Preface

This is a focused exercise book in algebra.

Facility in algebra is important for any student who wants to study advanced mathematics or science. An algebraic expression is a carrier of information. Sometimes it is easy to extract the information from the form of the expression; sometimes the information is latent, and the expression has to be altered to yield it up. Thus, students must learn to manipulate algebraic expressions judiciously with a sense of strategy. This sense of working towards a goal is lacking in many textbook exercises, so that students fail to gain a sense of the coherence of mathematics and so find it difficult if not impossible to acquire any significant degree of skill.

Pell's equation seems to be an ideal topic to lead college students, as well as some talented and motivated high school students, to a better appreciation of the power of mathematical technique. The history of this equation is long and circuituous. It involved a number of different approaches before a definitive theory was found. Numbers have fascinated people in various parts of the world over many centuries. Many puzzles involving numbers lead naturally to a quadratic Diophantine equation (an algebraic equation of degree 2 with integer coefficients for which solutions in integers are sought), particularly ones of the form $x^2 - dy^2 = k$, where d and k are integer parameters with d nonsquare and positive. A few of these appear in Chapter 2. For about a thousand years, mathematicians had various ad hoc methods of solving such equations, and it slowly became clear that the equation $x^2 - dy^2 = 1$ should always have positive integer solutions other than $(x, y) = (1, 0)$. There were some partial patterns and some quite effective methods of finding solutions, but a complete theory did not emerge until the end of the eighteenth century. It is unfortunate that the equation is named after a seventeenth-century English mathematician, John Pell, who, as far as anyone can tell, had hardly anything to do with it. By his time, a great deal of spadework had been done by many Western European mathematicians. However, Leonhard Euler, the foremost European mathematician of the eighteenth century, who *did* pay a lot of attention to the equation, referred to it as "Pell's equation" and the name stuck.

In the first three chapters of the book the reader is invited to explore the situation, come up with some personal methods, and then match wits with early Indian and

European mathematicians. While these investigators were pretty adept at arithmetic computations, you might want to keep a pocket calculator handy, because sometimes the numbers involved get pretty big. Just try to solve $x^2 - 61y^2 = 1$! So far there is not a clean theory for the higher-degree analogues of Pell's equation, although a great deal of work was done on the cubic equation by such investigators as A. Cayley, P.H. Daus, G.B. Mathews, and E.S. Selmer in the late nineteenth and early twentieth century; the continued fraction technique seems to be so special to the quadratic case that it is hard to see what a proper generalization might be. As sometimes happens in mathematics, the detailed study of particular cases becomes less important and research becomes more focused on general structure and broader questions. Thus, in the last fifty years, the emphasis has been on the properties of larger classes of Diophantine equations. Even the resolution of the Fermat Conjecture, which dealt with a particular type of Diophantine equation, by Andrew Wiles was done in the context of a very broad and deep study. However, this should not stop students from going back and looking at particular cases. Just because professional astronomers have gone on to investigating distant galaxies and seeking knowledge on the evolution of the universe does not mean that the backyard amateur might not find something of interest and value about the solar system.

The subject of this book is not a mathematical backwater. As a recent paper of H.W. Lenstra in the *Notices of the American Mathematical Society* and a survey paper given by H.C. Williams at the Millennial Conference on Number Theory in 2000 indicate, the efficient generation of solutions of an ordinary Pell's equation is a live area of research in computer science. Williams mentions that over 100 articles on the equation have appeared in the 1990s and draws attention to interest on the part of cryptographers. Pell's equation is part of a central area of algebraic number theory that treats quadratic forms and the structure of the rings of integers in algebraic number fields. Even at the specific level of quadratic Diophantine equations, there are unsolved problems, and the higher-degree analogues of Pell's equation, particularly beyond the third, do not appear to have been well studied. This is where the reader might make some progress.

The topic is motivated and developed through sections of exercises that will allow the student to recreate known theory and provide a focus for algebraic practice. There are several *explorations* that encourage the reader to embark on individual research. Some of these are numerical, and often require the use of a calculator or computer. Others introduce relevant theory that can be followed up on elsewhere, or suggest problems that the reader may wish to pursue.

The opening chapter uses the approximations to the square root of 2 to indicate a context for Pell's equation and introduce some key ideas of recursions, matrices, and continued fractions that will play a role in the book. The goal of the second chapter is to indicate problems that lead to a Pell's equation and to suggest how mathematicians approached solving Pell's equation in the past. Three chapters then cover the core theory of Pell's equation, while the sixth chapter digresses to draw out some connections with Pythagorean triples. Two chapters embark on the study of higher-degree analogues of Pell's equation, with a great deal left to the reader to pursue. Finally, we look at Pell's equation modulo a natural number.

I have used some of the material of this book in a fourth-year undergraduate research seminar, as well as with talented high school students. It has also been the basis of workshops with secondary teachers. A high school background in mathematics is all that is needed to get into this book, and teachers and others interested in mathematics who do not have (or have forgotten) a background in advanced mathematics may find that it is a suitable vehicle for keeping up an independent interest in the subject. Teachers could use it as a source of material for their more able students.

There are nine chapters, each subdivided into sections. Within the same chapter, Exercise z in Section y is referred to as *Exercise* $y.z$; if reference is made to an exercise in a different chapter x, it will be referred to as *Exercise* $x.y.z$. The end of an exercise may be indicated by ♠ to distinguish it from explanatory text that follows. Within each chapter there are a number of *Explorations*; these are designed to raise other questions that are in some way connected with the material of the exercises. Some of the explanations may be thought about, and then returned to later when the reader has worked through more of the exercises, since occasionally later work may shed additional light. It is hoped that these explorations may encourage students to delve further into number theory. A glossary of terms appears at the end of the book.

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