# Preface

Though we did not know it at the time, this book's genesis began with the arrival of Cris Calude in New Zealand. Cris has always had an intense interest in algorithmic information theory. The event that led to much of the recent research presented here was the articulation by Cris of a seemingly innocuous question. This question goes back to Solovay's legendary manuscript [371], and Downey learned of it during a visit made to Victoria University in early 2000 by Richard Coles, who was then a postdoctoral fellow with Calude at Auckland University. In effect, the question was whether the Solovay degrees of left-computably enumerable reals are dense.

At the time, neither of us knew much about Kolmogorov complexity, but we had a distinct interest in it after Lance Fortnow's illuminating lectures [148] at Kaikoura[1] in January 2000. After thinking about Calude's question for a while, and eventually solving it together with André Nies [116], we began to realize that there was a huge and remarkably fascinating area of research, whose potential was largely untapped, lying at the intersection of computability theory and the theory of algorithmic randomness.

We also found that, while there is a truly classic text on Kolmogorov complexity, namely Li and Vitányi [248], most of the questions we were in-

[1]Kaikoura was the setting for a wonderful meeting on computational complexity. There is a set of lecture notes [112] resulting from this meeting, aimed at graduate students. Kaikoura is on the east coast of the South Island of New Zealand, and is famous for its beauty and for tourist activities such as whale watching and dolphin, seal, and shark swimming. The name "Kaikoura" is a Maori word meaning "eat crayfish", which is a fine piece of advice.

terested in either were open, were exercises in Li and Vitányi with difficulty ratings of about 40-something (out of 50), or necessitated an archaeological dig into the depths of a literature with few standards in notation[2] and terminology, marked by relentless rediscovery of theorems and a significant amount of unpublished material. Particularly noteworthy among the unpublished material was the aforementioned set of notes by Solovay [371], which contained absolutely fundamental results about Kolmogorov complexity in general, and about initial segment complexity of sets in particular. As our interests broadened, we also became aware of important results from Stuart Kurtz' PhD dissertation [228], which, like most of Solovay's results, seemed unlikely ever to be published in a journal. Meanwhile, a large number of other authors started to make great strides in our understanding of algorithmic randomness.

Thus, we decided to try to organize results on the relationship between algorithmic randomness and computability theory into a coherent book. We were especially thankful for Solovay's permission to present, in most cases for the first time, the details from his unpublished notes.[3] We were encouraged by the support of Springer in this enterprise.

Naturally, this project has conformed to Hofstadter's Law: It always takes longer than you expect, even when you take into account Hofstadter's Law. Part of the reason for this delay is that a large contingent of gifted researchers continued to relentlessly prove theorems that made it necessary to rewrite large sections of the book.[4] We think it is safe to say that the study of algorithmic randomness and dimension is now one of the most active areas of research in mathematical logic. Even in a book this size, much has necessarily been left out. To those who feel slighted by these omissions, or by inaccuracies in attribution caused by our necessarily imperfect historical knowledge, we apologize in advance, and issue a heartfelt invitation to write their own books. Any who might feel inclined to thank us will find a suggestion for an appropriate gift on page 517.

This is *not* a basic text on Kolmogorov complexity. We concentrate on the Kolmogorov complexity of sets (i.e., infinite sequences) and cover only as much as we need on the complexity of finite strings. There is quite a lot of background material in computability theory needed for some of the more sophisticated proofs we present, so we do give a full but, by necessity, rapid refresher course in basic "advanced" computability theory. This material

[2] We hope to help standardize notation. In particular, we have fixed upon the notation for Kolmogorov complexity used by Li and Vitányi: $C$ for plain Kolmogorov complexity and $K$ for prefix-free Kolmogorov complexity.

[3] Of course, Li and Vitányi used Solovay's notes extensively, mostly in the exercises and for quoting results.

[4] It is an unfortunate consequence of working on a book that attempts to cover a significant portion of a rapidly expanding area of research that one begins to hate one's most productive colleagues a little.

should not be read from beginning to end. Rather, the reader should dip into Chapter 2 as the need arises. For a fuller introduction, see for instance Rogers [334], Soare [366], Odifreddi [310, 311], or Cooper [79].

We will mostly avoid historical comments, particularly about events predating our entry into this area of research. The history of the evolution of Kolmogorov complexity and related topics can make certain people rather agitated, and we feel neither competent nor masochistic enough to enter the fray. What seems clear is that, at some stage, time was ripe for the evolution of the ideas needed for Kolmogorov complexity. There is no doubt that many of the basic ideas were implicit in Solomonoff [369], and that many of the fundamental results are due to Kolmogorov [211]. The measure-theoretic approach was pioneered by Martin-Löf [259]. Many key results were established by Levin in works such as [241, 242, 243, 425] and by Schnorr [348, 349, 350], particularly those using the measure of domains to avoid the problems of plain complexity in addressing the initial segment complexity of sets. It is but a short step from there to prefix-free complexity (and discrete semimeasures), first articulated by Levin [243] and Chaitin [58]. Schnorr's penetrating ideas, only some of which are available in their original form in English, are behind much modern work in computational complexity, as well as Lutz' approach to effective Hausdorff dimension in [252, 254], which is based on martingales and orders. As has often been the case in this area, however, Lutz developed his material without being too aware of Schnorr's work, and was apparently the first to explicitly connect orders and Hausdorff dimension. From yet another perspective, martingales, or rather supermartingales, are essentially the same as continuous semimeasures, and again we see the penetrating insight of Levin (see [425]).

We are particularly pleased to present the results of Kurtz and Solovay mentioned above, as well as hitherto unpublished material from Steve Kautz' dissertation [200] and the fundamental work of Antonin Kučera. Kučera was a real pioneer in connecting computability and randomness, and we believe that it is only recently that the community has really appreciated his deep intuition.

Algorithmic randomness is a highly active field, and still has many fascinating open questions and unexplored directions of research. Recent lists of open questions include Miller and Nies [278] and the problem list [2] arising from a workshop organized by Hirschfeldt and Miller at the American Institute of Mathematics in 2006. Several of the questions on these lists have already been solved, however, with many of the solutions appearing in this book. We will mention a number of open questions below, some specific, some more open ended. The pages on which these occur are listed in the index under the heading *open question*.

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