## Contents

<table>
<thead>
<tr>
<th>Preface</th>
<th>vii</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1 Let’s Count!</strong></td>
<td>1</td>
</tr>
<tr>
<td>1.1 A Party</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Sets and the Like</td>
<td>4</td>
</tr>
<tr>
<td>1.3 The Number of Subsets</td>
<td>9</td>
</tr>
<tr>
<td>1.4 The Approximate Number of Subsets</td>
<td>14</td>
</tr>
<tr>
<td>1.5 Sequences</td>
<td>15</td>
</tr>
<tr>
<td>1.6 Permutations</td>
<td>17</td>
</tr>
<tr>
<td>1.7 The Number of Ordered Subsets</td>
<td>19</td>
</tr>
<tr>
<td>1.8 The Number of Subsets of a Given Size</td>
<td>20</td>
</tr>
<tr>
<td><strong>2 Combinatorial Tools</strong></td>
<td>25</td>
</tr>
<tr>
<td>2.1 Induction</td>
<td>25</td>
</tr>
<tr>
<td>2.2 Comparing and Estimating Numbers</td>
<td>30</td>
</tr>
<tr>
<td>2.3 Inclusion-Exclusion</td>
<td>32</td>
</tr>
<tr>
<td>2.4 Pigeonholes</td>
<td>34</td>
</tr>
<tr>
<td>2.5 The Twin Paradox and the Good Old Logarithm</td>
<td>37</td>
</tr>
<tr>
<td><strong>3 Binomial Coefficients and Pascal’s Triangle</strong></td>
<td>43</td>
</tr>
<tr>
<td>3.1 The Binomial Theorem</td>
<td>43</td>
</tr>
<tr>
<td>3.2 Distributing Presents</td>
<td>45</td>
</tr>
<tr>
<td>3.3 Anagrams</td>
<td>46</td>
</tr>
<tr>
<td>3.4 Distributing Money</td>
<td>48</td>
</tr>
</tbody>
</table>
3.5 Pascal’s Triangle ................................................. 49
3.6 Identities in Pascal’s Triangle ................................. 50
3.7 A Bird’s-Eye View of Pascal’s Triangle ..................... 54
3.8 An Eagle’s-Eye View: Fine Details ......................... 57

4 Fibonacci Numbers ............................................ 65
4.1 Fibonacci’s Exercise ............................................ 65
4.2 Lots of Identities .............................................. 68
4.3 A Formula for the Fibonacci Numbers ....................... 71

5 Combinatorial Probability .................................... 77
5.1 Events and Probabilities ..................................... 77
5.2 Independent Repetition of an Experiment .................. 79
5.3 The Law of Large Numbers ................................. 80
5.4 The Law of Small Numbers and the Law of Very Large Numbers ............................................. 83

6 Integers, Divisors, and Primes ............................... 87
6.1 Divisibility of Integers ...................................... 87
6.2 Primes and Their History ................................... 88
6.3 Factorization into Primes .................................. 90
6.4 On the Set of Primes ....................................... 93
6.5 Fermat’s “Little” Theorem ................................ 97
6.6 The Euclidean Algorithm ................................. 99
6.7 Congruences .................................................. 105
6.8 Strange Numbers .......................................... 107
6.9 Number Theory and Combinatorics ...................... 114
6.10 How to Test Whether a Number is a Prime? .......... 117

7 Graphs .................................................................... 125
7.1 Even and Odd Degrees ................................... 125
7.2 Paths, Cycles, and Connectivity ......................... 130
7.3 Eulerian Walks and Hamiltonian Cycles ............... 135

8 Trees .................................................................. 141
8.1 How to Define Trees ...................................... 141
8.2 How to Grow Trees ....................................... 143
8.3 How to Count Trees? .................................... 146
8.4 How to Store Trees ....................................... 148
8.5 The Number of Unlabeled Trees ....................... 153

9 Finding the Optimum ......................................... 157
9.1 Finding the Best Tree .................................... 157
9.2 The Traveling Salesman Problem ....................... 161

10 Matchings in Graphs ........................................ 165
10.1 A Dancing Problem ................. 165
10.2 Another matching problem .......... 167
10.3 The Main Theorem ................. 169
10.4 How to Find a Perfect Matching .... 171

11 Combinatorics in Geometry .... 179
11.1 Intersections of Diagonals ........ 179
11.2 Counting regions ............. 181
11.3 Convex Polygons ........... 184

12 Euler’s Formula .............. 189
12.1 A Planet Under Attack ........ 189
12.2 Planar Graphs ............. 192
12.3 Euler’s Formula for Polyhedra .... 194

13 Coloring Maps and Graphs ... 197
13.1 Coloring Regions with Two Colors .... 197
13.2 Coloring Graphs with Two Colors .... 199
13.3 Coloring graphs with many colors .... 202
13.4 Map Coloring and the Four Color Theorem .. 204

14 Finite Geometries, Codes, Latin Squares, and Other Pretty Creatures .. 211
14.1 Small Exotic Worlds ............ 211
14.2 Finite Affine and Projective Planes ... 217
14.3 Block Designs ............... 220
14.4 Steiner Systems ............. 224
14.5 Latin Squares .............. 229
14.6 Codes .................. 232

15 A Glimpse of Complexity and Cryptography ... 239
15.1 A Connecticut Class in King Arthur’s Court .... 239
15.2 Classical Cryptography ........ 242
15.3 How to Save the Last Move in Chess ... 244
15.4 How to Verify a Password—Without Learning it ... 246
15.5 How to Find These Primes ..... 246
15.6 Public Key Cryptography ..... 247

16 Answers to Exercises .................. 251