partial_x^\alpha \Gamma_v(x, t, y, 0)| \leq Ct^{-3/2}, \quad T > t > 0, \quad |\alpha| = 1, \quad x, y \in \mathbb{R}^2,
\]
by similar arguments as in (i) of §2.4.2. In [Maekawa 2008b] under the assumption that \( \sup_{t > 0} t^{\frac{1}{2}} \| v \|_\infty(t) < \infty \) and \( \text{div } v(t) = 0 \) (but the special structure for the velocity \( v \) of \( v = K * \omega \) is not assumed there), the Hölder continuity in (ii) of the lemma is obtained by establishing pointwise Gaussian lower bounds for fundamental solutions.

Finally, we will prove that if \( |m| \) is sufficiently small, the weak solution satisfying (2.12a) is unique. As in the case of the heat equation, the uniqueness of the weak solution shows that the limit function \( \overline{\omega} \) agrees with \( mg \), which is the weak solution with initial value \( m\delta \). By this result, we can prove the asymptotic formula (2.10). As a first step to prove the uniqueness we see that \( \overline{\omega} \) satisfies the following integral equation.

### 2.5.3 Integral Equation Satisfied by Weak Solutions

**Proposition.** Assume that the pair of functions \((\overline{\omega}, \overline{u})\) is a weak solution of (2.7) and (2.8) with initial value \( m\delta \), where \( m \in \mathbb{R} \). Moreover, we assume that \( \overline{\omega} \) and \( \overline{u} \) are smooth on \( \mathbb{R}^2 \times (0, \infty) \) and \( \overline{\omega} \) satisfies (2.12a). Then \((\overline{\omega}, \overline{u})\) satisfies
\[
\overline{\omega}(t) = mG_t - \int_0^t \text{div} \left( e^{(t-s)A} \overline{u}(s) \overline{\omega}(s) \right) ds \quad \text{in } \mathbb{R}^2
\]
for \( t > 0 \).
Remark. If \( \overline{w} \) satisfies (2.12a), the velocity \( \overline{u} \) defined by \( \overline{u} = K \ast \overline{w} \) satisfies (2.12b) automatically. By Remark 6.3.5, \( \overline{u} \) is smooth on \( \mathbb{R}^2 \times (0, \infty) \), and satisfies \( \partial_t^\beta \partial_x^\beta (K \ast \overline{w}) = K \ast (\partial_t^\beta \partial_x^\beta \overline{w}) \). (Here \( b = 0, 1, 2, \ldots \), and \( \beta \) is a multi-index.) Therefore by using the Calderón–Zygmund inequality, the Hardy–Littlewood–Sobolev inequality, and the Gagliardo–Nirenberg inequality as in the proof of (ii) and (iii) of Theorem 2.4.1, we obtain (2.12b) for \( \overline{u} = K \ast \overline{w} \).

Proof. First we set \( h(s) = -\overline{u}(s) \overline{w}(s) \). Regarding \( h \) as a given function, we consider \( \overline{w} \) as a weak solution of \( \partial_t \overline{w} - \Delta \overline{w} = \text{div} \, h \) on \( \mathbb{R}^2 \times (0, \infty) \) with initial value \( m \delta \) (see Definition 4.3.4). We shall apply Theorem 4.4.3. Since \( \overline{u} \) and \( \overline{w} \) are smooth in \( \mathbb{R}^n \times (0, \infty) \), \( h \) is also smooth in \( \mathbb{R}^n \times (0, \infty) \). By the estimate for the derivatives of \( \overline{w} \) (2.12a) and the estimate for the derivatives of \( \overline{u} \) (2.12b), we obtain

\[
\sup_{\delta \leq t \leq T} \| \partial_x^\beta \partial_t^\beta h(t) \|_{\infty} < \infty, \quad 0 < \delta < T.
\]

Moreover, \( \sup_{t > 0} t^{1/2} \| \overline{w}(t) \|_{\infty} < \infty \) by (2.12b) and \( \sup_{t > 0} \| \overline{w}(t) \|_{1} < \infty \) by (2.12a) imply \( \sup_{t > 0} t^{1/2} \| h(t) \|_{1} < \infty \). Since \( \sup_{t > 0} \| \overline{w}(t) \|_{1} < \infty \), we can apply Theorem 4.4.3, and the integral equality in the proposition is proved.

Here we used the smoothness of the weak solution for \( t > 0 \) and estimate (2.12a). But for the proof of the proposition, instead of the smoothness for \( \overline{w} \) and (2.12a), it is sufficient to assume the local integrability of \( \overline{w} \) on \( \mathbb{R}^2 \times (0, \infty) \) and \( \sup_{t > 0} t^{-1-p} \| \overline{w}(t) \|_{p} < \infty \) for each \( p \) with \( 1 \leq p \leq \infty \).

This follows from the fact that a weak solution of this type is always smooth in \( t > 0 \) and satisfies (2.12a); see [Giga Miyakawa Osada 1988].

2.5.4 Uniqueness of Solutions of Limit Equations

Theorem. For a positive constant \( c_0 \), there exists a (small) positive number \( m_0 \) such that the following statement is satisfied. Let \( (\overline{w}, \overline{u}) \) be a weak solution of (2.7) and (2.8) with initial value \( m \delta \). Assume that \( \overline{w} \) and \( \overline{u} \) are smooth on \( \mathbb{R}^2 \times (0, \infty) \) and satisfy (2.12a) and

\[
\sup_{t > 0} t^{1/4} \| \overline{w}(t) \|_{4/3} \leq c_0 |m|.
\]

If \( |m| < m_0 \), then \( \overline{w} = mg \) on \( \mathbb{R}^2 \times (0, \infty) \). Here \( g(x, t) = G_t(x) \) denotes the Gauss kernel.

Proof. First we assume that for \( i = 1, 2 \), \( (\omega_i, u_i) \) are weak solutions of (2.7) and (2.8) with initial value \( m \delta \) such that \( \omega_i \) is smooth on \( \mathbb{R}^2 \times (0, \infty) \) and satisfies (2.12a). We will show that \( \omega_1 \equiv \omega_2 \) on \( \mathbb{R}^2 \times (0, \infty) \). By \( \S 2.5.3 \), for any \( t > 0 \),

\[
\omega_i(t) = mg_t - \int_0^t \text{div} \left( e^{(t-s)\Delta} (u_i \omega_i)(s) \right) ds, \quad i = 1, 2,
\]
in \( \mathbb{R}^2 \). Set \( w = \omega_1 - \omega_2, \ v = u_1 - u_2 \). Then \( w \) satisfies
\[
  w(t) = \int_0^t \text{div} \ (e^{(t-s)\Delta}h_2(s)) ds \quad \text{in} \ \mathbb{R}^2, \ \ t > 0,
\]
\[
  h_2 = -u_1 w - v \omega_2 \quad \text{in} \ \mathbb{R}^2 \times (0, \infty).
\]

Using this integral equation for \( w \), we estimate \( \|w\|_{4/3}(t) \). By the \( L^p-L^q \) estimate for derivatives of the heat equation (§1.1.3), for \( p \) and \( q \) with \( 1 \leq q \leq p \leq \infty \), we obtain
\[
  \|\text{div} \ (e^{(t-s)\Delta}h_2)\|_p \leq \frac{C_1}{(t-s)^{\frac{1}{2}+\frac{1}{4}-\frac{1}{p}}} \|h_2\|_{q}(s), \quad 0 < s < t.
\]

(Here and in the sequel, \( C_j, \ j = 1, 2, 3 \), are constants independent of \( s, t, \omega_i \), and \( u_i \) (\( i = 1, 2 \)).) On the other hand, for \( v = K * w \) and \( u_i = K * \omega_i \), using the Hardy–Littlewood–Sobolev inequality (§6.2.1), we obtain
\[
  \|v\|_{r} \leq L_1(r)\|w\|_{p_1}(t), \quad \|u_i\|_{r}(t) \leq L_1(r)\|\omega_i\|_{p_1}(t),
\]
\[
  1/r = 1/p_1 - 1/2, \quad 1 < p_1 < 2, \quad i = 1, 2,
\]
for \( t > 0 \). Here \( L_1 = L_1(r) \) is a constant depending only on \( r \). Let us take the \( L^{4/3} \)-norm of both sides of the integral equation for \( w \). Then using these inequalities and the H"older inequality, i.e., \( \|u_1 w\|_1 \leq \|u_1\|_4\|w\|_{4/3} \) and \( \|v \omega_2\|_1 \leq \|v\|_4\|\omega_2\|_{4/3} \), we have
\[
  \|w\|_{4/3}(t) \leq C_1L_1(4) \int_0^t \frac{1}{(t-s)^{3/4}} \{\|\omega_1\|_{4/3}(s) + \|\omega_2\|_{4/3}(s)\} \|w\|_{4/3}(s) ds
\]
for \( t > 0 \). If \( \omega_i \) (\( i = 1, 2 \)) satisfies \( \sup_{t>0} t^{1/4}\|\omega_i\|_{4/3}(t) \leq c_0|m| \), then we obtain
\[
  \|w\|_{4/3}(t) \leq C_1L_1(4) \int_0^t \frac{1}{(t-s)^{3/4}} \frac{2c_0|m|}{s^{1/4}} \|w\|_{4/3}(s) ds
\]
for \( t > 0 \). We set \( t = \tau \) and multiply both sides by \( \tau^{1/4} \). Then, by taking the supremum of both sides on \( (0, t) \) with respect to \( \tau \), we obtain
\[
  \sup_{0<\tau<t} \tau^{1/4}\|w\|_{4/3}(\tau)
\]
\[
  \leq 2C_1L_1(4)c_0|m| \left\{ \sup_{0<\tau<t} \int_0^{\tau} \frac{\tau^{1/4}}{(\tau-s)^{3/4}s^{1/2}} ds \right\} \cdot \left\{ \sup_{0<\tau<t} \tau^{1/4}\|w\|_{4/3}(\tau) \right\}.
\]

(By assumption, \( \sup_{0<\tau<t} \tau^{1/4}\|w\|_{4/3}(\tau) \) is always finite. Hence the left-hand side of the inequality is finite.) Since
\[
  \int_0^{\tau} \frac{\tau^{1/4}}{(\tau-s)^{3/4}s^{1/2}} ds = \int_0^{1} (1-\rho)^{-3/4}\rho^{-1/2} d\rho =: C_2
\]
is a positive constant independent of $\tau$, setting $C_3 = 2C_1L_1(4)c_0C_2$, we have

$$\sup_{0<\tau<t} \tau^{1/4}\|w\|_{4/3}(\tau) \leq C_3|m| \sup_{0<\tau<t} \tau^{1/4}\|w\|_{4/3}(\tau).$$

Let $m_0$ be a positive number such that $0 < m_0 < C_3^{-1}$. Then we have $1 > C_3|m|$ for $|m| < m_0$, which gives $\sup_{0<\tau<t} \tau^{1/4}\|w\|_{4/3}(\tau) = 0$. Therefore we obtain $\|w\|_{4/3}(t) \equiv 0$, $t > 0$, that is, $\omega_1$ is identically equal to $\omega_2$ on $\mathbb{R}^2 \times (0, \infty)$, and $u_1$ is identically equal to $u_2$.

On the other hand, $(mg, K \ast (mg))$ is a weak solution of (2.7) and (2.8) with initial value $m\delta$ that satisfies (2.12a). (See the answer to Exercise 7.2.) Hence by the above uniqueness result (by setting $\omega_1 = \varpi$ and $\omega_2 = mg$), we obtain $\varpi = mg$. □

The uniqueness without the assumption of the smallness of $|m|$ was open for years. The difficulty is that the convective term $(\bar{u}, \nabla)\varpi$ cannot be regarded as small with respect to the diffusion term $\Delta\varpi$. Recently, Gallay and Wayne gave an affirmative answer to this uniqueness problem in [Gallay Wayne 2005]. The key idea there is to introduce the relative entropy as a Lyapunov function. The details will be discussed in §2.8.

### 2.5.5 Completion of the Proof of the Asymptotic Formula

Finally, we summarize the proof of the asymptotic formula (2.10) by rescaling methods. First assume that the initial vorticity $\omega_0$ is in $C_0(\mathbb{R}^2)$, and $(\omega, u)$ is a solution of (2.7), (2.8), and (2.9). Let $\{(\omega_k, \bar{\omega}_k)\}_{k \geq 1}$ be the rescaled family. Then each subsequence $\{\omega_{k(\ell)}\}_{\ell=1}^{\infty}$ of $\{\omega_k\}_{k \geq 1}$ (under suitable choice of subsequence $\{\omega_{k(\ell)}\}_{\ell=1}^{\infty}$) converges uniformly on $\mathbb{R}^n \times [\eta, 1/\eta]$ for any $\eta \in (0, 1)$. The limit function $\varpi$, together with the velocity $\bar{\omega} = \bar{K} \ast \varpi$, should be a weak solution of (2.7) and (2.8) with initial value $m\delta$ that satisfies (2.12a) and (2.12b). (This result is obtained in the first part of §2.5 and §2.5.1.) Here $m = \int \omega_0 dx$. By §2.5.2 and §2.5.3, using the uniqueness of solutions §2.5.4, $\varpi = mg$ follows. (Here $g(x, t) = G_t(x)$ denotes the Gauss kernel.) Hence the limit $\varpi$ is independent of the choice of the subsequence of $\{\omega_k\}$. By Exercise 1.4, for any $\eta \in (0, 1)$, $\{\omega_k\}$ converges uniformly to $mg$ on $\mathbb{R}^n \times [\eta, 1/\eta]$ as $k \to \infty$. Hence we obtain $\lim_{k \to \infty} \|\omega_k - mg\|_{\infty}(1) = 0$, and by §1.2.6, we get (2.10). This completes the proof of the theorem of asymptotic behavior of vorticities (§2.2.2).

The assumption that the absolute value of the total circulation $m$ is small is used only in the proof of the uniqueness of the limit equation. The main advantage of the rescaling method is that no matter how large $||\omega_0||_1$ is, we can prove show the asymptotic formula (2.10) if $|m|$ is sufficiently small.

Moreover, since the smallness assumption on $|m|$ is actually unnecessary in Theorem 2.5.4 (see §2.8), the asymptotic formula (2.10) is still valid without the smallness assumption.
2.6 Formation of the Burgers Vortex

Let us apply the asymptotic formula (2.10) to a problem of fluid mechanics. We consider an incompressible viscous flow whose velocity field is expressed by a sum of

- an axially symmetric flow $U$ without vortices,
- a two-dimensional flow $V$ whose vortex vector is parallel to the symmetric axis of $U$.

Here and in the sequel, $y_1$, $y_2$, and $y_3$ denote spatial variables, $\tau$ denotes the time variable, and the $y_3$-axis is taken as the axis of the symmetry. Consider $U$ as

$$U(y_1, y_2, y_3) = (-\alpha y_1, -\alpha y_2, 2\alpha y_3), \quad \alpha \in \mathbb{R}.$$ 

See Figure 2.1. If $\alpha > 0$, the flow concentrates on the $y_3$-axis of symmetry, and diverges to (plus and minus) infinity of the $y_3$-axis. Obviously, it satisfies

$$\text{div } U = 0, \quad \text{curl } U = 0, \quad \Delta U = 0.$$
2.6 Formation of the Burgers Vortex

Since \((U, \nabla)U = -\nabla P\), if we set \(P(y_1, y_2, y_3) = -\frac{\alpha^2}{2}(y_1^2, y_2^2, 4y_3^2)\), it is clear that the pair of functions \((U, P)\) is a stationary solution (i.e., it is independent of time) of the Navier–Stokes equations

\[
\frac{\partial u}{\partial \tau} - \nu \Delta u + (u, \nabla)u + \nabla p = 0, \quad \text{div } u = 0 \tag{2.13}
\]

in \(\mathbb{R}^3 \times (0, \infty)\). In this section, we consider the Navier–Stokes equations with density \(\rho_0 = 1\) and viscosity \(\nu > 0\). We assume that the unknown function \(u = u(y_1, y_2, y_3, \tau)\) in (2.13) has the form

\[
u = U + V. \tag{2.14}
\]

Since \(V\) denotes the velocity vector that expresses the two-dimensional flow, it is given by

\[
V(y_1, y_2, \tau) = (V^1(y_1, y_2, \tau), V^2(y_1, y_2, \tau), 0),
\]

and its vorticity vector is expressed by

\[
(0, 0, \Omega(y_1, y_2, \tau)), \quad \Omega = \frac{\partial V^2}{\partial y_1} - \frac{\partial V^1}{\partial y_2}.
\]

We are concerned with the behavior of \(\Omega(y, \tau)\) as \(\tau \to \infty\) in the case of \(\alpha > 0\). (Here, we write \(y = (y_1, y_2)\), and so \(\Omega(y, \tau)\) denotes \(\Omega(y_1, y_2, \tau)\). We use similar notation for other functions.) In the case of \(\alpha = 0\), the pair of functions \((\Omega, V)\) with \(\nu = 1\) satisfies the two-dimensional vorticity equations (2.7), (2.8), and (2.9) (see §2.1). Hence by the asymptotic formula (2.10), (if \(|m|\) is sufficiently small), \(\Omega\) asymptotically behaves like \(mg\) as \(\tau \to \infty\). Here we set

\[
m = \int_{\mathbb{R}^2} \Omega_0(y)dy, \quad \Omega_0(y) = \Omega(y, 0).
\]

(V also expresses the two-dimensional vector field \((V^1, V^2)\).) If \(\nu\) is a general positive constant, using the scaling transformation \((\tilde{\tau} = \nu \tau, \Omega = \Omega/\nu, \tilde{V} = V/\nu)\), we obtain the asymptotic formula \(\Omega \sim mg^\nu (\tau \to \infty)\), by considering the two-dimensional vorticity equations. Here we set \(g^\nu(y, \tau) = \frac{1}{4\pi \nu \tau} e^{-|y|^2/4\nu \tau}\).

In the case of \(\alpha > 0\), set

\[
\Omega_m(y) = \frac{m}{\pi \ell^2} e^{-|y|^2/\ell^2}, \quad \ell = \left(\frac{2\nu}{\alpha}\right)^{1/2},
\]

for \(y \in \mathbb{R}^2\). The above \(\Omega_m\) is called the Burgers vortex. In the following, we discuss the convergence of \(\Omega\) to \(\Omega_m\) when \(\tau\) goes to infinity.

2.6.1 Convergence to the Burgers Vortex

First we derive the equation for \(V\) from (2.13) and (2.14). Let \(U\) and \(P\) be as stated in the first part of this section. Since \((U, \nabla)U = -\nabla P\), \((V, \nabla)U = -\alpha V\), \((U, \nabla)V = -\alpha (y, \nabla)V\), we obtain
\[ \frac{\partial V}{\partial \tau} - \nu \Delta V - \alpha(y, \nabla) V - \alpha V + (V, \nabla)V + \nabla(p - P) = 0, \ \text{div} \ V = 0. \quad (S) \]

Considering \( V \) as a two-dimensional vector field and applying curl to both sides, we obtain

\[ \frac{\partial \Omega}{\partial \tau} - \nu \Delta \Omega - \alpha(y, \nabla) \Omega - 2\alpha \Omega + (V, \nabla)\Omega = 0 \quad \text{in} \ \mathbb{R}^2 \times (0, \infty) \quad (R) \]

for \( \Omega = \Omega(y, \tau), \ V = V(y, \tau), \ y \in \mathbb{R}^2, \ \text{and} \ \tau > 0. \) Here we used

\[ \text{curl} ((y, \nabla)V) = (y, \nabla)\Omega + \Omega, \quad y \in \mathbb{R}^2. \]

As in the calculation that leads to the vorticity equations from the Navier–Stokes equations, we obtain (R) and the Biot–Savart law \( V = K * \Omega \) on \( \mathbb{R}^2 \times (0, \infty) \) from (S). Here and in the sequel, we assume \( \nu = 1. \) If we transform (R) with \( V = K * \Omega \) by

\[
\begin{cases}
x = e^{\alpha \tau} y, & t = \int_0^\tau e^{2\alpha \sigma} \, d\sigma, \\
\omega(x, t) = e^{-2\alpha \tau} \Omega(y, \tau), & u(x, t) = e^{-\alpha \tau} V(y, \tau),
\end{cases}
\]

then we obtain the equation for \( \Omega. \) The pair of functions \( (\omega, u) \) is a solution of the vorticity equations (2.7), (2.8), and (2.9) with initial value \( \omega_0 \) given by \( \omega_0(x) = \Omega_0(x), \ x \in \mathbb{R}^2. \)

Assume that \( \Omega_0 \in C_0(\mathbb{R}^2) \) and take \( m_0 \) as in §2.2.2. Then by the asymptotic formula

\[
\lim_{t \to \infty} t \|\omega - mg\|_\infty(t) = 0, \quad |m| < m_0,
\]

we obtain

\[
\lim_{\tau \to \infty} t(\tau)e^{-2\alpha \tau}\|\Omega - m\Omega_*\|_\infty(\tau) = 0, \quad t(\tau) = (e^{2\alpha \tau} - 1)/(2\alpha).
\]

Here \( \Omega_* \) is a function such that \( g(x, t) = e^{-2\alpha \tau}\Omega_*(y, \tau), \) and since \( t = t(\tau) \) and \( x = e^{\alpha \tau} y, \) we have

\[
\Omega_*(y, \tau) = \frac{1}{\pi \ell^2(1 - e^{-2\alpha \tau})} \exp \left(-\frac{|y|^2}{\ell^2(1 - e^{-2\alpha \tau})}\right).
\]

From \( \lim_{\tau \to \infty} t(\tau)e^{-2\alpha \tau} = \ell^2/4, \) we obtain

\[
\lim_{\tau \to \infty} \|\Omega - m\Omega_*\|_\infty(\tau) = 0.
\]

On the other hand, since

\[
\lim_{\tau \to \infty} \|m\Omega_* - \overline{\Omega}_m\|_\infty(\tau) = 0,
\]
we also obtain \( \lim_{\tau \to \infty} \| \Omega - \Omega_m \|_{\infty}(\tau) = 0 \). Note that \( \Omega_m \) is a stationary (\( \tau \)-independent) solution of (R). In the case that \( \nu \) is a general positive constant, as in the case of \( \alpha = 0 \), we can reduce (R) to the case of \( \nu = 1 \), and similar results hold. Since the total circulation \( \int_{\mathbb{R}^2} \omega(x,t)dx \) is conserved and is equal to \( \int_{\mathbb{R}^2} \omega_0(x)dx \) (independent of \( t \)) (Proposition 2.2.4), for \( \tau \geq 0 \), it follows that \( \int_{\mathbb{R}^2} \Omega(y,\tau)dy = m \). Summarizing the above arguments, we finally obtain the following theorem.

**Theorem.** Let \( \Omega_0 \in C_0(\mathbb{R}^2) \) and set \( m = \int_{\mathbb{R}^2} \Omega_0(y)dy \). Assume that the pair of functions \( (\Omega,V) \) satisfies (R) and the Biot-Savart law \( V = K \ast \Omega \) on \( \mathbb{R}^2 \times (0,\infty) \). Moreover, assume that (i), (ii), and (iii) in Theorem 2.2.1 are satisfied by \( \Omega, \Omega_0, \) and \( V \), instead of \( \omega, \omega_0, \) and \( u \), respectively. Then there exists a (small) positive constant \( m_0 \) (that is independent of \( \alpha > 0, \Omega_0, \) and \( \nu \)) such that if \( |m|/\nu < m_0 \) then

\[
\lim_{\tau \to \infty} \| \Omega - \Omega_m \|_{\infty}(\tau) = 0.
\]

Moreover, the total circulation at \( \tau \geq 0 \) is \( \int_{\mathbb{R}^2} \Omega(y,\tau)dy = m \).

**Remark.** Since (2.10) is still valid (see Remark 2.2.2) without the smallness assumption on \( |m| \), the assertion of Theorem 2.6.1 is still valid without assuming that \( |m|/\nu \) is small. Thus the Burgers vortex is stable under two-dimensional perturbations even if it is large. As for the stability under three-dimensional perturbations, the linear stability is observed numerically by [Schmid Rossi 2004]. Recently, it was rigorously proved by [Gallay Maekawa] that the Burgers vortex is locally stable under three-dimensional perturbations independent of the value of the total circulation.

### 2.6.2 Asymmetric Burgers Vortices

The Burgers vortex is a simple model of tubelike structures that are observed in concentrated vorticity fields, and it is considered to represent the balance between the stretching effect by the axisymmetric straining flow and the diffusion effect through the action of viscosity. In real flows or numerical observations, however, such vortex tubes are not purely axisymmetric and usually have an elliptic core region; see, for example, [Kida Ohkitani 1992]. To explain this phenomenon as proposed in [Robinson Saffman 1984], we instead of (2.14) postulate that the unknown velocity field is of the form

\[
u = U_\lambda + V
\]

with

\[
U_\lambda = \left( -\frac{1 + \lambda}{2} y_1, -\frac{1 - \lambda}{2} y_2, y_3 \right),
\]

where \( \lambda \geq 0 \) is a fixed parameter. Note that the case \( \lambda = 0 \) corresponds to (2.14). The equation for \( \Omega = \frac{\partial V^2}{\partial y_1} - \frac{\partial V^1}{\partial y_2} \) then becomes
A stationary solution for (R') with \( \lambda \neq 0 \) is called an asymmetric (or nonaxisymmetric) Burgers vortex. Different from the (axisymmetric) Burgers vortex \( \mathcal{Q}_m \), an explicit representation is no longer available for an asymmetric Burgers vortex. Several properties of nonaxisymmetric Burgers vortices are numerically studied in [Robinson Saffman 1984], [Moffatt Kida Ohkitani 1994], and [Prochazka Pullin 1998] by changing two parameters: the total circulation \( m \) and the asymmetry parameter \( \lambda \). The first work in mathematical analysis on this problem was done by [Gallay Wayne 2006], [Gallay Wayne 2007]. In [Gallay Wayne 2007] the existence of an asymmetric Burgers vortex is proved for all \( m \) when \( \lambda \in [0, 1/2) \) is sufficiently small. Moreover, it is shown that for these values of parameters the asymmetric Burgers vortex is locally stable under two-dimensional perturbations. In [Gallay Wayne 2006] the existence of an asymmetric Burgers vortex is proved when \(|m|\) is sufficiently small depending on \( \lambda \in [0, 1) \); moreover, its local stability is obtained under three-dimensional perturbations. The results in [Gallay Wayne 2006], [Gallay Wayne 2007] have been substantially extended by [Maekawa 2009a], [Maekawa 2009b]. In [Maekawa 2009a] the existence of an asymmetric Burgers vortex and its local stability under two-dimensional perturbations are obtained for sufficiently large \(|m|\) when \( \lambda \in [0, 1/2) \). In [Maekawa 2009b] it is proved that an asymmetric Burgers vortex exists for any \( m \) and \( \lambda \in [0, 1) \). There seem to be no mathematical results on (R') for \( \lambda \geq 1 \). In particular, it seems that there are no stationary solutions to (R') that decay at spatial infinity if \( \lambda \geq 1 \).

In [Robinson Saffman 1984], [Moffatt Kida Ohkitani 1994], and [Prochazka Pullin 1998] it is observed that the isovorticity contour of an asymmetric Burgers vortex becomes more circular as \(|m|\) is increasing. This mechanism is explained in [Moffatt Kida Ohkitani 1994] by deriving a formal asymptotic expansion at large \(|m|\). This asymptotic expansion is rigorously proved by [Gallay Wayne 2007] for sufficiently small \( \lambda \in [0, 1/2) \) by [Maekawa 2009a] for all \( \lambda \in [0, 1/2) \) and finally by [Maekawa 2009b] for all \( \lambda \in [0, 1) \).

### 2.7 Self-Similar Solutions of the Navier–Stokes Equations and Related Topics

In this section we present recent developments mainly on self-similar solutions of the Navier–Stokes equations. We start with a brief history on behavior of vorticities at time infinity. Next we introduce the existence problem of solutions to the Navier–Stokes equations or the vorticity equations. Finally, we present the mentioned results on self-similar solutions to the Navier–Stokes equations.
2.7 Self-Similar Solutions of the Navier–Stokes Equations and Related Topics

2.7.1 Short History of Research on Asymptotic Behavior of Vorticity

The asymptotic formula (2.10) shows that the solution of the two-dimensional vorticity equations asymptotically converges to the rotationally symmetric self-similar solution at time infinity. This formula was first obtained by [Giga Kambe 1988]. In this paper the authors directly estimated $\omega - mg$ using the integral equation. But their argument required the smallness assumption of $\|\omega_0\|_1$. This result was improved using rescaling arguments in [Carpio 1994]. The advantage of this rescaling method is that we can relax the condition of the smallness of $\|\omega_0\|_1$ to the smallness of $|m|$ as we have seen in §2.2.2. Also in [Carpio 1994] the estimates as in §2.4.1 play essential roles, but the author used slightly weaker estimates there. For example, instead of §2.4.1(i), it is used that

$$\|\omega\|_q(t) \leq \frac{C}{t^{1-q/2}}\|\omega_0\|_1,$$

where the constant $C$ depends on $\|\omega_0\|_1$ nonincreasingly. This estimate is obtained in [Giga Miyakawa Osada 1988] by rewriting the Biot–Savart law to apply the results of [Osada 1987]. Later, the fundamental decay estimate in §2.3.1 was obtained by [Kato 1994, Ben-Artzi 1994]. In these works the estimates of §2.4.1 are also established. The contents of §2.3, except for a slight improvement in §2.3.6, is based on [Kato 1994]. (Another proof in §2.3.5 is due to [Ben-Artzi 1994].) The idea of the proof is based on the Nash–Moser method, which estimates fundamental solutions (in the case of the heat equation it is the Gauss kernel) of the second-order linear parabolic equations of divergence form (generalization of the heat equation) under an assumption that allows for singular coefficients in the equations. The fundamental work on this problem was done by [Nash 1958]. There are many references to the Nash–Moser method (see references below in this section). Here we only refer to the nice paper [Fabes Stroock 1986], since it is rather easy to read.

The method to obtain estimates of the velocity from the vorticity as in §2.4.1 is established in [Giga Miyakawa Osada 1988]. (Another proof in §2.4.1(ii) is given by [Ben-Artzi 1994].) The estimates of derivatives of vorticities §2.4.2 were obtained by [Kato 1994] in the case of $1 < p < \infty$. In this book we have proved them in a more elementary way, which covers the case $p = 1$ and $p = \infty$. In §2.4.3 we have proved the decay estimate at spatial infinity. We can also derive the same result using the pointwise estimate of fundamental solutions established in [Osada 1987] for general parabolic PDEs including (2.7). The method used in this book is more elementary, although the class of equations to which we can apply this argument will be restricted. Combining this with the estimate in [Kato 1994], we can improve the results in [Osada 1987]. For details, readers should refer to [Matsui Tokuno 1997]. In [Maekawa 2008a], spatial decay estimates for derivatives of solutions are obtained, which lead to the asymptotic behavior of derivatives of solutions by the above rescaling arguments.
In [Carpio 1994] there is no statement on the estimate of the limit function by the value $|m|$ as in §2.5.2, while one needs this estimate to prove the uniqueness of solutions to the limit equation. For this reason we have to use general results on parabolic PDEs by [Osada 1987] (§2.5.2). Since the details are complicated and beyond the scope of this book, we have omitted them. As mentioned at the end of §2.5.2, results in [Maekawa 2008b] simplify the proof of Lemma 2.5.2(ii) without assuming the special relation $v = K \ast \omega$. The pointwise estimates for fundamental solutions by Gaussian-like functions from above and below are called the Aronson estimates. These estimates were at first obtained by [Aronson 1968] for second order parabolic PDEs of divergence form (but without transport terms). It is known that the Aronson estimates lead to the Hölder continuity of fundamental solutions; for example, see [Fabes Stroock 1986]. As for the equations $(H_v)$, [Carlen Loss 1996] and [Matsui Tokuno 1997] obtained the pointwise Gaussian upper bounds for fundamental solutions, and in [Maekawa 2008b] the pointwise Gaussian lower bounds (and thus the Hölder continuity) are also established. For the Aronson estimates and the Hölder continuity of fundamental solutions to more general parabolic PDEs the reader is referred to [Fabes 1992], [Liskevich Samenov 2000], [Liskevich Zhang 2004], [Zhang 2004], [Zhang 2006], [Samenov 2006].

The uniqueness of solutions to the limit equations is essentially included in [Giga Miyakawa Osada 1988]. In [Giga Miyakawa Osada 1988] the time global solution of (2.7), (2.8), and (2.9) is constructed when the initial data is a general finite Radon measure. The uniqueness is proved in [Giga Miyakawa Osada 1988] under the assumption that the initial data is sufficiently regular in the sense that the singularity of Dirac delta type is sufficiently small. This result was slightly improved by [Kato 1994], but for a long time it remained an open problem whether the uniqueness of weak solutions holds when the total mass $|m|$ is large, even if the initial value is just $m \delta$. Recently the uniqueness of solutions with initial data as $m \delta$ was affirmatively proved in [Gallay Wayne 2005] and then in [Gallagher Gallay Lions 2005]. In [Gallay Wayne 2005] they introduced the relative entropy and showed that it is a Lyapunov function for the “flow” of the solution $\{\omega(t)\}_{t \geq 1}$, which leads to a characterization of solutions with initial data as $m \delta$. The details will be discussed in §2.8. In [Gallagher Gallay Lions 2005] another proof using the radial rearrangement of the vorticity is given. Using the results of [Gallay Wayne 2005], the uniqueness of solutions with general finite Radon measures as initial data was also proved by [Gallagher Gallay 2005].

In this chapter we have established the asymptotic formula (2.10) as elementarily as possible using scaling transformations. Readers can see how useful the detailed analysis of the linear equation $(H_v)$ is for the study of the nonlinear equation (2.7). Throughout the chapter the initial data $\omega_0$ has been assumed to be a continuous function with compact support. But this is just for simplicity and we can take the initial data in larger classes of functions. For example, if we define solutions appropriately, for any initial data
\( \omega_0 \) belonging to \( L^1(\mathbb{R}^2) \) we can prove the asymptotic formula (2.10) in §2.2.2 under the assumption \(|m| < m_0\) (see [Carpio 1994]). If we use the argument by [Gallay Wayne 2005], then this smallness assumption \(|m| < m_0\) is again removed.

The convergence to the Burgers vortex in §2.6 is obtained in [Kambe 1984], for rotationally symmetric \( \Omega \). In the case of the rotationally symmetric vortex, the result is reduced to the case of the heat equation (§2.2.5); hence, in order to obtain the desired convergence, the results in §1.1.4 are sufficient. Of course, we do not need to assume the smallness of \(|m|\). By the proof using the expression of solutions (§1.1.5), we have \( t\|u - mg\|_\infty(t) \leq C' t^{-1/2}, \; t > 0 \). In fact, we can show not only that \( \Omega \rightarrow \Omega_m (\tau \rightarrow \infty) \), but more precisely,

\[
\|\Omega - \Omega_m\|_\infty(\tau) = O(e^{-\alpha \tau}) \quad (\tau \rightarrow \infty).
\]

In the nonrotationally symmetric case, by [Giga Kambe 1988],

\[
\|\Omega - \Omega_m\|_\infty(\tau) = O(e^{-\alpha \sigma \tau}) \quad (\tau \rightarrow \infty)
\]

is proved for \( 0 \leq \sigma < 1 \) under the assumption that \( \|\Omega\|_1 \) is sufficiently small. This shows that \( \Omega \) converges exponentially to \( \Omega_m \). In [Gallay Wayne 2005] this exponential convergence of \( \Omega \) to \( \Omega_m \) is also verified for any \( m \) without a smallness assumption.

The transformation (2.7) from (R) is due to [Lundgren 1982, Kambe 1983]. But note that mathematically it is considered a transformation by similarity variables as stated in §2.7.3.

### 2.7.2 Problems of Existence of Solutions

The first mathematical approach to the initial value problem of the Navier–Stokes equations for general initial data was developed by [Leray 1933, Leray 1934a, Leray 1934b]. In [Leray 1934b], it is proved that in \( \mathbb{R}^3 \) if the \( L^2 \)-norm of the initial velocity is finite (in other words, if the initial kinetic energy is finite), then there exists a time-global weak solution of the Navier–Stokes equations. It is already known that if the initial velocity is sufficiently small in some sense or the spatial dimension is two, then the weak solution is smooth and unique. However, if the spatial dimension is three, for general initial data the uniqueness and smoothness of weak solutions are still open. For solvability problems including this famous open problem the reader is referred to the fundamental books [Ladyzhenskaya 1969, Temam 1977, Galdi 1994, Lions 1996], [Sohr 2001], [Chemin et al 2006] and the articles [Masuda 1985], [Yamazaki 1999], [Kozono 2002], [Camone 2004], [Hishida 2008]. Note that analysis of the Navier–Stokes equations covers a very broad field with generalizations in many different directions. As variants here we just point out recent results on the Navier–Stokes equations with more general boundary conditions or in the time-dependent domain [Saal 2006], [Saal 2007a], [Saal 2007b]. In the 1960s Kato and Fujita considered a good successive approximation method
to construct time local smooth solutions, which became a big milestone for solving the initial value problem of nonlinear partial differential equations; [Kato Fujita 1962, Kato 1996].

For the initial value problem of the two-dimensional vorticity equations, even if the initial vorticity is a continuous function with compact support, the $L^2$-norm of the initial velocity is not always finite. Hence we cannot directly obtain the existence of time global solutions of the vorticity equations from Leray’s results. In general, we have at least three methods to prove the existence of time global solutions of evolution equations:

(i) Extend time local solutions globally in time.
(ii) Approximate the problem by a problem for which the existence of time global solutions is easily obtained.
(iii) Use a fixed-point theorem.

Each method requires an a priori estimate, that is, we have to estimate how large a solution can be if it exists.

Fortunately, in the case of two-dimensional vorticity equations, we can uniquely construct a time global smooth solution $\omega$ by a successive approximation and the method (i). Indeed, the maximal existence time $T$ of the time local solution is estimated by the $L^p$-norm ($1 < p < 2$) of the initial vorticity $\omega_0$ as $T\frac{1}{p-1} \leq C\|\omega_0\|_p$, where the constant $C$ is independent of $\omega_0$. Hence, if $\|\omega\|_p(t_0)$ is bounded (for example bounded by $M$) for a solution $\omega$, then there exists a constant $T_M$, which depends only on $M$, such that the solution can be extended to the time $t_0 + T_M$. If the estimates as in §2.4.1 are valid for a time local solution, we can repeat this procedure, and the solution can be extended to any time as a smooth solution. In fact, in [Giga Miyakawa Osada 1988] a time global solution is constructed by this argument.

Finally, we consider the smoothness of weak solutions that is mentioned in the last part of §2.5.3. If $\|\omega\|_p(t_0) < \infty$ for each $p$ with $1 \leq p \leq \infty$, then there exists a smooth time global solution with the initial data $\omega(t_0)$ at initial time $t_0$ that satisfies (i), (ii), and (iii) in §2.2.1. If we can show that this solution and the weak solution coincide (namely, if we have the uniqueness of weak solutions), then the weak solution is smooth and satisfies (for example) (i), (ii), and (iii) in §2.2.1. Thus it suffices to prove the uniqueness of weak solutions with $L^p$-initial data. For $L^p$-initial data, different from the case that the initial data is the $\delta$ measure, we have in fact a good estimate for weak solutions near the initial time, which yields the uniqueness of weak solutions without smallness assumptions for initial data. In [Leray 1934b] a similar method as above is used to estimate the set of the time at which the weak solution in $\mathbb{R}^3$ can be singular.

In the next section we consider self-similar solutions of the Navier–Stokes equations.
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2.7.3 Self-Similar Solutions

As we have seen in this chapter, forward self-similar solutions play an important role in the asymptotic behavior of solutions. We can consider self-similar solutions to the Navier–Stokes equations as in the case of the vorticity equations. Assume that a pair of smooth functions \((u, p)\) satisfies the Navier–Stokes equations

\[
\partial_t u - \Delta u + (u, \nabla)u + \nabla p = 0, \quad \text{div } u = 0.
\]

in \(\mathbb{R}^n \times (0, \infty)\) \((n \geq 2)\). Here \(u = (u^1, \ldots, u^n)\) is an \(\mathbb{R}^n\)-valued function. Then for each \(\lambda > 0\) the pair of functions \((u(\lambda), p(\lambda))\) rescaled by

\[
u(\lambda x, \lambda^2 t), \quad p(\lambda x, \lambda^2 t), \quad x \in \mathbb{R}^n, \quad t > 0,
\]

is also a solution of the Navier–Stokes equations. In general, if a pair of functions \((u, p)\) in \(\mathbb{R}^n \times (0, \infty)\) (which is not necessarily a solution of the Navier–Stokes equations) satisfies

\[
u(\lambda x, \lambda^2 t) = v(x, t), \quad p(\lambda x, \lambda^2 t) = p(x, t), \quad x \in \mathbb{R}^n, \quad t > 0, \quad \lambda > 0,
\]

then the pair \((u, p)\) is called **forwardly self-similar**, and if \((u, p)\) is a solution of the Navier–Stokes equations, it is called a **forward self-similar solution**. For example, let \(g(x, t) = G_t(x)\) be the Gauss kernel and consider the associated velocity field \(u = K \ast g\). Then if we set the pressure field \(p\) as \(p = E \ast \sum_{i,j=1}^2 \partial_x_i \partial_x_j (u_i u_j)\), then \((u, p)\) is a forward self-similar solution of the Navier–Stokes equations.

In general, if \((u, p)\) is a self-similar solution, then by setting \(\lambda = 1/\sqrt{t}\), it is expressed as

\[
u(x, t) = \frac{1}{\sqrt{t}} u \left( \frac{x}{\sqrt{t}}, 1 \right), \quad p(x, t) = \frac{1}{t} p \left( \frac{x}{\sqrt{t}}, 1 \right).
\]

Hence, \((u, p)\) is forwardly self-similar if and only if it can be written in the form

\[
u(x, t) = \frac{1}{\sqrt{t}} U \left( \frac{x}{\sqrt{t}} \right), \quad p(x, t) = \frac{1}{t} P \left( \frac{x}{\sqrt{t}} \right),
\]

using a pair of functions \((U, P)\) on \(\mathbb{R}^n\) (where \(U\) is an \(\mathbb{R}^n\)-valued function). Thus it will be useful to consider the equations for \((U, P)\) instead of \((u, p)\). To derive the equations for \((U, P)\), first we transform the dependent variables as \(\hat{u}(x, t) = \sqrt{t} u(x, t), \hat{p}(x, t) = t p(x, t)\), and next transform the independent variables as \(y = x/\sqrt{t}\), and set \(\hat{u}(y, t) = \hat{u}(\sqrt{t} y, t), \hat{p}(y, t) = \hat{p}(\sqrt{t} y, t)\). Then we easily see that \((u, p)\) is forwardly self-similar if and only if \((\hat{u}, \hat{p})\) is independent of \(t > 0\), that is, if it depends only on \(y \in \mathbb{R}^n\). Now let us derive the equation that \((\hat{u}, \hat{p})\) satisfies when \((u, p)\) is a solution of the Navier–Stokes equations in \(\mathbb{R}^n \times (0, \infty)\). First, by the equalities
Note that \((u, w)\) as the velocity field in \(R^2\). We sometimes call new variables \((y, s, w, q)\) with respect to the rescaling \((2.16)\).

Let us rewrite the above transformation

\[
\begin{align*}
\partial_t \tilde{u}(y, t) &= \frac{1}{2\sqrt{t}} u(\sqrt{t}y, t) + \frac{1}{2} \sum_{i=1}^{n} y_i (\partial_x u)(\sqrt{t}y, t) + \sqrt{t} \partial_t u(x, t), \\
\partial_y \tilde{u}(y, t) &= t(\partial_x u)(\sqrt{t}y, t), \quad \Delta \tilde{u}(y, t) = t^{3/2} \Delta u(\sqrt{t}y, t), \\
\partial_y \tilde{p}(y, t) &= t^{3/2} (\partial_x p)(\sqrt{t}y, t),
\end{align*}
\]

if \((u, p)\) satisfies the Navier–Stokes equations in \(\mathbb{R}^n \times (0, \infty)\), then \((\tilde{u}, \tilde{p})\) satisfies

\[
t \partial_t \tilde{u} - \Delta \tilde{u} - \frac{1}{2} (y, \nabla) \tilde{u} - \frac{1}{2} \tilde{u} + (\tilde{u}, \nabla) \tilde{u} + \nabla \tilde{p} = 0, \quad \text{div} \tilde{u} = 0, \quad t > 0, \quad y \in \mathbb{R}^n.
\]

Moreover, since \(t \partial_t = \partial_s\), if we transform as \(s = \log t\) and set \(w(y, s) = \tilde{u}(y, e^s)\), \(q(y, s) = \tilde{p}(y, e^s)\), then \((w, q)\) satisfies

\[
\partial_s w - \Delta w - \frac{1}{2} (y, \nabla) w - \frac{1}{2} w + (w, \nabla) w + \nabla q = 0, \quad \text{div} w = 0, \quad s \in \mathbb{R}, \quad y \in \mathbb{R}^n \quad (S')
\]

(conversely, if \((w, q)\) satisfies \((S')\), then \((u, p)\) satisfies the Navier–Stokes equations in \(\mathbb{R}^n \times (0, \infty)\), which can be seen by performing the above calculation inversely). We sometimes call new variables \((y, s, w, q)\) similarity variables with respect to the rescaling \((2.16)\).

Let us rewrite the above transformation

\[
s = \log t, \quad y = x/\sqrt{t}, \quad w(y, s) = \sqrt{t} u(x, t), \quad q(y, s) = t p(x, t).
\]

Note that \((w, q)\) is related to \((u, p)\) as \(w(y, s) = e^{s/2} u(y e^{s/2}, e^s)\), \(q(y, s) = e^s p(y e^{s/2}, e^s)\).

The equation \((S')\) is nothing but the equation \((S)\) in §2.6.1 with \(n = 2\), \(\alpha = 1/2\), and \(\nu = 1\) under a suitable choice of \(p\). The transformation from \((S)\) to the Navier–Stokes equations (the vorticity equations) is essentially the same as the transformation by the above similarity variables.

We have now established the equations for \((U, P)\). Indeed, since \((U, P)\) is independent of \(s\) in the similarity variables, \((u, p)\) is a forward self-similar solution if and only if \(U = U(y)\) and \(P = P(y)\) satisfy

\[
- \Delta U - \frac{1}{2} (y, \nabla) U - \frac{1}{2} U + (U, \nabla) U + \nabla P = 0, \quad \text{div} U = 0, \quad y \in \mathbb{R}^n, \quad (E)
\]

in \(\mathbb{R}^n\). This equation is just the one that stationary solutions of \((S')\) satisfy.

Are there any forward self-similar solutions except for \(u = K * g\)? In fact, many special solutions are already known. We refer to [Okamoto 1997] for the construction of special solutions and their properties including backward self-similar solutions (we will mention backward self-similar solutions later).

Let us consider the forward self-similar solution \(u = K * g\). If it is regarded as the velocity field in \(\mathbb{R}^3\), then its initial vorticity concentrates on an axis through the origin and is zero outside the axis. Can we construct a self-similar solution whose initial vorticity concentrates on half-lines through the origin,
and is zero outside of them? This problem is studied in [Giga Miyakawa 1989], where small self-similar solutions are constructed by analyzing the vorticity equations directly instead of equation (E).

In [Carpio 1994], it is proved that if the initial vorticity is small, then the solution asymptotically converges to one of the above self-similar solutions.

The initial velocity $u_0$ of a self-similar solution is a function homogeneous of degree $-1$, i.e., $u_0(\lambda x) = u_0(x)$ ($\lambda > 0, x \in \mathbb{R}^n$), which is easily seen if the initial vorticity is $g$. The $L^p$-norm of such a function is not finite except when it is identically zero. For example, $1/|x|$ does not satisfy $\| |x|^{-1} \|_p < \infty$ for any $p$ ($1 \leq p \leq \infty$). So we cannot use classical existence theorems of solutions in $L^p$ spaces, and we need to introduce alternative function spaces that include these homogeneous functions. This is the reason that Morrey spaces are used in the analysis of the Navier–Stokes equations in [Giga Miyakawa 1989]. After this work, the Navier–Stokes equations in Morrey spaces were studied also by [Kato 1994] and [Taylor 1992], and completed by [Kozono Yamazaki 1994]. In [Kozono Yamazaki 1995], relations with self-similar solutions are also considered. Because of the important applications to forward self-similar solutions, the Navier–Stokes equations have been studied in several function spaces that include functions homogeneous of degree $-1$ (other than the Morrey spaces). In [Cannone Meyer Planchon 1994], [Cannone 1995, Cannone 1997], and [Cannone Planchon 1996], Besov spaces are used to construct forward self-similar solutions. In [Meyer 1999] forward self-similar solutions are obtained in Lorentz spaces, and in [LeJan Sznitman 1997] the Navier–Stokes equations are studied by probabilistic arguments in pseudomeasure spaces that include self-similar solutions. A simpler proof of the construction of forward self-similar solutions in pseudomeasure spaces is given in [Cannone Planchon 2000], where harmonic analysis plays an essential role.

There are many studies concerning decay properties of solutions to the Navier–Stokes equations at time or space infinity. Here we refer only to [Miyakawa 1996, Miyakawa 1997, Miyakawa 1998].

Next we consider backward self-similar solutions. Let $u^{(\lambda)}$ and $p^{(\lambda)}$ be rescaled functions of $u$ and $p$ as in (2.16). But in this case we assume that $u$ and $p$ are defined in $\mathbb{R}^n \times (-\infty, 0)$. If

$$u^{(\lambda)}(x, t) = u(x, t), \quad p^{(\lambda)}(x, t) = p(x, t), \quad x \in \mathbb{R}^n, \quad t < 0, \quad \lambda > 0,$$

holds, then $(u, p)$ is called backwardly self-similar. Moreover, if $(u, p)$ is a solution of the Navier–Stokes equations, it is called a backward self-similar solution. As with forward self-similar solutions, if $(u, p)$ is backwardly self-similar, then we can write

$$u(x, t) = \frac{1}{\sqrt{-t}} U \left( \frac{x}{\sqrt{-t}} \right), \quad p(x, t) = \frac{1}{-t} P \left( \frac{x}{\sqrt{-t}} \right).$$

By rewriting the Navier–Stokes equations in the similarity variables

$$y = x/\sqrt{-t}, \quad w(y, s) = \sqrt{-t}u(x, t), \quad q(y, s) = tp(x, t), \quad s = -\log(-t),$$
we see that the functions \( w = w(y, s) \) \((= e^{-s/2} u(y e^{-s/2}, e^{-s}))\) and \( q = q(y, s) \) \( (= e^{-s} q(y e^{-s/2}, e^{-s}))\) satisfy, instead of \((S')\),
\[
\frac{\partial w}{\partial s} - \Delta w + \frac{1}{2}(y, \nabla) w + \frac{1}{2} w + (w, \nabla) w + \nabla q = 0, \quad \text{div } w = 0, \quad s \in \mathbb{R}, \quad y \in \mathbb{R}^n.
\]
This is just the case of \( \alpha = -\frac{1}{2} \) and \( \nu = 1 \) in the equation \((S)\) in §2.6.1. The pair \((u, p)\) is a backward self-similar solution if and only if \((U, P)\) satisfy
\[
-\Delta U + \frac{1}{2}(y, \nabla) U + \frac{1}{2} U + (U, \nabla) U + \nabla p = 0, \quad \text{div } U = 0, \quad y \in \mathbb{R}^n. \quad (L)
\]
This equation is called Leray’s equation, since in [Leray 1934b] the authors suggested the idea of proving the existence of a solution \((u, p)\) that diverges to infinity in finite time by constructing a backward self-similar solution. Let \((U, P)\) be a smooth solution of \((L)\) with \(U(0) \neq 0\). For \(T > 0\), set
\[
 u(x, t) = \frac{1}{\sqrt{T-t}} U\left(\frac{x}{\sqrt{T-t}}\right), \quad p(x, t) = \frac{1}{T-t} P\left(\frac{x}{\sqrt{T-t}}\right).
\]
Then \((u, p)\) is a solution of the Navier–Stokes equations in the interval \((0, T)\), but \(u(0, t)\) diverges to infinity as \(t \to T\) (this is called “blowup” at time \(T\)). Usually weak solutions are constructed under the assumption that the initial data \(u_0 = u(x, 0)\) has the finite energy \(\|u_0\|_2 < \infty\) and that they satisfy the energy inequality
\[
\|u\|_2^2(t) + 2 \int_0^t \|\nabla u\|_2^2(s) ds \leq \|u_0\|_2^2, \quad t > 0.
\]
Are there any self-similar solutions satisfying the energy inequality? If such solutions exist, we can construct a weak solution that loses regularities in finite time. In the case of \(n = 2\) every weak solution satisfying the energy inequality is shown to be smooth for all time, so the blowup does not occur. Hence there is no solution \((U, P)\) of \((L)\) with the above properties (we can prove this directly by multiplying both sides of \((L)\) by \(U\) and performing integration by parts). When \(n = 3\), by the energy inequality we have \(\|U\|_2 < \infty\) and \(\|\nabla U\|_2 < \infty\). Then by the Sobolev inequality we obtain \(\|U\|_6 < \infty\) (moreover, the Hölder inequality yields \(\|U\|_3 < \infty\)). The problem is whether there exists \((U, P)\) satisfying \((L)\). For this problem, it is proved in [Necas Růžička Šverák 1996] that any weak solution of \((L)\) with \(U \in L^3 \cap W^{1,2}_{\text{loc}}\) must be identically zero. Later in [Málek Nečas Pokorný Schonbek 1999] it is shown by another approach that any weak solution of \((L)\) belonging to \(W^{1,2}\) is a trivial function. This is extended by [Miller O’Leary Schonbek 2001], in which the nonexistence of pseudo (backward) self-similar solutions is obtained. Although backward self-similar solutions discussed in the above papers (if they exist) are assumed to decay at spatial infinity, [Tsai 1998] relaxed this condition and proved that any
solution of (L) belonging to $W^{1,2}_{\text{loc}}$ is a constant function. The existence of backward self-similar solutions is discussed also for other equations related to the Navier–Stokes equations. For example, in [Guo Jiang 2006] it is proved that there are no backward self-similar solutions to the isothermal compressible Navier–Stokes equations. Moreover, [Chae 2007a] showed the nonexistence of self-similar blowing-up solutions to the three-dimensional Euler equations. Related to these results, recently [Chae 2007b] showed that asymptotically self-similar blowup does not occur for solutions to the Navier–Stokes equations or the Euler equations. See also [Chae preprint] for a simplified proof. These results are extended to cover equations in magnetohydrodynamics in [Chae 2008], [Chae 2009].

Hence Leray’s idea of using backward self-similar solutions does not give the construction of blowup solutions. But it does not mean the nonexistence of blowup solutions. As for relations between the smoothness of solutions of the Navier–Stokes equations and backward self-similar solutions, we refer to [Kozono 1997], [Kozono Sohr 1996], [Kozono 1998], [Escauriaza Seregin Sverak 2003]. We also refer to [Cannone 2004], in which several topics related to the Navier–Stokes equations (including the topic of self-similar solutions) are discussed using tools of harmonic analysis.

In Section 3 we will see that backward self-similar solutions are deeply connected with blowup phenomena in some nonlinear partial differential equations.

### 2.8 Uniqueness of the Limit Equation for Large Circulation

In this section, we shall prove that a weak solution of (2.7)–(2.8) with initial data $m\delta$ is unique, provided that the vorticity $\omega$ satisfies the Gaussian estimate

$$
\frac{C_1}{t} e^{-|x|^2/C_2 t} \leq \omega(x, t) \leq \frac{C'_1}{t} e^{-|x|^2/C'_2 t}, \quad x \in \mathbb{R}^2, \quad t > 0,
$$

with some positive constants $C_1, C_2, C'_1, C'_2$ independent of $x, t$. As in §2.4 this estimate yields (2.12a), (2.12b). Our main statement in this section is summarized as follows.

### 2.8.1 Uniqueness of Weak Solutions

**Theorem.** Let the pair $(\omega, u)$ be a weak solution of (2.7)–(2.8) with initial data $m\delta$, where $m > 0$. Assume that $\omega$ and $u$ are smooth in $\mathbb{R}^2 \times (0, \infty)$ and satisfy (2.16) (so that (2.12a) and (2.12b) hold). Then $\omega = mg$.

As proved by H. Osada [Osada 1987] (see also [Giga Miyakawa Osada 1988]), $\Gamma_\nu(x, t, 0, 0)$ in Lemma 2.5.2 satisfies the Gaussian estimate (2.16) with constants depending only on $M_1 = \sup_{0 < t < \infty} \|\omega\|_1(t)$ if $v = K \ast \omega$. This
estimate is inherited by the rescaled limit \((\bar{\omega}, \bar{u})\) of a subsequence of \(\{(\omega_k, u_k)\}\) as \(k \to \infty\), so \(\bar{\omega}\) satisfies (2.16). By the above uniqueness theorem one is able to conclude that \(\bar{\omega} = mg\) without assuming that \(|m|\) is small. We argue in the same way as in §2.5.5 and obtain (2.10) without assuming that \(|m|\) is small.

We shall prove this theorem in the rest of this section.

2.8.2 Relative Entropy

The main idea of the proof is to use a relative entropy function with respect to \(g\) for (2.7)–(2.8) of the form

\[
H(g, \omega)(t) = \int_{\mathbb{R}^2} \omega(x, t) \log \left( \frac{\omega(x, t)}{g(x, t)} \right) \, dx.
\]

This quantity is monotone decreasing in time if \((\omega, u)\) is a solution of (2.7)–(2.8).

**Theorem.** Let the pair \((\omega, u)\) be a smooth solution of (2.7)–(2.8) in \(\mathbb{R}^2 \times I\), where \(I\) is an open interval in \((0, \infty)\). Assume that there exist positive constants \(C_1, C_2, C'_1, C'_2\) that satisfy

\[
C_1 e^{-|x|^2/C_2} \leq \omega(x, t) \leq C'_1 e^{-|x|^2/C'_2} \quad \text{for} \quad t \in I, \quad x \in \mathbb{R}^2. \tag{2.17}
\]

Then

\[
\frac{d}{dt} H(g, \omega)(t) = -\int_{\mathbb{R}^2} \left| \nabla \left( \frac{\omega}{g} \right) \right|^2 \frac{g^2}{\omega} \, dx, \quad t \in I. \tag{2.18}
\]

If \(H(g, \omega)(t)\) is a constant on \(I\), then \(\omega = \hat{m}g\) in \(\mathbb{R}^2 \times I\) with some constant \(\hat{m} > 0\).

**Proof.** As in §2.4.2, from the estimate (2.17) it follows that

\[
\|\partial^b_\beta \partial_x^\omega\|_p(t), \quad \|\partial^b_\beta \partial_x^u\|_q(t)
\]

are bounded on any compact time interval of \(I\) for all \(p \in [1, \infty] , q \in (2, \infty)\), \(b = 0, 1, 2, \ldots\), and all multi-indices \(\beta\). These bounds and (2.17) justify all calculations below. For example, (2.17) guarantees that \(H(g, \omega)(t)\) is a well-defined convergent quantity for \(t \in I\).

We differentiate under the integral sign to observe that

\[
\frac{d}{dt} H(g, \omega)(t) = \int \partial_t \omega \log \left( \frac{\omega}{g} \right) \, dx - \int \omega \frac{\partial_t g}{g} \, dx + \int \partial_t \omega \, dx;
\]

all integration in this proof is over the whole plane, so we suppress the region of integration. We use (2.7) and \(\partial_t g = \Delta g\) and observe that the last term vanishes by integration by parts, so that
\[
\frac{d}{dt} H(g, \omega)(t) = \int \left( \Delta \omega \log \left( \frac{\omega}{g} \right) - \frac{\omega}{g} \Delta g \right) dx - \int \log \left( \frac{\omega}{g} \right) \text{div}(u \omega) dx
\]
\[
=: I + II,
\]
where we have invoked the property \(\text{div}(u \omega) = (u, \nabla)\omega\). Integrating by parts, we obtain
\[
I = - \int \langle \nabla \omega, \nabla (\omega/g) \rangle \frac{\omega}{g} dx + \int \langle \nabla (\omega/g), \nabla g \rangle dx
\]
\[
= \int \left\langle \nabla \left( \frac{\omega}{g} \right), \nabla \left( \frac{g}{\omega} \right) \right\rangle \omega dx - \int \left| \nabla \left( \frac{\omega}{g} \right) \right|^2 \frac{g^2}{\omega} dx.
\]
Again integrating by parts yields
\[
II = \int u \omega \left( \frac{\nabla \omega}{\omega} - \frac{\nabla g}{g} \right) dx = \int \left( (u, \nabla) \omega - (u, \nabla) g \frac{\omega}{g} \right) dx
\]
\[
= \int \text{div}(u \omega) dx + \frac{1}{2t} \int \langle x, u(x,t) \rangle \omega(x,t) dx,
\]
where the explicit form of \(g = e^{-|x|^2/4t(4\pi t)^{-1}}\) is invoked. The first term vanishes by integration by parts. Since \(u = K * \omega\), the second term also vanishes by the next lemma. This implies (2.18). If \(H\) is constant in \(I\), then by (2.18), \(\omega = \hat{m}(t)g\) with \(\hat{m}\) independent of \(x\). Since \(\int \omega dx\) is independent of \(t\) (§2.2.4), \(\hat{m}\) is also independent of \(t\). We thus conclude that \(\omega = \hat{m}g\) for \(t \in I\).

**Lemma.** Let \(\omega\) and \(\tilde{\omega}\) be functions on \(\mathbb{R}^2\) such that \(|\omega|^{2+\varepsilon}, |\tilde{\omega}|^{2+\varepsilon}, (|x| + 1)\omega, (|x| + 1)\tilde{\omega}\) are integrable on \(\mathbb{R}^2\) for some \(\varepsilon > 0\). Let \(B\) be the bilinear form defined by
\[
B(\omega, \tilde{\omega}) = \int \langle x, \nabla \tilde{\omega} E * \omega \rangle \tilde{\omega} \, dx
\]
Then \(B(\omega, \tilde{\omega}) = -B(\tilde{\omega}, \omega)\). In particular, \(B(\omega, \omega) = 0\).

**Proof.** By definition
\[
-2\pi B(\omega, \tilde{\omega}) = \iint \left\langle x, \frac{(x-y)}{|x-y|^2} \right\rangle \omega(y)\tilde{\omega}(x) \, dx \, dy,
\]
where \(x^\perp = (x_2, -x_1)\). (By our assumption the integrand is integrable on \(\mathbb{R}^2 \times \mathbb{R}^2\), so we may change the order of integration by Fubini’s theorem (§7.2.2).)

The right-hand side equals
\[
\iint \frac{\langle x-y, (x-y)^\perp \rangle}{|x-y|^2} \omega(y)\tilde{\omega}(x) \, dx \, dy + \iint \frac{\langle y, (x-y)^\perp \rangle}{|x-y|^2} \omega(y)\tilde{\omega}(x) \, dx \, dy
\]
\[
= 0 + 2\pi B(\tilde{\omega}, \omega).
\]
The proof is now complete. \(\square\)
2.8.3 Boundedness of the Entropy

**Lemma.** Let the pair \((\omega, u)\) be a smooth solution of (2.7)–(2.8) in \(\mathbb{R}^2 \times (0, \infty)\) satisfying the Gaussian estimate (2.16). Then

\[
H_0 = \lim_{t \to 0} H(g, \omega)(t) \quad \text{and} \quad H_\infty = \lim_{t \to \infty} H(g, \omega)(t)
\]

exist as finite values.

**Proof.** By Theorem 2.8.2 the function \(H(t) = H(g, \omega)(t)\) is nonincreasing on \((0, \infty)\). So it suffices to prove that \(H(t)\) is bounded on \((0, \infty)\). By estimate (2.16),

\[
H(t) \leq \int \omega \log(4\pi C'_1 e^{-|x|^2/C''_2 t} / e^{-|x|^2/4t}) dx
\]

\[
\leq \int \omega \left\{ \left( -\frac{|x|^2}{C''_2 t} + \frac{|x|^2}{4t} \right) + \max(0, \log(4\pi C'_1)) \right\} dx.
\]

Applying (2.16) and changing the variable of integration as \(y = x/t^{3/2}\), we see that \(H(t)\) is bounded from above, since \(y^2 e^{-y^2/C''_2} \) is integrable on \(\mathbb{R}^2\). Similarly, one is able to prove that \(H(t)\) is bounded from below. \(\square\)

2.8.4 Rescaling

We rescale \((\omega, u)\) as before by

\[
\omega_k(x, t) = k^2 \omega(kx, k^2 t),
\]

\[
\bar{u}_k(x, t) = ku(kx, k^2 t),
\]

where \((\omega, u)\) is the solution of Theorem 2.8.1. If \((\omega, u)\) satisfies (2.12a), (2.12b), the rescaled pair \((\omega_k, u_k)\) satisfies (2.12a), (2.12b) with the same bound independent of \(k > 0\). As argued at the beginning of §2.5, applying the Ascoli–Arzelà theorem (§5.2.5), we see that for any subsequence \(\{\omega_k(\ell)\}_{\ell=1}^{\infty}\) \((k(\ell) \to \infty)\) (respectively, \(k(\ell) \to 0\)) there are a subsequence \(\{\omega_{k(\ell)(i)}\}\) and a limit \(\sigma_\infty\) (resp. \(\sigma_0\)) such that \(\omega_{k(\ell)(i)}\) converges to \(\sigma_\infty\) (resp. \(\sigma_0\)) locally uniformly in \(\mathbb{R}^2 \times (0, \infty)\) with its derivatives. When \(k \to 0\), differently from the case \(k \to \infty\) we are unable to apply §2.4.3, so we do not claim uniform convergence in \(\mathbb{R}^2 \times [\eta, 1/\eta]\) as \(k \to 0\). Estimates (2.12a) hold for \(\sigma_0, \sigma_\infty\) and also (2.12b) holds for \(u_0 = K * \sigma_0, u_\infty = K * \sigma_\infty\). As in §2.5.1, \((\sigma_0, u_0)\) and \((\sigma_\infty, u_\infty)\) are smooth solutions of (2.7)–(2.8) in \(\mathbb{R}^2 \times (0, \infty)\) satisfying (2.12a), (2.12b).

**Proposition.** Let \(\sigma_0\) and \(\sigma_\infty\) be functions defined as above. Then

\[
H(g, \sigma_0)(t) = H_0 \quad \text{and} \quad H(g, \sigma_\infty)(t) = H_\infty
\]

for all \(t \in (0, \infty)\). In particular, \(\sigma_0 = m_0 g, \sigma_\infty = m_\infty g\) with constants \(m_0\) and \(m_\infty\).
Proof. Since \( g_k = g \), we easily see that

\[
H(g, \omega_k)(t) = H(g, \omega)(k^2 t),
\]

so that

\[
\lim_{k \to 0} H(g, \omega_k)(t) = H_0, \quad \lim_{k \to \infty} H(g, \omega_k)(t) = H_\infty.
\]

Since \( \omega_k \) satisfies (2.16) with a constant independent of \( k \), one is able to prove that

\[
\lim_{k \to 0} H(g, \omega_k)(t) = H(g, \sigma_0)(t), \quad \lim_{k \to \infty} H(g, \omega_k)(t) = H(g, \sigma_\infty)(t)
\]

by Lebesgue’s dominated convergence theorem (§7.1.1); the way to estimate the integrand is the same as in §2.8.3. The last statement follows from Theorem 2.8.2.

\[ \Box \]

2.8.5 Proof of the Uniqueness Theorem

We are now in position to prove Theorem 2.8.1. Let \((\omega, u)\) be a (weak) solution of (2.7)–(2.8) satisfying (2.16) with initial data \( m\delta \). Then one is able to prove that \( \int \omega \, dx = m \) by Proposition 2.5.3. This implies that \( \int \omega_k \, dx = m \), which yields \( \int \sigma_0 \, dx = \int \sigma_\infty \, dx = m \). Since \( \sigma_\infty = m_\infty g \), \( \sigma_0 = m_0 g \) in Proposition 2.8.4, we conclude that \( m_\infty = m_0 = m \), since \( \int g \, dx = 1 \). Thus \( H_0 = H_\infty \). By Theorem 2.8.2 this implies that \( \omega = m' g \) with some \( m' \in \mathbb{R} \). However, \( \int \omega \, dx = m \), so \( m' = m \). We now conclude that \( \omega = mg \) and the assertion follows.

\[ \Box \]

Remark. The main result (Theorem 2.8.1) is due to [Gallay Wayne 2005], where the authors studied rescaled vortex equations \((R)\) in §2.6.1 and \( V = K \ast \Omega \) (with \( \alpha = 1/2, \nu = 1 \)) for \((\Omega, V)\) with

\[
\Omega(y, \tau) = e^{\tau} \omega(e^{\tau/2} y, e^{\tau}), \quad V(y, \tau) = e^{\tau/2} u(e^{\tau/2} y, e^{\tau}).
\]

The quantity \( H(g, \omega) \) is transformed as

\[
H(\Omega) = \int \Omega \log(\Omega/e^{-|y|^2/4}) \, dy.
\]

To prove Theorem 2.8.1 they instead studied the dynamical system \((R)\) with \( V = K \ast \Omega \) for \( \tau \in \mathbb{R} \) instead of the rescaled functions \( \omega_k \) directly.

In their paper they further studied asymptotic expansions, not only for the leading term of \( \omega \) as \( t \to \infty \), but also the second term by spectral analysis for \((R)\).
2.8.6 Remark on Asymptotic Behavior of the Vorticity

In §2.2.2 we estimate the difference only in the $L^\infty$-norm. However, it is possible to replace this by the $L^p$-norm. We just state the results currently available (also for the velocity) without proofs.

**Theorem.** For $\omega_0 \in L^1(\mathbb{R}^2)$ let $(\omega, u)$ be the solution of (2.7)–(2.9). Then it satisfies

\[
\lim_{t \to \infty} t^{1-\frac{1}{p}} \|\omega - mg\|_p(t) = 0, \quad 1 \leq p \leq \infty, \quad (2.19)
\]

\[
\lim_{t \to \infty} t^{\frac{1}{2} - \frac{1}{p}} \|u - mK \ast g\|_q(t) = 0, \quad 2 < q \leq \infty, \quad (2.20)
\]

for $m = \int_{\mathbb{R}^2} \omega_0 \, dx$.

The result (2.20) for $u$ follows from $\omega$ as in the proof of Theorem (ii), (iii) of §2.4.1. Results (2.19) for general $p$, $1 \leq p < \infty$, follow from a Rellich-type compactness theorem instead of the Ascoli–Arzelà-type theorem. A full proof using $(R)$ is given in [Gallay Wayne 2005]. Convergence of higher derivatives was also shown by [Maekawa 2008a].

**Exercises 2**

2.1 (§2.1.2) Prove formulas (2.3.1) and (2.3.2).
2.2 (§2.3.4) Calculate $\sum_{j=1}^{\infty} j/2^j$.
2.3 (§2.3.5, §2.3.7) For $f \in C(\mathbb{R}^n)$, show that $\lim_{r \to \infty} \|f\|_r = \|f\|_\infty$. Here we assume that there exists an $r_0$ ($1 \leq r_0 < \infty$) with $\|f\|_{r_0} < \infty$. (To show this, it suffices to assume (Lebesgue) measurability. We need not assume continuity.)
2.4 (§2.3.5) For $1 \leq q \leq \infty$, prove that $\|f\|_q \leq \|f\|_1^{1/q} \|f\|_\infty^{1-1/q}$, where $f \in C(\mathbb{R}^n)$. (For a more general case, see Exercise 6.2.)
2.5 (§2.3.6) In Lemma 2.3.4, show that if $y_\rho \leq N_\rho$, $\rho = 2^k$, then for sufficiently large $m$,

\[
(y_s(t))^{1/s} \leq \left(\frac{4}{a}\right)^{1/\rho} N_\rho^{1/\rho} t^{-1/\rho + 1/s}, \quad t > 0,
\]

where $s = 2^m \geq \rho$.
2.6 (§2.5) Assume that $\lim_{k \to \infty} f_k(x) = f(x)$, $x \in \mathbb{R}^n$, (i.e., that $f_k$ converges pointwise to $f$ on $\mathbb{R}^n$) and $f_k$, $f$ are (Lebesgue) integrable (it may be assumed that the functions are continuous). Under these assumptions, show that

\[
\|f\|_q \leq \lim_{k \to \infty} \|f_k\|_q, \quad 1 \leq q \leq \infty.
\]

(Hint: Use Fatou’s lemma from §7.1.2.)
2.7 (§2.3) Extend the estimates in Theorem 2.3.6 to $n$-dimensional space to prove $L^p$-$L^q$ estimates (like (1.5)) for the heat equation without using the representation formula.

2.8 (§2.3)
(i) Extend Lemma 2.3.2 to $n$-dimensional space (for Exercise 2.7). Prove in particular that

$$2 \int_0^t \| \nabla \omega \|_2^2(s) ds \leq \| \omega_0 \|_2^2.$$

(ii) If $v = 0$, then

$$\| \nabla \omega \|_2(t) \leq C t^{-1/2} \| \omega_0 \|_2, \quad t > 0,$$

with some constant $C$. (One may take $C = 1/\sqrt{2}$.)

(iii) If $v = 0$, then

$$\| \partial_\alpha^\alpha \omega \|_2(t) \leq C' t^{-1} \| \omega_0 \|_2, \quad t > 0, \quad |\alpha| = 2,$$

with some constant $C'$.

Hints:
(i) Integrate the identity in the lemma over the time interval $(0, t)$.
(ii) By scaling it suffices to prove the estimate only at $t = 1$. The estimate (i) implies that there is a set $J \subset (0, 1)$ whose Lebesgue measure is at least $1/2$ such that $\| \nabla \omega \|_2(s) \leq \| \omega_0 \|_2$ for $s \in J$. Since $\| \nabla \omega \|_2(1) \leq \| \nabla \omega \|_2(s)$ for the heat equation we have $\| \nabla \omega \|_2(1) \leq \| \omega_0 \|_2$. One may modify this argument in order to get $\| \nabla \omega \|_2(1) \leq C \| \omega_0 \|_2$ for any $C > 1/\sqrt{2}$.
(iii) Use (ii) twice to get $\| \partial_\alpha^\alpha \omega \|_2(t) \leq C(t/2)^{-1/2} \| \nabla \omega \|_2(t/2) \leq C^2(t/2)^{-1} \| \omega_0 \|_2$. 

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