Dynamic Modeling of Musculoskeletal Motion
A Vectorized Approach for Biomechanical Analysis in Three Dimensions
Yamaguchi, G.T.
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the art of dynamics it was difficult to determine which method was the best to use to derive the equations for a particular problem. Newtonian Methods, D’Alembert’s Method, energy methods based on the works of Hamilton and Lagrange, etc., were all capable and available, but with the right choice a seemingly complex problem could be solved rather simply. Secondly, some of the methods required imaginary perturbations of exactly the right size to be introduced. One had to carry the perturbations along in the mathematics and know from intuition or experience when the effects of the perturbations could be neglected, and when they had to be accounted for. Finally, most of these methods required the usage of differential and integral calculus. While not always difficult when problems were simple, the geometry of more complex systems in which the objects exhibited strong interactions made the calculus tedious, difficult, and error prone.

Kane’s Method differs from most of these classical methods in the following ways:

- The method is vector-based, and therefore is extremely well suited for 3-D analyses;
- Vector cross and dot products are used to determine velocities and accelerations rather than calculus;
- The dynamic equations for an $n$ degree of freedom system are formed out of simplified forms of the forces, moments, and torques, called generalized forces $F_r$ ($r = 1, 2, \ldots, n$), rather than the actual forces, moments, and torques;
- Inertial forces and inertial torques are also incorporated in simplified form as generalized inertia forces $F^* r$;
- Generalized forces and generalized inertia forces are simply added together to create $n$ dynamic equations describing the motions of the $n$ degrees of freedom,

$$F_r + F^*_r = 0.$$  \hspace{1cm} (1.4)

Though other authors emphasize vector methods to determine velocities and accelerations (for example, Merriam & Kraige, 1977), Kane invented the concepts of the generalized speed, partial velocity, partial angular velocity, generalized active force, and generalized inertia force. These provided the key simplifications enabling forces and torques having no influence on the dynamic equations to be eliminated early in the analysis. Early elimination of these “noncontributing forces and torques” greatly simplified the mathematics and enabled problems having greater complexity to be handled.

The generalized forces are scalars formed by taking dot products of the vector forces and particular portions of the velocity vectors called “partial
Figure 1.1. Planar two-link kinematic chain with an endpoint force. Rigid bodies $A$ and $B$, and the ground reference frame $N$ are joined together with frictionless pins at points $A_o$ and $B_o$. Triads of mutually perpendicular unit vectors define vector component directions for each of these three rigid body reference frames. $\hat{a}_1$ remains parallel to $\overline{A_oB_o}$ (a line joining points $A_o$ and $B_o$) as body $A$ rotates by angle $q_1$, and $\hat{b}_1$ remains parallel to line $B_oC_o$, as both bodies $A$ and $B$ rotate. Angle $q_2$ is defined to be the counterclockwise rotation angle from $\hat{a}_1$ to $\hat{b}_1$. Mass centroids $A^*$ and $B^*$ for the respective bodies are located at distances $\rho_A$ and $\rho_B$ from their proximal ends. A torque, $\tau_{N/A}$, is exerted by $N$ on $A$, and another torque, $\tau_{A/B}$, is exerted by $A$ on $B$. An endpoint force of arbitrary direction and magnitude, $\vec{F} = f_1\hat{n}_1 + f_2\hat{n}_2$ is exerted by an external influence at the endpoint of the linkage, $C_o$. 
be,

\[ N^A_{\mathbf{q}^A^*} = \rho_A \dot{q}_1 \ddot{a}_2 \]  
\[ N^B_{\mathbf{q}^B_0} = \ell_A \dot{q}_1 \ddot{a}_2 \]  
\[ N^B_{\mathbf{q}^B^*} = \ell_A \dot{q}_1 \ddot{a}_2 + \rho_B (\dot{q}_1 + \dot{q}_2) \ddot{b}_2 \]  
\[ N^C_{\mathbf{q}^C_0} = \ell_A \dot{q}_1 \ddot{a}_2 + \ell_B (\dot{q}_1 + \dot{q}_2) \ddot{b}_2 . \]

Deriving dynamic equations often appears to be difficult because finding expressions for the accelerations of the mass centers can be arduous. However, in a “kinematic chain” of linked rigid bodies, a vectorized method of deriving the accelerations can be done using a step by step process that uses vector cross products instead of differential calculus. Vector cross products require the analyst to copy text strings and then to simplify their products using nothing more than algebra. After only a few lines of computation the accelerations \( N^A_{\mathbf{q}^A^*} \) and \( N^B_{\mathbf{q}^B^*} \) of mass locations \( A^* \) and \( B^* \), with respect to reference frame \( N \), can be derived as,

\[ N^A_{\mathbf{q}^A^*} = -\rho_A \dot{q}_1^2 \ddot{a}_1 + \rho_A \ddot{q}_1 \ddot{a}_2 \]  
\[ N^B_{\mathbf{q}^B^*} = -\ell_A \dot{q}_1^2 \ddot{a}_1 + \ell_A \dot{q}_1 \ddot{a}_2 - \rho_B (\dot{q}_1 + \dot{q}_2)^2 \ddot{b}_1 + \rho_B \dot{q}_1 \ddot{b}_2 . \]

Now that the kinematics is complete, a comparison of the two methods can proceed. Kane’s Method is presented first, as it refers to Figure 1.1.

1.3.1.1 KANE’S METHOD

In Kane’s approach, quantities called partial angular velocities and partial velocity vectors must be generated directly from angular velocity expressions and velocity expressions. The partial angular velocities are important quantities to define for bodies that rotate in response to applied torques. Likewise, the partial velocities are important, and come from the velocities of points which change in response to forces acting through them. The “partial” quantities are picked out of the expressions of a body’s angular velocity or a point’s velocity. Doing so is easy to learn, and is much like learning how to pick cherries from a cherry tree. All one needs to do to become a successful cherry picker is to learn to differentiate the cherries from the leaves and branches!

As an example, the angular velocity of body \( A \) with respect to reference frame \( N \) has already been given in Equation 1.14, and can be factored into the following form,

\[ N^{\mathbf{\omega}A} = (\ddot{a}_3) u_1 + (\ddot{b}) u_2 , \]

where the quantities \( u_i \equiv \dot{q}_i, (i = 1, 2) \). \( u_1 \) is known as the first generalized speed of the system, and \( u_2 \) is called the second generalized speed. Most of the time, \( n \) generalized speeds will be defined for a system having \( n \) degrees
This is begun by summing together all forces acting upon the outermost link $B$,

$$
\vec{F}_B + \vec{F}^{B*} = 0. 
$$

(1.37)

$\vec{F}^{B*}$ is the inertial force which resists the acceleration, and must be summed together with the endpoint force $\vec{F}$ and the joint reaction force $\vec{F}_{A/B}$ according to D'Alembert's Principle contained in Equation 1.2. Equation 1.37 becomes

$$
\vec{F} + \vec{F}_{A/B} - m_B g \hat{n}_2 - m_B N^{aB*} = 0, 
$$

(1.38)

which allows the unknown joint reaction force to be solved,

$$
\vec{F}_{A/B} = -\vec{F} + m_B g \hat{n}_2 + m_B N^{aB*}. 
$$

(1.39)

If there were more than two links in the system, one would proceed inward using D'Alembert's Principle to compute the rest of the joint reaction forces. For the next link, the joint reaction force $-\vec{F}_{A/B}$ would be summed together with the other forces acting upon it. This force is equal and opposite to the force acting on the outermost link because it is a reactive force. The force between the ground and the first link (here, $\vec{F}_{N/A}$) does not need to be determined because the moment of the ground reaction force can always be computed about the point of ground contact (here, point $A_o$) and thus be made zero.

Once the joint reaction forces are determined, the analysis proceeds from the ground outward. Moments about the proximal (inward) ends are summed together with torques acting upon each free body of the system. For body $A$, torque $\vec{\tau}_{N/A}$ acts at its lower end and torque $-\vec{\tau}_{A/B}$ acts at its upper end. The moments are cross products of position vectors and forces. For body $A$, the moments are computed from a common point, $A_o$ and summed together with the torques,

$$
\vec{M}_{A_o} = I_A N^{aA} ,
$$

(1.40)

or,

$$
\vec{\tau}_{N/A} - \vec{\tau}_{A/B} + \left( \vec{p}^{A_o A*} \times m_A \left( -g \hat{n}_2 - N^{aA*} \right) \right) 
+ \left( \vec{p}^{A_o B_o} \times \left( -\vec{F}_{A/B} \right) \right) = I_A N^{aA}. 
$$

(1.41)

Equation 1.40 is the first dynamic equation, as $N^{aA}$ contains the second time derivative of angle $q_1$.

The second dynamic equation is determined in the same manner, this time summing moments of forces acting on body $B$ about point $B_o$,

$$
\vec{M}_{B_o} = I_B N^{aB};
$$

(1.42)

$$
\vec{\tau}_{A/B} + \left( \vec{p}^{B_o B*} \times m_B \left( -g \hat{n}_2 - N^{aB*} \right) \right) 
+ \left( \vec{p}^{B_o C_o} \times \vec{F} \right) = I_B N^{aB}. 
$$

(1.43)
ments needed to create a specific movement. For example, an optimal criterion function might minimize total muscle power while still achieving the desired motions of the limbs. Though this application is far from fully developed, it affords musculoskeletal modelers a non-invasive means of exploring the movement control strategies utilized by the central nervous system to coordinate multijoint movements.

1.5 EXERCISES

1. The dynamic equations for the two link planar linkage determined by the Newton-Euler method were given symbolically in Equations 1.40 and 1.42. Complete the derivations of the two dynamic equations of motion. Use the following expressions for the joint torques,

\[ \tau_{N/A} = T_1\dot{a}_3 \]  \hspace{1cm} (1.44)

\[ \tau_{A/B} = T_2\dot{b}_3. \]  \hspace{1cm} (1.45)

The answers are given below.

\[ \left( I_A + \rho_A^2 m_A + \ell_A^2 m_B + \ell_A \rho_B m_B c_2 \right) \ddot{q}_1 + \left( \ell_A \rho_B m_B c_2 \right) \ddot{q}_2 \]

\[ = \left( T_1 - T_2 \right) - g \left( \rho_A m_A c_1 + \ell_A m_B c_1 \right) \]

\[ + \ell_A \rho_B m_B s_2 \left( \dot{q}_1 + \dot{q}_2 \right)^2 + \ell_A \left( f_2 c_1 - f_1 s_1 \right) \]  \hspace{1cm} (1.46)

\[ \left( I_B + \rho_B^2 m_B + \ell_A \rho_B m_B c_2 \right) \ddot{q}_1 + \left( I_B + \rho_B^2 m_B \right) \ddot{q}_2 \]

\[ = T_2 + g \rho_B m_B \left( s_1 s_2 - c_1 c_2 \right) - \left( \ell_A \rho_B m_B s_2 \right) \dot{q}_1^2 \]

\[ + \ell_B \left( - f_1 c_1 c_2 + s_1 s_2 + f_2 \left( - s_1 s_2 + c_1 c_2 \right) \right) \]  \hspace{1cm} (1.47)

2. In Figure 1.2, the free body diagram for the two link planar linkage is shown. The angle of the second link, \( q_2 \), is defined as a joint angle between the \( \dot{a}_1 \) and \( \dot{b}_1 \) axes, as shown in Figure 1.1. If, instead, angle \( q_2 \) was defined as a segmental angle between the \( \dot{n}_1 \) and \( \dot{b}_1 \) axes, the dynamic equations of motion will change.

Using the Newton-Euler approach, and using the same torque expressions given in Problem 1, find the dynamic equations for the two link planar linkage when segmental angles are employed for segments \( A \) and \( B \). The answers are given on the next page.
\[
\begin{align*}
\left( I_A + \rho_A^2 m_A + \ell_A^2 m_B \right) \ddot{q}_1 + \left( \ell_A\rho_B m_B c_{21} \right) \ddot{q}_2 &= \left( T_1 - T_2 \right) - g \left( \ell_A m_B + \rho_A m_A \right) c_1 \\
&\quad + \left( \ell_A\rho_B m_B s_{21} \right) \dot{q}_2^2 + \ell_A \left( f_2 c_1 - f_1 s_1 \right) \\
\left( \ell_A\rho_B m_B c_{21} \right) \dot{q}_1 + \left( I_B + \rho_B^2 m_B \right) \ddot{q}_2 &= T_2 - g \rho_B m_B c_2 - \left( \ell_A\rho_B m_B s_{21} \right) \dot{q}_1^2 \\
&\quad + \ell_B \left( f_2 c_2 - f_1 s_2 \right)
\end{align*}
\]

where \( c_{21} = \cos(q_2 - q_1) \) and \( s_{21} = \sin(q_2 - q_1) \).

References


2.1.5 FORCE-VELOCITY PROPERTY

In addition to the dependence of active muscle force on length, the rate at which a muscle fiber changes length influences the magnitude of the active tension developed by the muscle (Figure 2.12). When activated muscle shortens (called a concentric contraction), the resulting force is less than that observed during isometric contractions. The opposite is true in cases where the muscle is lengthening eccentrically. It is as if the muscle’s contractile machinery acts in parallel with a fluid “damper” which generates a force that is proportional to, but opposite in direction to, the velocity. An empirical relation by Hill (1938) relating the tensile force, \( F^M \), to the muscle shortening velocity, \( v^M \), was found that adequately matches data for cardiac and skeletal muscle,

\[
\left( F^M + a \right) \left( v^M + b \right) = \left( F^M_o + a \right) b, \tag{2.31}
\]

which is a hyperbola described by asymptotes at \( F^M = -a \) and \( v^M = -b \). \( F^M_o \) is the optimal muscle force as defined in Section 2.1.4.

2.1.5.1 ACTIVATION SCALING AND NORMALIZATION

Subtracting the passive contributions from the isometric tension-length curves yields a family of length-dependent curves of active force development. If we

Figure 2.12. Contractile force development as a function of muscle fiber velocity, where positive velocity \( v^M (t) \) is defined as shortening velocity \( v^M \equiv -\dot{\ell}^M \). When \( v^M \geq 0 \), the “f – v” curve follows a hyperbola with asymptotes at \( v^M = -b \) and \( F^M = -a \). The left-hand side of the curve indicates that force production during eccentric (lengthening) activity exceeds that produced isometrically \( (v^M = 0) \), while the right-hand side shows that force diminishes during concentric contractions. Maximum shortening velocity \( v_{max} \approx 10 f^M_o / \text{sec.} \)
Figure 3.3.  A. A system of three rigid links. If each joint has one degree of freedom, the system has a total of three degrees of freedom. B. When the endpoint $P$ is rigidly pinned to the "ground", the system becomes a four bar linkage having only one degree of freedom.

To resolve this problem, motion constraints are distinguished from configurational constraints by examining the lowest order equation that can be written to physically describe the system. In the example just stated, if the zeroth order equation (i.e., no time derivatives) describes an actual physical relationship between the system variables, then the system is said to be subject to a configurational constraint. It is called a configurational constraint even though the first derivative of that equation is also mathematically correct. On the other hand, if the first order equation describes an actual physical relationship between the velocities, but the corresponding zeroth order equation does not, then the system is said to be subject to a motion constraint. In other words, systems subject to configurational constraints also have constraints to their motions, but these are not called motion constraints. Systems subject to real motion constraints are not necessarily subject to corresponding configurational constraints.

3.1.2 EXAMPLE – THE FOUR-BAR LINKAGE

Three degrees of freedom uniquely define the configuration of the three-link planar system shown in Figure 3.3A when the endpoint $P$ of link $C$ is free. A set of angular measures defining the orientations of links $A$, $B$, and $C$ is one set of generalized coordinates that uniquely specify the system configuration.

If, however, the endpoint is pinned to the base frame $N$ as in Figure 3.3B, a fourth link is created $(N)$ and the endpoint $P$ is no longer free to move with respect to $N$. In this case, point $P$ has lost its ability to move in the $\hat{x}$ (horizontal) and in the $\hat{y}$ (vertical) directions – minus two degrees of freedom in all. Hence the four-bar linkage has only one degree of freedom because two of
Figure 3.4. Depiction of the four bar linkage model of the knee proposed by Strasser (1917). The anterior cruciate (ACL) and posterior cruciate (PCL) ligaments remain extended and nearly constant in length as the joint is flexed. Because they do not go slack, they are considered to be two of the rigid links. The bony interconnections between the attachments to the tibia and the femur form the other two links.

the three original degrees of freedom were lost by pinning the endpoint. In fact, only one degree of freedom is needed to uniquely define the configuration of the system, such as the angle $q_1$ shown. The other angles, $q_2$ and $q_3$, are functions ($\mathcal{F}$ and $\mathcal{G}$) of the lengths $\ell_A$, $\ell_B$, and $\ell_C$ of links $A$, $B$, and $C$, respectively, and the single variable angle $q_1$,

$$q_2 = \mathcal{F}(q_1)$$

$$q_3 = \mathcal{G}(q_1).$$

The study of four-bar linkages encompasses enough material to form an entire graduate course on the subject. Strasser (1917) even used a four-bar linkage to describe the mechanical action of the anterior and posterior cruciate ligaments of the knee (Figure 3.4). The anterior cruciate was considered to be one link, and the posterior cruciate ligament another. Even though they were not rigid, it was observed that they remained taut and maintained the same length throughout the range of flexion and extension. The bony spans between the connections on the femur and tibia formed the remaining two links. Though the crossed four-bar model of the knee has been disproven, the Strasser model did establish the primary mechanical interactions during flexion and extension. The model has even been extended to make four-bar polycentric hinges (a hinge with a moving center of rotation) for prosthetic knee joints.

\footnote{This model does not account for the smaller three dimensional motions that are evident. For instance, during weight bearing, the geometry of the articulating surfaces greatly affects the knee motions.}
where $\vec{a}$, $\vec{b}$, and $\vec{c}$ are any three noncoplanar vectors, and $\alpha$, $\beta$, and $\gamma$ are three scalar measure numbers. Similarly, any vector $\vec{u}$ may be decomposed into component vectors in any three noncoplanar directions. The most common decomposition utilizes a basis (i.e., a set of three noncoplanar vectors) comprised of mutually perpendicular unit vectors,

$$\vec{u} = \delta \hat{e}_1 + \epsilon \hat{e}_2 + \phi \hat{e}_3,$$

(3.7)

where $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ are any three mutually perpendicular unit vectors, and $\delta$, $\epsilon$, and $\phi$ are three scalar measure numbers as before.

Mathematically, any three noncoplanar vectors can be used to define a basis in three dimensional space. Hereafter in this text, however, a set of basis vectors will refer to the most commonly used basis, which is a set of mutually perpendicular unit vectors arranged in right-handed fashion (Figure 3.6). For example, if rigid body $B$ has a reference frame affixed to it comprised of basis vectors $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$, the vectors are usually considered to have unit length, are mutually perpendicular, and arranged so the cross product $\vec{b}_1 \times \vec{b}_2$ equals $\vec{b}_3$ according to the right-hand rule (Figure 3.7). If the fingers of the right hand are curled in the same direction as the circular arrow, the extended thumb will yield the direction vector. In the same way, a counterclockwise circular arrow will be used here to indicate a vector pointing out of the drawing, and a clockwise circular arrow will be used to show a vector pointing perpendicularly into the paper (Figure 3.8).

![Figure 3.6. Mutually perpendicular unit vectors.](image)

![Figure 3.7. Right hand rule for vector cross products. When the fingers of the right hand are initially aligned with vector $\vec{b}_1$, and curled toward $\vec{b}_2$ through the acute angle, then $\vec{b}_3$ in a right handed system points in the same direction as the outstretched right thumb.](image)
3.4.1 DIRECTION COSINES IN PLANAR MOTIONS

Now, let's include a rotation at the knee as well as the previous rotation at the ankle. If rigid body B rotates by angle \( q_2 \) from rigid body A, and A rotates from N as defined before, then additional direction cosines may be defined. Figure 3.13 depicts such a case when the motions of the femur (B) and the shank (A) are coplanar. The following direction cosine tables now describe the simple rotation from N to A, and the simple rotation from A to B:

\[
\begin{align*}
N_R^A & | \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \\
\hat{n}_1 & c_1 & -s_1 & 0 \\
\hat{n}_2 & s_1 & c_1 & 0 \\
\hat{n}_3 & 0 & 0 & 1
\end{align*}
\]  

(3.15)

\[
A_R^B & | \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \\
\hat{a}_1 & c_2 & -s_2 & 0 \\
\hat{a}_2 & s_2 & c_2 & 0 \\
\hat{a}_3 & 0 & 0 & 1
\]  

(3.16)

*Helpful hint: Note that angle \( q_1 \) is shown positively in Figure 3.12, and negatively in Figure 3.13. The direction cosine table is identical for both positive and negative angles, but the numerical values of \( s_1 \) and \( -s_1 \) are different. In order to make sure the direction cosine tables are derived with the correct sign in front of each sine function, the analyst should rotate the basis vectors (mentally or on paper) until the angle between them is positive and less than 90 degrees.*
Angle $q_1$ is defined positively when the rotation of $\hat{p}_1'$ is counterclockwise from $\hat{f}_1$, consistent with the angular representation for the $\hat{f}_3, \hat{p}'_3$ axis. The view is defined with the $\hat{f}_3, \hat{p}'_3$ unit vectors pointing out of the paper (directly at the viewer), and with the angle $q_1$ positive and less than 90 degrees.

$$ FRF' \begin{array}{ccc} \hat{p}_1' & \hat{p}_2' & \hat{p}_3' \\ \hat{f}_1 & c_1 & -s_1 & 0 \\ \hat{f}_2 & s_1 & c_1 & 0 \\ \hat{f}_3 & 0 & 0 & 1 \end{array} \quad (3.44)$$

A negative rotation corresponds to pelvic list (drooping of the pelvis to the left in the frontal plane) or hip adduction. Abduction of the hip occurs with a positive rotation angle.

The position of the viewer's eye looking along the next rotation axis ($\hat{p}_1''$) is also shown on the left side of Figure 3.15C. If the eye were placed in this position looking toward the hip joint, the $\hat{p}_2'$ axis will appear to be pointing directly downward, and the $\hat{p}_3'$ axis will point to the right as shown shaded in grey in Figure 3.15D.

\textit{ii) Rotation about the common $\hat{p}_1', \hat{p}_1''$ axis.} The next intermediate orientation $F_{0'}$ is obtained via a second simple rotation about the common $\hat{p}_1', \hat{p}_1''$ axis (Figure 3.15D). A drawing of the basis vectors looking directly at the common $\hat{p}_1', \hat{p}_1''$ axis is made as an aid to the derivation of the table elements.

$$ FR_{0'} \begin{array}{ccc} \hat{p}_1'' & \hat{p}_2'' & \hat{p}_3'' \\ \hat{p}_1' & 1 & 0 & 0 \\ \hat{p}_2' & 0 & c_2 & -s_2 \\ \hat{p}_3' & 0 & s_2 & c_2 \end{array} \quad (3.45)$$

When $q_1$ is small, a positive value of the rotation angle $q_2$ approximates hip extension. In the unusual event that $q_1$ is large, it would be incorrect to refer to $q_2$ as the hip extension angle. This is because hip extension is usually defined in the vertical sagittal plane, and if the hip is abducted or adducted the rotation axis $\hat{p}_1'$, which is perpendicular to the $\hat{p}_2', \hat{p}_3'$ plane, will not be perpendicular to the vertical sagittal plane. The position of the viewer's eye looking along the last rotation axis ($\hat{p}_3'), \hat{p}_3''$) is shown on the upper right side of Figure 3.15D. With the eye placed in this position looking toward the hip joint, the $\hat{p}_1''$ axis will appear to be pointing downward, and the $\hat{p}_1'$ axis will point to the left.
in one reference frame into coordinates measured in another reference frame
is shown in Figure 3.17. The reference frame \( N \) has origin \( N_o \) and a third
reference frame \( C \) has its origin \( C_o \) some distance from \( N_o \). The vector \( N \vec{p}^{C_o} \)
(expressed in the basis vectors of reference frame \( N \)) defines the position of \( C_o \)
in \( N \). The position vector \( C \vec{p}^{P_i} \) from the origin of \( C \) to point \( P_i \) (expressed in
the basis vectors of \( C \)) is related to \( N \vec{p}^{P_i} \) by either of the following equations,

\[
N \vec{p}^{P_i} = N R_C C \vec{p}^{P_i} + N \vec{p}^{C_o} \tag{3.53}
\]

\[
C \vec{p}^{P_i} = C R C_N \left( N \vec{p}^{P_i} - N \vec{p}^{C_o} \right) \tag{3.54}
\]

where \( N \vec{p}^{P_i} = \alpha_3 \hat{n}_1 + \beta_3 \hat{n}_2 + \gamma_3 \hat{n}_3 \), \( C \vec{p}^{P_i} = \alpha_1 \hat{c}_1 + \beta_1 \hat{c}_2 + \gamma_1 \hat{c}_3 \), and
\( N \vec{p}^{C_o} = (dx) \hat{n}_1 + (dy) \hat{n}_2 + (dz) \hat{n}_3 \).

It is understood from the discussion following Equation 3.52 that the rotation-
matrix-to-vector multiplication only serves to re-express vectors from one
reference frame to another, changing the measure numbers but not changing the
original vector’s magnitude or direction. Parentheses are included to ensure that
vectors are expressed in the proper bases before they are added together. Equation
3.53 is the more commonly used of the two expressions. Equation 3.54 can be
useful when it is more convenient to work in the basis vectors of \( C \).

Numerically, the operation in Equation 3.53 is expressed as:

\[
\begin{bmatrix}
\alpha_3 \\
\beta_3 \\
\gamma_3
\end{bmatrix} = \left( N R_C \begin{bmatrix}
\alpha_1 \\
\beta_1 \\
\gamma_1
\end{bmatrix} \right) + \begin{bmatrix}
dx \\
dy \\
dz
\end{bmatrix} \tag{3.55}
\]

Note that the matrix premultiplication by \( N R_C \) serves to change the measure
numbers to a common basis prior to adding the translation. This explains
why the translation by adding \([dx \ dy \ dz]^T\) in Equation 3.53 is performed
after the matrix-vector multiplication. A common computational trick adds a
trivial fourth equation \( 1 = 1 \) to the three equations expressed in matrix form in
Equation 3.55. This allows the rotation by change of basis and translation to
be accomplished in a single matrix-vector multiplication,

\[
\begin{bmatrix}
\alpha_3 \\
\beta_3 \\
\gamma_3 \\
1
\end{bmatrix} = N T_C \begin{bmatrix}
\alpha_1 \\
\beta_1 \\
\gamma_1 \\
1
\end{bmatrix} \tag{3.56}
\]

where the \( 3 \times 3 \) rotation matrix \( N R_C \) has been imbedded into the \( 4 \times 4 \)
transformation matrix, \( N T_C \),

\[
N T_C = \begin{bmatrix}
N R_C & \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \\
0 & 0 & 0 & 1
\end{bmatrix} \tag{3.57}
\]
(b) Rotation of A from N by angle $\theta_1$ about the common $\hat{n}_1$, $\hat{a}_1$ axis; rotation of B from A by angle $\theta_2$ about the common $\hat{a}_2$, $\hat{b}_2$ axis; and rotation of C from B by angle $\theta_3$ about the common $\hat{b}_3$, $\hat{c}_3$ axis.

4. The Euler rotations were defined in Section 3.5.2 as a sequence of rotations, $\hat{z} - \hat{x} - \hat{z}$ or $\hat{3} - \hat{1} - \hat{3}$. It is also common to define these as $\hat{z} - \hat{y} - \hat{z}''$. Find the direction cosines for this alternate sequence of Euler rotations.

5. Find the Euler angles commonly referred to as $\alpha$, $\beta$, $\gamma$ where $\alpha = q_1$, $\beta = q_2$, $\gamma = q_3$ using the definitions of Section 3.5.2, given the following instantaneous values of the rotation matrix:

$$FRP = \begin{bmatrix} 0.826358 & 0.563079 & 0.008593 \\ 0.394664 & -0.568180 & -0.722090 \\ -0.401710 & 0.600097 & -0.691750 \end{bmatrix} \quad (3.58)$$

Either the matrix from Problem 4 or Equation 3.48 may be used to find the answers.

6. Sometimes there is a need to separately digitize the front and back views of an object or a set of markers. For instance, a laser scanner can only see one side of an object at a time to digitize a solid object, or a set of cameras might only be able to see the markers placed on one side of an opaque body.

Devise a procedure (transformation) that will convert the “back” set of marker coordinates into the reference frame of the “front” set. Assume a minimum of three markers are visible in both the “front” and “back” sets, and that these markers have different coordinates in the two views.

**Note:** This problem requires some foreknowledge regarding the numerical solution of $n$ equations and $n$ unknowns when the equations are expressed in linear system form, $A\bar{x} = \bar{b}$. Students unfamiliar with this may gain some insights by reading Section 7.1.1.

**References**


are either parallel or perpendicular. Any unit vector dotted with itself yields a scalar value of 1 (i.e., $\hat{a}_i \cdot \hat{a}_i = 1$), and any vector dotted with a vector that is perpendicular to itself yields a scalar value of 0, (i.e., $\hat{a}_i \cdot \hat{a}_j = 0$, where $i \neq j$).

When dot products are desired in mixed bases, the operation is simplified through the use of the direction cosine tables. Suppose basis vector $\hat{a}_i$ is to be dotted with the basis vector $\hat{b}_j$. Let's assume we have the table of direction cosines relating reference frames $A$ and $B$, written so that the basis vectors $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ are listed on the left hand side of the table while the basis vectors $\hat{b}_1$, $\hat{b}_2$, and $\hat{b}_3$ are listed on top of the table,

<table>
<thead>
<tr>
<th>$A_{RB}$</th>
<th>$\hat{b}_1$</th>
<th>$\hat{b}_2$</th>
<th>$\hat{b}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{a}_1$</td>
<td>$r_{11}$</td>
<td>$r_{12}$</td>
<td>$r_{13}$</td>
</tr>
<tr>
<td>$\hat{a}_2$</td>
<td>$r_{21}$</td>
<td>$r_{22}$</td>
<td>$r_{23}$</td>
</tr>
<tr>
<td>$\hat{a}_3$</td>
<td>$r_{31}$</td>
<td>$r_{32}$</td>
<td>$r_{33}$</td>
</tr>
</tbody>
</table>

The dot product of $\hat{a}_i$ with $\hat{b}_j$ may then be simply read off the table as the element $r_{ij}$ listed in the $i^{th}$ row and the $j^{th}$ column of the table.

**Proof.** Since the vector $\hat{a}_i$ can be expressed using the basis vectors $\hat{b}_1$, $\hat{b}_2$, and $\hat{b}_3$, we can form the dot product of $\hat{a}_i$ with $\hat{b}_j$ in a common basis,

$$\hat{a}_i = r_{i1} \hat{b}_1 + r_{i2} \hat{b}_2 + r_{i3} \hat{b}_3,$$

and thus,

$$\hat{a}_i \cdot \hat{b}_j = (r_{i1} \hat{b}_1 + r_{i2} \hat{b}_2 + r_{i3} \hat{b}_3) \cdot \hat{b}_j = r_{i1} (\hat{b}_1 \cdot \hat{b}_j) + r_{i2} (\hat{b}_2 \cdot \hat{b}_j) + r_{i3} (\hat{b}_3 \cdot \hat{b}_j),$$

and for the usual case where the basis vectors $\hat{b}_1$, $\hat{b}_2$, and $\hat{b}_3$ are mutually perpendicular,

$$\hat{b}_i \cdot \hat{b}_j = 0 \quad (i \neq j)$$

$$\hat{b}_i \cdot \hat{b}_j = 1 \quad (i = j).$$
oriented in the $\hat{a}_3$, $\hat{b}_3$, and $\hat{c}_3$ directions, respectively. Using $m_A$, $m_B$, and $m_C$ as the masses, the potential energy ($PE$) of the system relative to the origin of $N$ is computed by finding the elevations of the mass centers. These elevations are easily derived as the dot products of $\hat{n}_2$ with $^{N}\vec{p}^{A*}$, $^{N}\vec{p}^{B*}$, and $^{N}\vec{p}^{C*}$,

$$PE = \hat{n}_2 \cdot g \left[ m_A \, ^{N}\vec{p}^{A*} + m_B \, ^{N}\vec{p}^{B*} + m_C \, ^{N}\vec{p}^{C*} \right]$$

(4.15)

$$= \hat{n}_2 \cdot \left[ g m_A (\rho_A \hat{a}_3) + \hat{n}_2 \cdot g m_B (\ell_A \hat{a}_3 + \rho_B \hat{b}_3) + \hat{n}_2 \cdot g m_C (\ell_A \hat{a}_3 + \mathbf{A} B \hat{b}_3 + \mathbf{C} \hat{c}_3) \right]$$

(4.16)

$$= g (m_A \rho_A + m_B \ell_A + m_C \ell_A) (\hat{n}_2 \cdot \hat{a}_3) + g m_B \rho_B (\hat{n}_2 \cdot \hat{b}_3) + g m_C \rho_C (\hat{n}_2 \cdot \hat{c}_3).$$

(4.17)

Finally, if the rotation matrices $^{N}R^A$, $^{N}R^B$, and $^{N}R^C$ are known, the dot products $\hat{n}_2 \cdot \hat{a}_3$, $\hat{n}_2 \cdot \hat{b}_3$, and $\hat{n}_2 \cdot \hat{c}_3$, can be read directly from the tables. That is, $\hat{n}_2 \cdot \hat{a}_3$ is the scalar element of $^{N}R^A$ residing in the second ($\hat{n}_2$) row, third ($\hat{a}_3$) column, $\hat{n}_2 \cdot \hat{b}_3$ is the scalar element of $^{N}R^B$ residing in the second row, third column, and $\hat{n}_2 \cdot \hat{c}_3$ is the scalar element of $^{N}R^C$ residing in the second row, third column.

### 4.1.2 THE VECTOR CROSS PRODUCT

While the dot product may be defined in mixed bases, the vector cross product must be performed using a common basis (Figure 4.3). Using two arbitrary vectors $\vec{v}$ and $\vec{w}$ defined in the basis vectors of $A$,

$$\vec{v} = v_1 \hat{a}_1 + v_2 \hat{a}_2 + v_3 \hat{a}_3$$

(4.18)

$$\vec{w} = w_1 \hat{a}_1 + w_2 \hat{a}_2 + w_3 \hat{a}_3,$$

(4.19)

Figure 4.3. Cross product of vectors $\vec{v}$ and $\vec{w}$. The direction of the cross product is determined using the right hand rule (Figure 3.7).
4.1.4 MOMENT OF FORCE, TORQUE

Equation 4.21 is most commonly used to compute the moment of a force about a particular point. Since force is a vector quantity, we can find the moment $\vec{M}_f$ of force $\vec{F}$ about any point $P$ using the definition above,

$$\vec{M}_f = \vec{P}_Q \times \vec{F},$$

where $Q$ is any point along the line of action of the force $\vec{F}$ (Figure 4.4). Note that the computation of a moment first requires the designation of a point $P$ about which the moment is to be computed. That is to say, the moment of force is point specific and changes depending upon the location of point $P$.

Torque is a special type of moment, defined as the moment of a couple, where a couple is defined as a set of $n$ forces having no net resultant, i.e.,

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n = 0.$$  (4.23)

Torque, then, is the algebraic sum of the moments of forces $\vec{F}_1$ through $\vec{F}_n$ about a common point $P$ (see Figure 4.5). An interesting property of torque is that torque is invariant regardless of which point $P$ is chosen. Since any point can be chosen, a judicious choice of point $P$ will yield the same result and can save the analyst a lot of work (see the Example following).

Torque and moment of force are therefore different, although in common usage the terms “torque” and “moment” are often interchanged incorrectly. In Kane’s words, “torque is a moment, but a moment is not a torque” – the torque vector is point independent but the moment vector is intimately associated with a specific point (Kane and Levinson, 1985).
Figure 4.6. Free body diagram of the forearm and hand acted upon by gravity ($m_B\vec{g}$). The biceps muscle inserts at point $I$, and exerts a tensile force $\vec{F}_b$ directed toward the origin of the muscle. Joint reaction force $\vec{R}$ is the summation of all joint contact forces exerted by the distal humerus on the proximal ulna. $\vec{R}$ is applied on the ulna at point $E$. The inertial force $-m_B\vec{a}^{B^*}$ opposes the acceleration of point $B^*$ in reference frame $\text{N}$. Position vector $\vec{r}_b$ is defined from $E$ to $I$. The moment of the biceps muscle about $E$ is simply $\vec{r}_b \times \vec{F}_b$. However, the torque exerted on the forearm and hand is the sum of the moments of all of the forces about a single point.

4.1.4.1 EXAMPLE – TORQUE EXERTED ON A RIGID BODY BY A MUSCLE CONTRACTION

Referring to Figure 4.6, it is easy to define the *moment* of the *biceps* muscle about the elbow, $E$,

$$\vec{M}_b = \vec{r}_b \times \vec{F}_b. \quad (4.24)$$

In fact, computing the moment about any specific point requires only that a vector be defined from that point to a point along the line of action of the force.

The *torque* created on the forearm and hand segment ($B$) is the sum of the moments of all the forces acting about a common point. The moments of all the forces are required, because every force shown, including the inertial force $-m_B\vec{a}^{B^*}$, is needed to create a couple,

$$\vec{F}_b + \vec{R} + m_B\vec{g} - m_B\vec{a}^{B^*} = 0, \quad (4.25)$$

and a couple is needed to compute the torque. By D’Alembert’s principle, the inertial force is treated as if it were any other external force or body force applied directly to the system.

When computing torques, the process is simplified because the *torque will be the same no matter what point it is computed about*. Therefore, since any point can be chosen to compute the individual moments about, a point that lies along the line of action of an unknown force is usually chosen. This causes the moment of the unknown force to be zero, and allows the torque to be computed without requiring the force to be determined beforehand.
and insertion areas are indicated by points $O$ and $I$, respectively. Measured from point $C$, the position vectors locating $O$ and $I$ are,

\begin{align}
\mathbf{p}^{CI} &= i_1 \mathbf{j}_1 + i_2 \mathbf{j}_2 + i_3 \mathbf{j}_3 \\
\mathbf{p}^{CO} &= o_1 \mathbf{j}_1 + o_2 \mathbf{j}_2 + o_3 \mathbf{j}_3.
\end{align}

What is the force required in the *gluteus medius* muscle needed to statically counterbalance the gravitational moment about point $C$ in the $\mathbf{j}_3$ direction?

The moment of the *gluteus medius* muscle about point $C$ is,

\[ M_{\text{gmmed}} = \mathbf{p}^{CO} \times \mathbf{F}, \]

where the force $\mathbf{F}$ is of unknown magnitude. The magnitude of $\mathbf{F}$ can be solved for if an expression can be written for its moment about $C$. But since the magnitude of $\mathbf{F}$ is not known yet, the procedure to compute the unknown force is not immediately obvious.

However, we can formulate an expression for $\mathbf{F}$ that incorporates an unknown magnitude $F$ and a known direction $\mathbf{k}$, where $\mathbf{k}$ is a unit vector pointing along the line of action from $O$ to $I$,

\[ \mathbf{k} = \frac{\mathbf{p}^{OI}}{|\mathbf{p}^{OI}|}. \]

The unknown force $\mathbf{F}$ is then,

\[ \mathbf{F} = F \mathbf{k}, \]

where a straight line of action is assumed. Figure 4.7 shows the vector diagram that relates the pertinent vectors of the problem to each other. Referring to the diagram, and starting from point $C$, one can get to point $I$ either directly or via point $O$,

\[ \mathbf{p}^{CO} + \mathbf{p}^{OI} = \mathbf{p}^{CI} \]
As with angular velocity vectors, the angular acceleration of body $A$ in reference frame $B$ is related to the angular acceleration of body $B$ in $A$,

$\overset{\cdot\cdot}{B}\alpha_{A} = -A\omega^{B}.$ \hspace{1cm} (4.76)

### 4.3 VECTOR CALCULUS VIA CROSS PRODUCTS

#### 4.3.1 DIFFERENTIATION OF A VECTOR USING ANGULAR VELOCITY VECTORS

Any vector $\vec{v}$ fixed in reference frame $B$ can be differentiated with respect to time in reference frame $A$ using the angular velocity of $B$ in $A$ rather than performing direct differentiation as defined in Equation 4.48,

$\frac{A\overset{\cdot}{v}}{dt} = A\omega^{B} \times \vec{v}.$ \hspace{1cm} (4.77)

This relationship is extremely useful, as it provides an algebraic alternative to the normal process of direct time differentiation using calculus.

**Proof.** An expression for the left-hand side of Equation 4.77 is found first. Again, $\vec{v}$ is a vector fixed in reference frame $B$ which is rotating with respect to reference frame $A$. Since $A\omega^{B}$ is a vector quantity, it can be expressed in terms of a simple rotation over an infinitesimally small time interval $\Delta t$,

$A\omega^{B} = \dot{\eta} \hat{\eta},$ \hspace{1cm} (4.78)
\[ = N_a^Q + N_\omega^B \times \mathbf{P}^{QP} + N_\omega^B \times \left( N_\omega^B \times \mathbf{P}^{QP} \right). \]  

(4.138)

### 4.3.3.1 Example – A η-Link Planar Kinematic Chain

A planar chain of rigid links is pinned together such that all of the rotation axes are parallel to the \( \hat{b}_3 \) direction, as shown in Figure 4.10. Each rigid link \( B^{(i)} \) (\( i = 1, 2, 3, \ldots, \eta \)) has a set of mutually perpendicular basis vectors \( \hat{b}_1^{(i)}, \hat{b}_2^{(i)}, \hat{b}_3^{(i)} \), arranged so that the \( \hat{b}_i^{(i)} \) vector points from the joint closest to the ground (point \( P_{i-1} \)) to the joint further away from the ground (point \( P_i \)). Angles \( q_i \) are measured between the adjacent \( \hat{b}_i^{(i)} \) and \( \hat{b}_i^{(i-1)} \) axes, and the distances between adjacent joints are given as \( \ell_i \). To find the velocities and accelerations of each joint, the following procedure is followed using the formulas for the velocity and acceleration of two points on a rigid body.

**Position vectors:**

\[ \mathbf{p}_1 = \ell_1 \hat{b}_1 \quad (4.139) \]
\[ \mathbf{p}_2 = \ell_2 \hat{b}_2 \quad (4.140) \]
\[ \mathbf{p}_3 = \ell_3 \hat{b}_3 \quad (4.141) \]
\[ \vdots = \vdots \]
\[ \mathbf{p}_\eta = \ell_\eta \hat{b}_1^{(\eta)} \quad (4.142) \]

**Angular velocities:**

\[ B_\omega^B' = \dot{q}_1 \hat{b}_3 \quad (4.143) \]
\[ B_\omega^B'' = B_\omega^B' + B_\omega^B\hat{b}_3'' \quad (4.144) \]
\[ = \dot{q}_1 \hat{b}_3 + \dot{q}_2 \hat{b}_3^2 \quad (4.145) \]
\[ = (\dot{q}_1 + \dot{q}_2) \hat{b}_3 \quad (4.146) \]
\[ B_\omega^B''' = B_\omega^B'' + B_\omega^B\hat{b}_3''' \quad (4.147) \]
\[ = (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \hat{b}_3 \quad (4.148) \]
\[ \vdots = \vdots \]
\[ B_\omega^B^{(\eta)} = B_\omega^B(\eta-1) + B^{(\eta-1)}\omega B^{(\eta)} \quad (4.149) \]
\[ = \left( \sum_{i=1}^{\eta} \dot{q}_i \right) \hat{b}_3^{(\eta)} \quad (4.150) \]

**Velocities:**

\[ B_0^{P_0} = 0 \quad (4.151) \]
Figure 4.10. A planar kinematic chain of $\eta$ links.
\[
\begin{align*}
B_0^P_1 & = B_0^P_0 + B_0^\omega B' \times \vec{p}_{P_0}^P P_1 \\
B_0^P_2 & = B_0^P_1 + B_0^\omega B'' \times \vec{p}_{P_1}^P P_2 \\
\vdots & = \vdots \\
B_0^P_\eta & = B_0^P_{\eta-1} + B_0^\omega B^{(\eta)} \times \vec{p}_{P_{\eta-1}}^P P_\eta
\end{align*}
\] (4.152)

\[
\begin{align*}
\dot{B}_0^P_0 & = 0 \\
\dot{B}_0^P_1 & = \dot{B}_0^P_0 + \dot{B}_0^\omega B' \times \left( B_0^\omega B' \times \vec{p}_{P_0}^P P_1 \right) \\
& + \dot{B}_0^\omega B' \times \vec{p}_{P_0}^P P_1 \\
\dot{B}_0^P_2 & = \dot{B}_0^P_1 + \dot{B}_0^\omega B'' \times \left( B_0^\omega B'' \times \vec{p}_{P_1}^P P_2 \right) \\
& + \dot{B}_0^\omega B'' \times \vec{p}_{P_1}^P P_2 \\
\vdots & = \vdots \\
\dot{B}_0^P_\eta & = \dot{B}_0^P_{\eta-1} + \dot{B}_0^\omega B^{(\eta)} \times \left( B_0^\omega B^{(\eta)} \times \vec{p}_{P_{\eta-1}}^P P_\eta \right) \\
& + \dot{B}_0^\omega B^{(\eta)} \times \vec{p}_{P_{\eta-1}}^P P_\eta
\end{align*}
\] (4.156)

\[
\begin{align*}
\ddot{B}_0^P_0 & = 0 \\
\ddot{B}_0^P_1 & = \ddot{B}_0^P_0 + \ddot{B}_0^\omega B' \times \left( B_0^\omega B' \times \vec{p}_{P_0}^P P_1 \right) \\
& + \ddot{B}_0^\omega B' \times \vec{p}_{P_0}^P P_1 \\
\ddot{B}_0^P_2 & = \ddot{B}_0^P_1 + \ddot{B}_0^\omega B'' \times \left( B_0^\omega B'' \times \vec{p}_{P_1}^P P_2 \right) \\
& + \ddot{B}_0^\omega B'' \times \vec{p}_{P_1}^P P_2 \\
\vdots & = \vdots \\
\ddot{B}_0^P_\eta & = \ddot{B}_0^P_{\eta-1} + \ddot{B}_0^\omega B^{(\eta)} \times \left( B_0^\omega B^{(\eta)} \times \vec{p}_{P_{\eta-1}}^P P_\eta \right) \\
& + \ddot{B}_0^\omega B^{(\eta)} \times \vec{p}_{P_{\eta-1}}^P P_\eta
\end{align*}
\] (4.157)

\[
\begin{align*}
4.3.3.2 \quad \text{EXAMPLE – VELOCITY AND ACCELERATION OF THE FOOT DURING A KICK}
\end{align*}
\]

A chain of rigid links representing the thigh (A), shank (B), and foot (C) is actively swinging toward a ball in a kicking motion. The velocity of the hip joint \( H \) is known from experimental measurements and is assumed to be fairly constant from kick to kick. If the angular velocities of bodies A, B, and C are varied, what will the velocity of the contact point \( F^* \) of the foot be with respect to the inertial reference frame \( N^* \)?

The velocities are computed by propagating distally from the point of known velocity, which is the hip joint in this example.

\[
\begin{align*}
N_0^\omega K & = N_0^\omega H + N_0^\omega A \times \vec{p}^H K \\
N_0^\omega A & = N_0^\omega K + N_0^\omega B \times \vec{p}^K A \\
\dot{N}_0^\omega F^* & = N_0^\omega A + N_0^\omega C \times \vec{p}^A F^*
\end{align*}
\] (4.161)

\[
\begin{align*}
\dot{N}_0^\omega K & = \dot{N}_0^\omega H + \dot{N}_0^\omega A \times \left( N_0^\omega A \times \vec{p}^H K \right) + \dot{N}_0^\omega A \times \vec{p}^H K \\
\ddot{N}_0^\omega K & = \ddot{N}_0^\omega H + \ddot{N}_0^\omega A \times \left( N_0^\omega A \times \vec{p}^H K \right) + \ddot{N}_0^\omega A \times \vec{p}^H K
\end{align*}
\] (4.162)

\[
\begin{align*}
\dot{N}_0^\omega A & = \dot{N}_0^\omega K + \dot{N}_0^\omega B \times \vec{p}^K A \\
\ddot{N}_0^\omega A & = \ddot{N}_0^\omega K + \ddot{N}_0^\omega B \times \vec{p}^K A
\end{align*}
\] (4.163)

The accelerations may also be found in the same manner.

\[
\begin{align*}
\dot{N}_0^\omega K & = \dot{N}_0^\omega H + \dot{N}_0^\omega A \times \left( N_0^\omega A \times \vec{p}^H K \right) + \dot{N}_0^\omega A \times \vec{p}^H K \\
\ddot{N}_0^\omega K & = \ddot{N}_0^\omega H + \ddot{N}_0^\omega A \times \left( N_0^\omega A \times \vec{p}^H K \right) + \ddot{N}_0^\omega A \times \vec{p}^H K
\end{align*}
\] (4.164)
Figure 4.12. Figure and definitions to illustrate one point moving on a rigid body (Equation 4.167). The moving point, \( P \), is depicted as a ladybug crawling upon the hand (body \( B \)). Point \( \bar{B} \) is the point of the hand which is coincident with \( P \). \( \bar{B} \) is shown as a shadow underneath the bug, but in actuality is considered to be stationary with respect to \( B \) and coincident with the bug. \( P \) has velocity \( \dot{B}P \).

\( \bar{B} \) is the point of body \( B \) that the point \( P \) happens to coincide with at the instant under consideration, and \( \dot{N}\bar{B}P \) is the velocity of \( \bar{B} \) in \( N \) (see Figure 4.12). Similarly, the acceleration of \( P \) in \( N \) is,

\[
\dot{N}\bar{B}P = \dot{N}\bar{B} + B\ddot{a}P + 2 \dot{N}\omega^B \times \dot{B}P.
\] (4.168)

The last term \( (2 \dot{N}\omega^B \times \dot{B}P) \) represents the coriolis acceleration due to the angular velocity of \( B \) in \( N \) and the velocity of point \( P \) in \( B \). The coriolis force exerted on a mass \( m \) at \( P \) is equal to \(-2m(\dot{N}\omega^B \times \dot{B}P)\). There is neither a coriolis acceleration nor a coriolis force if the angular velocity of the body \( B \) in \( N \) or the linear velocity of the point \( P \) on body \( B \) are zero. The proof of Equation 4.168 is from Kane and Levinson (1985).

Proof. Let \( B \) be a rigid body having angular velocity \( \dot{N}\omega^B \) in reference frame \( N \). Two points belonging to body \( B \) are defined in Figure 4.12. \( \bar{B} \) is the point instantaneously coincident with the moving point \( P \), and \( \bar{B} \) which is another point of \( B \). Position vectors from origin \( O \) fixed in reference frame \( N \) are also defined in the figure. Since \( \bar{B} \) and \( P \) happen to be coincident at the instant of time under consideration, the velocity of \( P \) in \( N \) is,

\[
\dot{N}\bar{B}P = \frac{d}{dt} \left( \dot{N}\bar{B} + \dot{p}\bar{B} \right) \quad (4.169)
\]

\[
= \dot{N}\bar{B} + \frac{d}{dt} \left( \dot{p}\bar{B} \right) + \dot{N}\omega^B \times \dot{p}\bar{B} \quad (4.170)
\]

The time rate of change of the position vector from \( \bar{B} \) to \( \bar{B} \) is nonzero, because \( \bar{B} \) is continually being redefined as point \( P \) moves on body \( B \),

\[
\frac{d}{dt} \left( \dot{p}\bar{B} \right) = \dot{B}\bar{B}P, \quad (4.171)
\]

and thus,

\[
\dot{N}\bar{B}P = \dot{N}\bar{B} + B\dot{a}P + \dot{N}\omega^B \times \bar{B} \quad (4.172)
\]
\( \vec{B} \) is the point of body \( B \) that the point \( P \) happens to coincide with at the instant under consideration, and \( \dot{N}_B \vec{B} \) is the velocity of \( \vec{B} \) in \( N \) (see Figure 4.12). Similarly, the acceleration of \( P \) in \( N \) is,

\[
\ddot{N}_B \vec{B} = \dot{N}_B \vec{B} + \dot{B}_P + 2 \dot{N}_B B \times \dot{B}_P .
\] (4.168)

The last term (\( 2 \dot{N}_B B \times \dot{B}_P \)) represents the coriolis acceleration due to the angular velocity of \( B \) in \( N \) and the velocity of point \( P \) in \( B \). The coriolis force exerted on a mass \( m \) at \( P \) is equal to \(-2m(\dot{N}_B B \times \dot{B}_P)\). There is neither a coriolis acceleration nor a coriolis force if the angular velocity of the body \( B \) in \( N \) or the linear velocity of the point \( P \) on body \( B \) are zero. The proof of Equation 4.168 is from Kane and Levinson (1985).

**Proof.** Let \( B \) be a rigid body having angular velocity \( \dot{N}_B B \) in reference frame \( N \). Two points belonging to body \( B \) are defined in Figure 4.12. \( \vec{B} \) is the point instantaneously coincident with the moving point \( P \), and \( \vec{B} \) which is another point of \( B \). Position vectors from origin \( O \) fixed in reference frame \( N \) are also defined in the figure. Since \( \vec{B} \) and \( P \) happen to be coincident at the instant of time under consideration, the velocity of \( P \) in \( N \) is,

\[
\dot{N}_B \vec{B} = \frac{d}{dt} \left( \dot{N}_B \vec{B} + \vec{B} \cdot \vec{B} \right) \] (4.169)

\[
= \dot{N}_B \vec{B} + \frac{dB}{dt} \left( \vec{B} \cdot \vec{B} \right) + \dot{N}_B B \times \dot{B}_P . \] (4.170)

The time rate of change of the position vector from \( \vec{B} \) to \( \vec{B} \) is nonzero, because \( \vec{B} \) is continually being redefined as point \( P \) moves on body \( B \),

\[
\frac{d}{dt} \left( \vec{B} \cdot \vec{B} \right) = B_P , \] (4.171)

and thus,

\[
\dot{N}_B \vec{B} = \dot{N}_B \vec{B} + B_P + \dot{N}_B B \times \dot{B}_P . \] (4.172)
Figure 4.13. A seven degree of freedom model of the upper extremity. This model features the radius and ulna as separate and distinct rigid bodies so that pronation and supination can be realistically modeled. In order to highlight supination angle $q_6$, the diagrams depict shoulder angles $q_1 = q_2 = q_3 = 0$, elbow angle $q_4 = 0$, and wrist adduction angle $q_5 = 0$. 
are two massless “intermediate” reference frames. The third rotation about the shoulder $q_3$ is external rotation, and occurs about the longitudinal axis of the humerus $\hat{a}_3$ and $\hat{a}_3$. Direction cosines from $N$ to $A$ are given by

\[
N\hat{R}^A = \begin{array}{ccc}
\hat{n}_1 & c_2c_3 & -c_2s_3 & s_2 \\
\hat{n}_2 & s_1s_2c_3 + c_1s_3 & -s_1s_2s_3 + c_1c_3 & -s_1c_2 \\
\hat{n}_3 & -c_1s_2c_3 + s_1s_3 & c_1s_2s_3 + s_1c_3 & c_1c_2
\end{array}
\] (4.181)

Elbow flexion is represented as a simple rotation by angle $q_4$ about the common $\hat{a}_2$, $\hat{b}_2$ axis,

\[
A\hat{R}^B = \begin{array}{ccc}
\hat{b}_1 & \hat{b}_2 & \hat{b}_3 \\
\hat{a}_1 & c_4 & 0 & s_4 \\
\hat{a}_2 & 0 & 1 & 0 \\
\hat{a}_3 & -s_4 & 0 & c_4
\end{array}
\] (4.182)

To accurately model the rotation of the radius about the ulna, the fulcrum of pronation and supination is determined as the line between the center of the concave proximal end of the radius and the distal end of the ulna (Kapandji, 1982). A positive, fixed rotation by angle $\phi$ about the common $\hat{b}_1$, $\hat{c}_1$ axis brings unit vector $\hat{c}_3$ of intermediate reference frame $C'$ parallel to this line.\(^6\) Note that angle $\phi$ is a property of the geometry of the system, and is not a degree of freedom, as it is a fixed rotation angle. Supination occurs when reference frame $C$ rotates by positive angle $q_5$ about the common $\hat{c}_3$, $\hat{c}_3$ axis, and pronation occurs when $q_5$ is negative. $\sin(\phi)$ and $\cos(\phi)$ are represented as $s_\phi$ and $c_\phi$, respectively, in the following tables,

\[
B\hat{R}^{C'} = \begin{array}{ccc}
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 \\
\hat{b}_1 & 1 & 0 & 0 \\
\hat{b}_2 & 0 & c_\phi & -s_\phi \\
\hat{b}_3 & 0 & s_\phi & c_\phi
\end{array}
\] (4.183)

\[
B\hat{R}^{C} = \begin{array}{ccc}
\hat{c}_1 & \hat{c}_2 & \hat{c}_3 \\
\hat{b}_1 & c_5 & -s_5 & 0 \\
\hat{b}_2 & c_\phi s_5 & c_\phi c_5 & -s_\phi \\
\hat{b}_3 & s_\phi s_5 & s_\phi c_5 & c_\phi
\end{array}
\] (4.184)

In order to realign the wrist flexion axis ($\hat{a}_2$), an inverse rotation by angle $-\phi$ about the common $\hat{c}_1$, $\hat{a}_1'$ axis is required. Because wrist adduction occurs

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\(^6\)In practice, and depending on the geometry of the arm segment being modeled, sometimes two fixed rotations are needed to bring $\hat{c}_3$ parallel, but one rotation is usually sufficiently accurate.
the centroid of the wrist, and a point about which to compute muscle moments. Point $P$ is any point on the hand at which the velocity is computed, and is given three arbitrary coordinates with respect to point $D_o$.

\[
\begin{align*}
\tilde{p}^{A_oB_o} &= \ell_{A} \hat{a}_3 \\
\tilde{p}^{B_oA} &= \delta \hat{a}_2 + r \hat{c}_3 \\
\tilde{p}^{A_Co} &= 0 \\
\tilde{p}^{C_oB_1} &= \ell_{C,B} \hat{c}_3 \\
\tilde{p}^{C_oC_1} &= \ell_{C_2} \hat{c}_2 + \ell_{C_3} \hat{c}_3 \\
\tilde{p}^{B_1C_1} &= \tilde{p}^{C_oC_1} - \tilde{p}^{C_oB_1} \\
\tilde{p}^{C_oD_o} &= \tilde{p}^{C_oB_1} + \frac{1}{2} \tilde{p}^{B_1C_1} \\
\tilde{p}^{D_oP} &= \ell_{D_1} \hat{d}_1 + \ell_{D_2} \hat{d}_2 + \ell_{D_3} \hat{d}_3
\end{align*}
\] (4.205)

The formula relating the velocities of two points fixed in a rigid body are used for the majority of the velocity calculations. The velocity of point $C_o$ in $N$, however, uses the formula for the velocity of one point moving on a rigid body, since the proximal end of the radius glides upon the capitulum with speed $r \dot{q}_4$ and direction $\hat{c}_1$ ($\hat{b}_1$). Relative to rigid body $A$, then,

\[
\begin{align*}
\hat{A}_o^{C_o} &= r \dot{q}_4 \hat{c}_1 \\
&= r \dot{q}_4 \hat{b}_1 \\
&= r \dot{q}_4 (c_4 \hat{a}_1 - s_4 \hat{a}_3).
\end{align*}
\] (4.213)

\[
\begin{align*}
\end{align*}
\] (4.214)

\[
\begin{align*}
\end{align*}
\] (4.215)

**Figure 4.14.** The radiohumeral joint and the radius, shown from the lateral side. The proximal end of the radius glides upon the capitulum as the elbow is flexed or extended. Thus, point $C_o$ has a nonzero velocity in reference frame $A$. Its velocity in reference frame $N$ is computed using the formula for one point moving on a rigid body.
The accelerations are found via a logical progression from point \( A_o \),

\[
\begin{align*}
N_\alpha^A A_o &= 0 \\
N_\alpha^A B_o &= N_\alpha^A A_o + N_\alpha^A \times \left( N_\alpha^A \times \vec{p}^A A_o B_o \right) \\
N_\alpha^A C_o &= N_\alpha^A \dot{A} + A_\alpha^C + 2 N_\alpha^A \times A \vec{v} C_o \\
&= \left( N_\alpha^B B_o + N_\alpha^A \times \left( N_\alpha^A \times \vec{p}^B B_o \dot{A} \right) \right) \\
&\quad + N_\alpha^A \times \vec{p}^B B_o A \\
&\quad + A_\alpha^C A + 2 N_\alpha^A \times A \vec{v} C_o \\
\end{align*}
\] (4.235)

where the equation for the acceleration of two points fixed on a rigid body was used to expand the acceleration of \( \ddot{A} \) in reference frame \( \dot{A} \) in terms of known quantities. The remaining accelerations are computed using the accelerations of two points in a rigid body formula,

\[
\begin{align*}
N_\alpha^D D_o &= N_\alpha^D C_o + N_\alpha^C \times \left( N_\alpha^C \times \vec{p}^C C_o D_o \right) \\
&\quad + N_\alpha^C \times \vec{p}^C C_o D_o \\
N_\alpha^P &= N_\alpha^P D_o + N_\alpha^P \times \left( N_\alpha^P \times \vec{p}^P D_o \right) \\
&\quad + N_\alpha^P \times \vec{p}^P D_o P \\
\end{align*}
\] (4.236) (4.237)

4.5 EXERCISES

1. Find the potential energy of a four-link planar linkage using the definitions of Figure 4.10. Start by developing direction cosine matrices beginning with the ground \( \dot{N} \) reference frame and working upward. Assume that the links have joints separated by lengths \( \ell_i \) and masses \( m_i \) \((i = 1, 2, 3, 4)\) centered between the joints.

2. A weight \( W \) is hung on an inextensible string of length \( \ell \) to form a planar pendulum. If the vertical angle of the string is \( \theta \), find the velocity of \( W \) by taking the cross product of the pendulum's angular velocity and the appropriate position vector.

3. Prove that the moment of a couple is the same regardless of the point chosen to compute the moments about. It may be helpful to consider the following cases:

(a) A couple composed of two coplanar forces, with any point \( P \) contained in the same plane.
(b) A couple composed of two coplanar forces, with a non-coplanar point $P$.

(c) A couple composed of any number of non-coplanar forces, with any point $P$.

4. Estimate the force $F_b$ in the biceps muscle required to hold a cup full of liquid in a static posture similar to Figure 4.2. The muscle force should balance the gravitational moment of the forearm and hand about the elbow. Estimate muscle origin and insertion positions and limb dimensions by direct measurement on a volunteer.

5. Find the angular velocity of the pelvis following the three rotations depicted in Figures 3.12, 3.13, and 3.14.

6. This problem refers to Figure 4.7, and uses the $\hat{z} - \hat{x} - \hat{z}'$ Euler rotations of the pelvis derived in Chapter 3 (Figure 3.15 and Equation 3.48).

Let point $O$ represent the “origin” of the gluteus medius muscle, and point $I$ represent its insertion on the greater trochanter. More precisely, these points are located at the centroids of the attachment areas. Let the origin of pelvic reference frame $P$ be at the centroid of the acetabulum (point $C$), and let the following position vectors be used to locate $O$ and $I$ from point $C$,

$$\vec{p}^{CO} = o_1\hat{p}_1 + o_2\hat{p}_2 + o_3\hat{p}_3 \quad (4.238)$$
$$\vec{p}^{CI} = i_1\hat{f}_1 + i_2\hat{f}_2 + i_3\hat{f}_3 \quad (4.239)$$

(a) Find the position vector pointing from $O$ to $I$, or $\vec{p}^{OI}$. Express your answer in either $P$ or $F$ basis vectors.

(b) Show that the velocity of the muscle, $\vec{v}^{OI}$, can be found by finding the first time derivative of $P\vec{p}^I$.

(c) Find $P\vec{v}^I$ by direct differentiation of $P\vec{p}^I$ (in the $P$ reference frame).

(d) Find $P\vec{v}^I$ by relating the time derivatives of $P\vec{p}^I$ in two reference frames,

$$\frac{P d(P\vec{p}^I)}{dt} = \frac{F d(P\vec{p}^I)}{dt} + P\omega^F \times P\vec{p}^I \quad (4.240)$$

7. The following are simplified forms of the position vectors locating mass centers $A^*$, $B^*$, and $C^*$ of the seven degree of freedom arm model depicted in Section 4.4.

$$\vec{p}^{A^*A} = \rho_{A_3}\hat{a}_3 \quad (4.241)$$
$$\vec{p}^{B^*B} = \rho_{B_3}\hat{b}_3 \quad (4.242)$$
$$\vec{p}^{C^*C} = \rho_{C_3}\hat{c}_3 \quad (4.243)$$
good working definition for the ICR between reference frames A and B is the point instantaneously having zero velocity in both reference frames. Figure 5.4 shows, for example, a pathway of the ICR for the femur moving into flexion, with respect to a fixed tibia.

The reader is cautioned that the ICR is an approximation of relative motion that substitutes a hypothetical (two-dimensional) rotation in place of a (two or three-dimensional) transformation having both rotation and translation. In other words, it first presumes that the relative motion is completely planar and revolute. This is okay during a smooth, planar joint rotation, but it is objectionable when translations accompany the joint rotation because the ICR location is often translated outside the physiological joint. In the case where one bone quickly and suddenly slides upon another during the rotational motion, the ICR is translated infinitely far away.

A second objection many biomechanists have with the ICR is that it is a very difficult concept to apply. For example, as the the time sampling interval decreases, it becomes increasingly difficult to draw accurate perpendicular bisectors. Also, no measurement is without some inaccuracy, and the effects of inherent point localization errors magnifies the difficulty of determining the intersection point of the two (probably inaccurate) perpendicular bisectors. Soudan et al. (1979) described the difficulty in locating the ICR using the standard Method of Reuleaux, and provided a more accurate “tangent method” (see Figure 5.5). However, even the tangent method requires accurate knowledge of intersections defining the location of the ICR. Knowledge of the poloids is considered to be kinematically equivalent to knowing the relative motions between the two rigid bodies.
Figure 5.5. Two methods for finding the instantaneous center of rotation (ICR) for a single degree of freedom joint. A. The Method of Reuleaux uses the positions (points $P$ and $Q$) of two markers or landmarks at time $t$. At time $t + \Delta t$, $P$ and $Q$ are found at positions $P'$ and $Q'$. Straight connecting lines are drawn between $P$ and $P'$, and between $Q$ and $Q'$. Then, perpendicular bisectors are drawn from the connecting lines. The location where the perpendicular bisectors intersect is the approximate, time-averaged location of the ICR ($\times$) for the time interval $\Delta t$. B. The Tangent Method uses the instantaneous velocities of points $P$ and $Q$. Lines are drawn through $P$ and $Q$ in directions perpendicular to the velocity vectors $N \overrightarrow{P'}$ and $N \overrightarrow{Q'}$. Where the perpendicular lines cross is the instantaneous location of the ICR ($\times$).

Figure 5.6. ICR locations during slipping (I), sliding (II), and rolling (III), for a wheel moving relative to the ground. A. Slipping is defined when the wheel rotates but its axle $B'$ remains stationary. The ICR location(s) can be considered to be any point along the wheel axis, here shown at the hub of the wheel ($\times$). B. In sliding, the wheel translates without rotation. In this case, the location of the ICR is undefined, as it is at $\pm \infty$. C. When a wheel rolls upon the ground, the points of the wheel and the ground that are in contact have a common velocity of zero. In this case, the ICR is located at the contact point ($\times$).
settle for a smooth curve drawn between successive marker positions sampled at discrete intervals of time.

Though the ICR is rather limited as a descriptor of joint motion, the ICR has been used to describe gross orthopaedic problems in joints. For instance, Nordin and Frankel (1989) describe how distraction and compression of the knee can be diagnosed based upon the relative positions of the ICR and the articulating joint surface. To illustrate how the ICR defines the kinematic relationships between two contacting bodies, consider two rigid bodies $A$ and $B$ moving relative to each other. For clarity, we will first consider one body $B$ to be a wheel, and the other body $A$ to be a horizontal surface (Figure 5.7A). For the purposes of discussion, $B$ and $A$ contact each other at points $\hat{B}$ of rigid body $B$ and $\hat{A}$ of rigid body $A$. The center of the wheel $B$ is point $B^*$. Figure 5.7 provides a physiological analogy to these cases.

- **Case I.** If a wheel was revolving ($^A\omega^B \neq 0$) but its axle remained fixed relative to the ground ($^A\vartheta^B^* = 0$), then the ICR of the wheel relative to the ground is at the hub of the wheel. This case may be described as **slipping**, and is characterized by *rotation without translation*.

- **Case II.** If a wheel purely translates relative to the ground ($^A\vartheta^B^* \neq 0$ and $^A\omega^B = 0$), then the location of the ICR is undefined, as it is located vertically upward or downward at a distance of infinity from the wheel-ground interface. This case is called **sliding**, and is characterized by *translation without rotation*.
Case III. When the velocities of the contacting points \( B \) and \( A \) are equal, \( \mathbf{N}^B = \mathbf{N}^A \) where \( N \) is a third reference frame, this defines the condition for rolling without slip at the interface. When a wheel rolls on the ground without slipping, the ICR is at the point of ground-wheel contact, because the part of the wheel touching the ground must have the same velocity as the part of the ground touching the wheel.

Usually, in biomechanical joints, the ICR will be located away from the area of contact between two bones, but not infinitely far away as in Case II. This means that the bone is not purely rolling, nor purely translating. In smooth joint motions such an ICR pathway is likely to describe a mixture of rolling, slipping and sliding. The farther away the point of contact and the ICR are, the more sliding, slipping, or translating will tend to be evident. In “jerky” joint motions such as a smooth rotation interrupted by an abrupt translation, the ICR will abruptly translate away from the smooth ICR pathway. Thus the ICR is potentially a very sensitive indicator of joint derangement.

It is possible this sensitivity can be exploited for clinical uses provided a long list of details are adequately addressed. For instance, the motions of the bones themselves (not the positions of markers mounted to the skin) should be used to determine the ICR pathway. If the motion is planar, the images must be viewed exactly perpendicular to the plane containing the motion. In the past, X-rays were sometimes used for this and it was difficult to keep the axis of joint rotation parallel to the path of the X-ray beam. Finally, if the motion is not completely planar, the ICR should actually be replaced by an instantaneous axis of rotation, or IAR. The IAR replaces a 3-D transformation by an equivalent rotation in the same way that the ICR does this for 2-D motions. If a 2-D approximation to 3-D motion is desired, one must first define the plane containing the primary components of the motion. The ICR is the intersection point of the IAR with this plane. Because the IAR is unlikely to remain perpendicular to this plane throughout the motion, the plane’s location is usually fixed along the axes of the long bones and along the midline of the joint.

5.4.2 THE MOMENT PRODUCED BY A SPANNING FORCE

When two rigid bodies joined by a SDOF joint are spanned by a tension producing element (muscle-tendon actuator or ligament), the forces applied to each rigid body exert equal and opposite moments on the respective bodies. Consider the case of two bodies \( A \) and \( B \) in contact at point \( P \) (Figure 5.8). If a tension-producing element spans the joint and attaches to point \( O \) of body \( A \) and \( J \) of body \( B \), the moment applied by the tensile force \( \mathbf{T} \) on \( A \) about the point \( P \) will be equal and opposite to the moment applied by the opposite tensile force \(-\mathbf{T}\) on \( B \) about \( P \).
the adjacent rigid bodies. Here, $\vec{F}$ is considered to point away from $EO$ and toward $EI$.

Sometimes, additional points along the musculotendon pathway are defined, called “via points.” Via points are defined to route the pathway along the outside of bony surfaces, particularly for joint extensors. While via points are not required to determine the muscle moment applied to the segments adjacent to the joint, the via points do help to define the length of the musculotendon actuator. The muscletendon path is usually approximated by straight lines between the via points, so that the resulting pathway is piecewise linear. Pandy (1999) provides additional details regarding via points and the general computation of joint moments. Good examples of musculotendon paths requiring via point definitions are found in Appendix B.

If multiple muscle activation levels are used to calculate $\vec{M}$ for a single joint configuration (as they are here), it is more convenient to define the “unit moment” $\vec{m}_u$ (the moment due to a unit force), so computations of the cross product above can be minimized. Let $\vec{p}^{EO\,EI}$ refer to the vector from the effective origin to the effective insertion. Then, the unit moment is,

$$\vec{m}_u = \vec{p} \times \frac{\vec{p}^{EO\,EI}}{||\vec{p}^{EO\,EI}||}.$$  \hfill (5.23)

Once $\vec{m}_u$ is known, the joint moment due to the musculotendon force is easily calculated given any magnitude of $\vec{F}$,

$$\vec{M} = \pm |\vec{F}| \vec{m}_u.$$  \hfill (5.24)

The sign of $\vec{M}$ is defined to be consistent with the angular definitions, with a positive moment causing a positive segmental or joint acceleration. $\vec{m}_u$ can also be used to define the (scalar) moment arm $\rho$ of $\vec{F}$ about $C$, which is a commonly used measure of the effectiveness of $\vec{F}$ in producing moments of

---

**Figure 5.10.** A muscle pathway spanning two rigid bodies $A$ and $B$ is represented by 3-segment line-of-action. The bodies are assumed to be in contact at a single point $C$. The origin is labeled point $O$, and the insertion, point $I$. Because the musculotendon pathway traverses around the bony prominences of the joint, an effective origin $EO$ and an effective insertion point $EI$ is defined for the purpose of computing muscle moments.

(The corner of the O was clipped off a bit, that’s all!)
position of body B relative to body A. This can be thought of as the rotational analog of work, much like the work done in twisting a torsional spring,

$$ W_{rotating} = \int_0^{\theta+\Delta\theta} |\vec{\tau}| \, d\theta. $$

(5.28)

Since energy is not stored or dissipated in the system, all of the work done in shortening the muscle will go into rotation, and these two work quantities will be equal to each other. For small changes in length $\Delta\ell$ and small changes in angle $\Delta\theta$, we can simplify these expressions to

$$ |\vec{F}| \Delta\ell = |\vec{\tau}| \Delta\theta, $$

(5.29)

or,

$$ \frac{\Delta\ell}{\Delta\theta} = \frac{|\vec{\tau}|}{|\vec{F}|}. $$

(5.30)

In the limit, as $\Delta\ell$ and $\Delta\theta$ approach infinitesimally small values, we can express Equation 5.30 using differentials,

$$ \frac{d\ell}{d\theta} = \frac{|\vec{\tau}|}{|\vec{F}|}. $$

(5.31)

At this point, another set of simplifying assumptions must be made. First, it is acknowledged that the effects of gravity are significant. Yet, gravitational contributions to the torque are ignored because only the muscle’s contributions to the torque are desired. Second, the muscle contraction is assumed to be performed under quasistatic conditions, so that any inertial forces due to accelerations of the limbs A and B are negligibly small. Making these assumptions simplifies the computation of the torque, which is computed by finding the moment of the muscle force and the joint reaction force. Third, it is assumed that the moments of the contact forces are zero about some point (either a focal point or a single contact point). By computing all moments about that point, the moment due to the joint reaction force is also eliminated. Only the muscle force remains, and the magnitude of the torque $|\vec{\tau}|$ can be expressed as,

$$ |\vec{\tau}| = |\vec{\tau} \times \vec{F}| $$

$$ = |\vec{\tau}| |\vec{F}| \sin \beta, $$

(5.32)  

(5.33)

where $\vec{r}$ is a vector from the contact point to any point along the line of action of the muscle force, and $\beta$ is the angle between $\vec{r}$ and $\vec{F}$.

Therefore, the equation becomes:

$$ \frac{d\ell}{d\theta} = \frac{|\vec{\tau}| |\vec{F}| \sin \beta}{|\vec{F}|} $$

(5.34)
of resistance to the joint velocities, and makes intuitive sense as the joint capsules are infused with synovial fluids. These should be added any time a particular joint approaches the limits of its range of motion.\footnote{For inverse dynamic analyses, the joint ranges of motion are known \textit{a-priori}, and thus it is easy to determine whether the passive moments should be added to a joint model. For dynamic simulations, they should always be added!} If the \( n \) degrees of freedom are defined in terms of segmental angles \( q_i \), the segmental angles must be converted into joint angles \( \theta \) and joint angular velocities \( \dot{\theta} \) by performing the proper calculations, \( \theta = q_i - q_j \) and angular velocities \( \dot{\theta} = \dot{q}_i - \dot{q}_j \) (see Figure 5.11). The user should also take care to include the contributions of joint moments properly within the set of dynamic equations expressed in the form of segmental torques.\footnote{See Section 6.6.6 in the next chapter for more on this subject.}

\textbf{Figure 5.11.} Segmental angles \((q_1, q_2)\) must be converted into joint angles for passive joint moment calculations. In this example, the knee angle \( \theta_{\text{knee}} = q_2 - q_1 \).

\textit{Figure 5.12.} Double exponential function used to generate the passive joint moment. “Breakpoints” \( \theta_1 \) and \( \theta_2 \) are prescribed well inside the joint’s range of motion. Torque magnitudes increase rapidly when \( \theta \) lies outside of the range \( \theta_1 < \theta < \theta_2 \). The odd symmetry of these curves serves to apply moments which restore the joint angle to its normal range of motion.
Using joint angles in radians and joint angular velocities in radians per second, the passive joint moments take the form,

\[ M_{i, \text{pass}}(\theta, \dot{\theta}) = k_1 e^{-k_2(\theta - \theta_1)} - k_3 e^{-k_4(\theta_2 - \theta)} - c_1 \dot{\theta}, \]

(5.38)

where \( M_{i, \text{pass}}(\theta) \) is the measure number of the joint flexion or extension moment expressed in \( N \cdot m \). For joint angle \( \theta_i \) and joint axis \( \hat{k}_i \), the scalar contributions are defined as,

\[ M_{i, \text{pass}}(\theta, \dot{\theta}) = \hat{M}_{i, \text{pass}}(\theta) \cdot \hat{k}_i. \]

(5.39)

With one equation, this function approximates the passive moment contributions provided by the joint structures at both extremes of joint motion. It provides for large restoring moments (i.e., the joint is stiff) when \( \theta \) falls outside the range set by the breakpoints \( \theta_1 \) and \( \theta_2 \) (when \( \theta < \theta_1 \) or \( \theta > \theta_2 \)). The exponents in the double exponential must be negative between the breakpoints, when \( \theta_1 < \theta < \theta_2 \). Otherwise, the passive moment will attain unrealistically large values within the joint range of motion. Note that \( k_2 \) and \( k_4 \) govern the sharpness of the break, and that \( \theta_1 \) and \( \theta_2 \) set the location of the breakpoints in \( \theta \) but \( \theta_1 \) and \( \theta_2 \) are well inside of the limits of the joint range of motion. Generally speaking, at \( \theta = \theta_1 \), the moment is approximately equal to \( k_1 \), and at \( \theta = \theta_2 \), the moment has an approximate magnitude of \( -k_3 \).

Figure 5.12 shows a typical curve of passive moment and the constants used to obtain it. When damping coefficients are not known, typical models have used a value of \( c_1 = 0.1 \), which provides 1 unit of torque for every 10 radians per second.

Example: Passive moment curves for flexion/extension in the lower extremity. Figure 5.13 shows the resultant curves of passive torque developed under quasistatic conditions (\( \dot{\theta} \approx 0 \)) from Equation 5.38 for the ankle, knee, and hip joints.

### 5.5.1 KNEE JOINT EXTENSION

The structure and function of the knee is a complex topic, because there are three bones which contribute to the mechanics of knee extension. Many excellent studies have been reported in the literature, and each has contributed to our understanding (see the Reference list for a selection). However, the mechanical interactions of the patella, tibia, and femur can be condensed into a single degree of freedom model with a few simplifications (Figure 5.14, Yamaguchi and Zajac, 1989). Because the quasistatic knee extension model serves as a good example of the preceding material, it is included briefly here. For more details, the reader is referred to the original paper.

Modification of the procedure for computing joint moments is required at the knee joint, because of the interactions of the patellofemoral and tibiofemoral
joints. Accuracy of the computed extensor moment arm is considered to be critical at the knee because large muscular forces are exerted across small extensor moment arms. Small absolute errors in moment arm thus yield large errors in extensor moment of force. In developing the SDOF knee model, what is desired is a model that incorporates the essential mechanics of knee extension without being too complex to use in whole-body simulations of movement.

Despite its simplicity, the model is able to reproduce the internal configuration of the joint (angles $\alpha$, $\beta$, and the positions of the tibiofemoral and patellofemoral contact points) at various flexion angles ($\theta$). Also, the model uses a static force balance to compute the reduction in the force transmitted from the quadriceps tendon to the patellar ligament. This allows the model to compensate for the differences between the “actual moment arm” ($m_{act}$) and the “effective extensor moment arm” ($m_{eff}$). As shown in Figure 5.14, $m_{act}$ is defined as the perpendicular distance from the centroid of the patellar ligament force to the tibiofemoral contact point, $m_{eff}$ is the apparent moment arm of the quadriceps muscle when tension is produced, and $m_{eff}$ is the moment arm that is desired during movement simulation studies, because knowledge of the effective moment arm allows the extension moment to be computed more easily from the quadriceps muscle tension $F_q = |\vec{F}_q|$. 

Figure 5.14. The planar, static model of the knee used to compute the knee effective extensor moment arm. Orientations of the patellar and patellar ligament axes ($\alpha$, $\beta$) are determined in order to balance the forces $\vec{F}_r$, $\vec{F}_pl$ and moments governing patellar equilibria given an applied quadriceps force and direction ($\vec{F}_q$ and $\theta_q$). Tibiofemoral joint motions are defined by a prescribed pathway of instantaneous centers of joint rotation to yield proper locations of tibiofemoral contact with joint flexion. Once the actual moment arm ($m_{act}$) is determined, the ratio of force transmitted through the patella ($|\vec{F}_{pl}|/|\vec{F}_q|$) is used to express the effective extensor moment arm in terms of the applied quadriceps force, $m_{eff} = m_{act}|\vec{F}_{pl}|/|\vec{F}_q|$. 

\[ m_{act} = \text{circle of radius } l_{pl} \]
Defining precisely how the knee extends is complicated by an axis of tibiofemoral joint rotation that moves and changes orientation with flexion and extension. It is generally agreed in the literature that the femur rolls backward on the tibial plateau during the initiation of flexion from an extended position. Because the length of the tibial plateau is only about half that of the corresponding articulating surface on the femoral condyles, the femur begins to slip increasingly until its backward motion is halted by the anterior and posterior cruciate ligaments. Because of this, the ICR should be located at the tibiofemoral contact at full extension, and should rise off of the surface with increasing knee flexion. Experimental data measuring the ICR location is not useful due to inaccuracies. An example of this was illustrated previously in Figure 5.4. The solution prescribes the tibiofemoral motion by creating a pathway of the ICR that delivers proper contact locations and interface conditions.

The key to understanding the action of the patellofemoral joint is realizing that the patella functions both as a spacer and a lever. The spacing function increases the displacement of the patellar ligament away from the tibiofemoral contact point $C_{TF}$ and increases $m_{act}$. However, because the patella rocks about its contact with the femur, it also functions as a force reducing lever to alter the force transmitted from the quadriceps tendon to the patellar ligament (Grood et al., 1984).

In the knee model, the patellar leveraging effect was computed by numerically solving the equations governing static equilibria of the patellar mechanism. The force balance equation includes the unknown patellofemoral joint reaction ($F_r = |F_r|$), Figure 5.14) and patellar ligament ($F_{pl} = |F_{pl}|$) tensions as functions of the unknown patellar and patellar ligament orientations ($\alpha$ and
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