

Contents

Preface	xiv
Acknowledgments	xvii
Introduction	xx
I Background	1
1 Preliminaries	2
1.1 Notation and conventions	2
1.2 Basics from measure theory	4
2 Computability Theory	7
2.1 Computable functions, coding, and the halting problem .	7
2.2 Computable enumerability and Rice's Theorem	11
2.3 The Recursion Theorem	13
2.4 Reductions	15
2.4.1 Oracle machines and Turing reducibility	16
2.4.2 The jump operator and jump classes	18
2.4.3 Strong reducibilities	19
2.4.4 Myhill's Theorem	21
2.5 The arithmetic hierarchy	23
2.6 The Limit Lemma and Post's Theorem	24

2.7	The difference hierarchy	27
2.8	Primitive recursive functions	28
2.9	A note on reductions	29
2.10	The finite extension method	31
2.11	Post’s Problem and the finite injury priority method . .	34
2.12	Finite injury arguments of unbounded type	39
	2.12.1 The Sacks Splitting Theorem	39
	2.12.2 The Pseudo-Jump Theorem	41
2.13	Coding and permitting	42
2.14	The infinite injury priority method	44
	2.14.1 Priority trees and guessing	44
	2.14.2 The minimal pair method	47
	2.14.3 High computably enumerable degrees	53
	2.14.4 The Thickness Lemma	56
2.15	The Density Theorem	58
2.16	Jump theorems	64
2.17	Hyperimmune-free degrees	67
2.18	Minimal degrees	70
2.19	Π_1^0 and Σ_1^0 classes	72
	2.19.1 Basics	72
	2.19.2 Π_n^0 and Σ_n^0 classes	75
	2.19.3 Basis theorems	77
	2.19.4 Generalizing the low basis theorem	81
2.20	Strong reducibilities and Post’s Program	82
2.21	PA degrees	84
2.22	Fixed-point free and diagonally noncomputable functions	87
2.23	Array noncomputability and traceability	93
2.24	Genericity and weak genericity	100

3 Kolmogorov Complexity of Finite Strings **110**

3.1	Plain Kolmogorov complexity	110
3.2	Conditional complexity	114
3.3	Symmetry of information	116
3.4	Information-theoretic characterizations of computability	117
3.5	Prefix-free machines and complexity	121
3.6	The KC Theorem	125
3.7	Basic properties of prefix-free complexity	128
3.8	Prefix-free randomness of strings	132
3.9	The Coding Theorem and discrete semimeasures	133
3.10	Prefix-free symmetry of information	134
3.11	Initial segment complexity of sets	136
3.12	Computable bounds for prefix-free complexity	137
3.13	Universal machines and halting probabilities	139
3.14	The conditional complexity of σ^* given σ	143
3.15	Monotone and process complexity	145

3.16	Continuous semimeasures and KM -complexity	150
4	Relating Complexities	154
4.1	Levin's Theorem relating C and K	154
4.2	Solovay's Theorems relating C and K	155
4.3	Strong K -randomness and C -randomness	161
	4.3.1 Positive results	161
	4.3.2 Counterexamples	162
4.4	Muchnik's Theorem on C and K	168
4.5	Monotone complexity and KM -complexity	169
5	Effective Reals	197
5.1	Computable and left-c.e. reals	197
5.2	Real-valued functions	202
5.3	Representing left-c.e. reals	203
	5.3.1 Degrees of representations	203
	5.3.2 Presentations of left-c.e. reals	206
	5.3.3 Presentations and ideals	208
	5.3.4 Promptly simple sets and presentations	215
5.4	Left-d.c.e. reals	217
	5.4.1 Basics	217
	5.4.2 The field of left-d.c.e. reals	219
II	Notions of Randomness	225
6	Martin-Löf Randomness	226
6.1	The computational paradigm	227
6.2	The measure-theoretic paradigm	229
6.3	The unpredictability paradigm	234
	6.3.1 Martingales and supermartingales	234
	6.3.2 Supermartingales and continuous semimeasures	238
	6.3.3 Martingales and optimality	239
	6.3.4 Martingale processes	241
6.4	Relativizing randomness	245
6.5	Ville's Theorem	246
6.6	The Ample Excess Lemma	250
6.7	Plain complexity and randomness	252
6.8	n -randomness	254
6.9	Van Lambalgen's Theorem	257
6.10	Effective 0-1 laws	259
6.11	Infinitely often maximally complex sets	260
6.12	Randomness for other measures and degree-invariance	263
	6.12.1 Generalizing randomness to other measures	263
	6.12.2 Computable measures and representing reals	265

7	Other Notions of Algorithmic Randomness	269
7.1	Computable randomness and Schnorr randomness	270
7.1.1	Basics	270
7.1.2	Limitations	275
7.1.3	Computable measure machines	277
7.1.4	Computable randomness and tests	279
7.1.5	Bounded machines and computable randomness	281
7.1.6	Process complexity and computable randomness	282
7.2	Weak n -randomness	285
7.2.1	Basics	285
7.2.2	Characterizations of weak 1-randomness	290
7.2.3	Schnorr randomness via Kurtz null tests	293
7.2.4	Weakly 1-random left-c.e. reals	295
7.2.5	Solovay genericity and randomness	296
7.3	Decidable machines	298
7.4	Selection revisited	301
7.4.1	Stochasticity	301
7.4.2	Partial computable martingales and stochasticity	303
7.4.3	A martingale characterization of stochasticity	308
7.5	Nonmonotonic randomness	309
7.5.1	Nonmonotonic betting strategies	309
7.5.2	Van Lambalgen’s Theorem revisited	313
7.6	Demuth randomness	315
7.7	Difference randomness	316
7.8	Finite randomness	318
7.9	Injective and permutation randomness	319
8	Algorithmic Randomness and Turing Reducibility	323
8.1	Π_1^0 classes of 1-random sets	324
8.2	Computably enumerable degrees	324
8.3	The Kučera-Gács Theorem	325
8.4	A “no gap” theorem for 1-randomness	327
8.5	Kučera coding	330
8.6	Demuth’s Theorem	333
8.7	Randomness relative to other measures	334
8.8	Randomness and PA degrees	336
8.9	Mass problems	344
8.10	DNC degrees and subsets of random sets	347
8.11	High degrees and separating notions of randomness	349
8.11.1	High degrees and computable randomness	349
8.11.2	Separating notions of randomness	350
8.11.3	When van Lambalgen’s Theorem fails	357
8.12	Measure theory and Turing reducibility	358
8.13	n -randomness and weak n -randomness	360
8.14	Every 2-random set is GL_1	363

8.15	Stillwell's Theorem	364
8.16	DNC degrees and autocomplexity	366
8.17	Randomness and n -fixed-point freeness	370
8.18	Jump inversion	373
8.19	Pseudo-jump inversion	376
8.20	Randomness and genericity	378
	8.20.1 Similarities between randomness and genericity	378
	8.20.2 n -genericity versus n -randomness	379
8.21	Properties of almost all degrees	381
	8.21.1 Hyperimmunity	381
	8.21.2 Bounding 1-generics	383
	8.21.3 Every 2-random set is CEA	386
	8.21.4 Where 1-generic degrees are downward dense	394
 III Relative Randomness		 403
9	Measures of Relative Randomness	404
9.1	Solovay reducibility	405
9.2	The Kučera-Slaman Theorem	408
9.3	Presentations of left-c.e. reals and complexity	411
9.4	Solovay functions and 1-randomness	412
9.5	Solovay degrees of left-c.e. reals	413
9.6	cl-reducibility and rK-reducibility	419
9.7	K -reducibility and C -reducibility	425
9.8	Density and splittings	427
9.9	Monotone degrees and density	432
9.10	Further relationships between reducibilities	433
9.11	A minimal rK-degree	437
9.12	Complexity and completeness for left-c.e. reals	439
9.13	cl-reducibility and the Kučera-Gács Theorem	441
9.14	Further properties of cl-reducibility	442
	9.14.1 cl-reducibility and joins	442
	9.14.2 Array noncomputability and joins	444
	9.14.3 Left-c.e. reals cl-reducible to versions of Ω	447
	9.14.4 cl-degrees of versions of Ω	454
9.15	K -degrees, C -degrees, and Turing degrees	456
9.16	The structure of the monotone degrees	459
9.17	Schnorr reducibility	462
10	Complexity and Relative Randomness for 1-Randoms Sets	464
10.1	Uncountable lower cones in \leq_K and \leq_C	464
10.2	The K -complexity of Ω and other 1-random sets	466
	10.2.1 $K(A \upharpoonright n)$ versus $K(n)$ for 1-random sets A	466
	10.2.2 The rate of convergence of Ω and the α function	467

10.2.3	Comparing complexities of 1-random sets	468
10.2.4	Limit complexities and relativized complexities	471
10.3	Van Lambalgen reducibility	473
10.3.1	Basic properties of the van Lambalgen degrees	474
10.3.2	Relativized randomness and Turing reducibility	475
10.3.3	vL-reducibility, K -reducibility, and joins	476
10.3.4	vL-reducibility and C -reducibility	478
10.3.5	Contrasting vL-reducibility and K -reducibility	479
10.4	Upward oscillations and \leq_K -comparable 1-random sets	481
10.5	LR-reducibility	489
10.6	Almost everywhere domination	495
11	Randomness-Theoretic Weakness	500
11.1	K -triviality	500
11.1.1	The basic K -triviality construction	501
11.1.2	The requirement-free version	502
11.1.3	Solovay functions and K -triviality	504
11.1.4	Counting the K -trivial sets	505
11.2	Lowness	507
11.3	Degrees of K -trivial sets	511
11.3.1	A first approximation: wtt-incompleteness	511
11.3.2	A second approximation: impossible constants	512
11.3.3	The less impossible case	514
11.4	K -triviality and lowness	518
11.5	Cost functions	526
11.6	The ideal of K -trivial degrees	529
11.7	Bases for 1-randomness	531
11.8	ML-covering, ML-cupping, and related notions	534
11.9	Lowness for weak 2-randomness	536
11.10	Listing the K -trivial sets	538
11.11	Upper bounds for the K -trivial sets	541
11.12	A gap phenomenon for K -triviality	550
12	Lowness and Triviality for Other Randomness Notions	554
12.1	Schnorr lowness	554
12.1.1	Lowness for Schnorr tests	554
12.1.2	Lowness for Schnorr randomness	559
12.1.3	Lowness for computable measure machines	561
12.2	Schnorr triviality	564
12.2.1	Degrees of Schnorr trivial sets	564
12.2.2	Schnorr triviality and strong reducibilities	568
12.2.3	Characterizing Schnorr triviality	569
12.3	Tracing weak truth table degrees	576
12.3.1	Basics	576
12.3.2	Reducibilities with tiny uses	576

12.3.3	Anti-complex sets and tiny uses	578
12.3.4	Anti-complex sets and Schnorr triviality	580
12.4	Lowness for weak genericity and randomness	581
12.5	Lowness for computable randomness	586
12.6	Lowness for pairs of randomness notions	590
13	Algorithmic Dimension	592
13.1	Classical Hausdorff dimension	592
13.2	Hausdorff dimension via gales	594
13.3	Effective Hausdorff dimension	596
13.4	Shift complex sets	601
13.5	Partial randomness	602
13.6	A correspondence principle for effective dimension	607
13.7	Hausdorff dimension and complexity extraction	608
13.8	A Turing degree of nonintegral Hausdorff dimension	611
13.9	DNC functions and effective Hausdorff dimension	618
13.9.1	Dimension in h -spaces	619
13.9.2	Slow-growing DNC functions and dimension	623
13.10	C -independence and Zimand's Theorem	627
13.11	Other notions of dimension	635
13.11.1	Box counting dimension	635
13.11.2	Effective box counting dimension	636
13.11.3	Packing dimension	638
13.11.4	Effective packing dimension	641
13.12	Packing dimension and complexity extraction	642
13.13	Clumpy trees and minimal degrees	645
13.14	Building sets of high packing dimension	648
13.15	Computable dimension and Schnorr dimension	654
13.15.1	Basics	654
13.15.2	Examples of Schnorr dimension	657
13.15.3	A machine characterization of Schnorr dimension	658
13.15.4	Schnorr dimension and computable enumerability	659
13.16	The dimensions of individual strings	662
IV	Further Topics	667
14	Strong Jump Traceability	668
14.1	Basics	668
14.2	The ideal of strongly jump traceable c.e. sets	672
14.3	Strong jump traceability and K -triviality: the c.e. case	680
14.4	Strong jump traceability and diamond classes	689
14.5	Strong jump traceability and K -triviality: the general case	700
15	Ω as an Operator	705

15.1	Introduction	705
15.2	Omega operators	707
15.3	A -1-random A -left-c.e. reals	709
15.4	Reals in the range of some Omega operator	711
15.5	Lowness for Ω	712
15.6	Weak lowness for K	713
	15.6.1 Weak lowness for K and lowness for Ω	714
	15.6.2 Infinitely often strongly K -random sets	715
15.7	When Ω^A is a left-c.e. real	716
15.8	Ω^A for K -trivial A	719
15.9	K -triviality and left-d.c.e. reals	722
15.10	Analytic behavior of Omega operators	722
16	Complexity of Computably Enumerable Sets	728
16.1	Barzdins' Lemma and Kummer complex sets	728
16.2	The entropy of computably enumerable sets	731
16.3	The collection of nonrandom strings	738
	16.3.1 The plain and conditional cases	738
	16.3.2 The prefix-free case	743
	16.3.3 The overgraphs of universal monotone machines	752
	16.3.4 The strict process complexity case	761
	References	767
	Index	797



<http://www.springer.com/978-0-387-95567-4>

Algorithmic Randomness and Complexity

Downey, R.G.; Hirschfeldt, D.R.

2010, XXVIII, 855 p., Hardcover

ISBN: 978-0-387-95567-4