Chapter 2
Framework

In this chapter we develop the general framework to be used in this book. The starting point for the discussion will be the standard white noise structures and how constructions of this kind can be given a rigorous treatment. White noise analysis can be addressed in several different ways. The presentation here is to a large extent influenced by ideas and methods used by the authors. In particular, we emphasize the use of multidimensional structures, i.e., the white noise we are about to consider will in general take on values in a multidimensional space and will also be indexed by a multidimensional parameter set.

2.1 White Noise

2.1.1 The 1-Dimensional, $d$-Parameter Smoothed White Noise

Two fundamental concepts in stochastic analysis are *white noise* and *Brownian motion*. The idea of white noise analysis, due to Hida (1980), is to consider white noise rather than Brownian motion as the fundamental object. Within this framework, Brownian motion will be expressed in terms of white noise.

We start by recalling some of the basic definitions and properties of the 1-dimensional white noise probability space. In the following $d$ will denote a fixed positive integer, interpreted as either the time-, space- or time–space dimension of the system we consider. More generally, we will call $d$ the *parameter dimension*. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing smooth ($C^\infty$) real-valued functions on $\mathbb{R}^d$. A general reference for properties of this space is Rudin (1973). $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space under the family of seminorms

$$
\|f\|_{k,\alpha} := \sup_{x \in \mathbb{R}^d} \{(1 + |x|^k)|\partial^\alpha f(x)|\},
$$

(2.1.1)
where $k$ is a non-negative integer, $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index of non-negative integers $\alpha_1, \ldots, \alpha_d$ and

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f \quad \text{where} \quad |\alpha| := \alpha_1 + \cdots + \alpha_d. \quad (2.1.2)$$

The dual $S' = S'(\mathbb{R}^d)$ of $S(\mathbb{R}^d)$, equipped with the weak-star topology, is the space of tempered distributions. This space is the one we will use as our basic probability space. As events we will use the family $\mathcal{B}(S'(\mathbb{R}^d))$ of Borel subsets of $S'(\mathbb{R}^d)$, and our probability measure is given by the following result.

**Theorem 2.1.1. (The Bochner–Minlos theorem)** There exists a unique probability measure $\mu_1$ on $\mathcal{B}(S'(\mathbb{R}^d))$ with the following property:

$$E[e^{i\langle \cdot, \phi \rangle}] := \int_{S'} e^{i\langle \omega, \phi \rangle} d\mu_1(\omega) = e^{-\frac{1}{2} \|\phi\|^2} \quad (2.1.3)$$

for all $\phi \in S(\mathbb{R}^d)$, where $\|\phi\|^2 = \|\phi\|_{L^2(\mathbb{R}^d)}^2$, $\langle \omega, \phi \rangle = \omega(\phi)$ is the action of $\omega \in S'(\mathbb{R}^d)$ on $\phi \in S(\mathbb{R}^d)$ and $E = E_{\mu_1}$ denotes the expectation with respect to $\mu_1$.

See Appendix A for a proof. We will call the triplet $(S'(\mathbb{R}^d), \mathcal{B}(S'(\mathbb{R}^d)), \mu_1)$ the 1-dimensional white noise probability space, and $\mu_1$ is called the white noise measure.

The measure $\mu_1$ is also often called the (normalized) Gaussian measure on $S'(\mathbb{R}^d)$. The reason for this can be seen from the following result.

**Lemma 2.1.2.** Let $\xi_1, \ldots, \xi_n$ be functions in $S(\mathbb{R}^d)$ that are orthonormal in $L^2(\mathbb{R}^d)$. Let $\lambda_n$ be the normalized Gaussian measure on $\mathbb{R}^n$, i.e.,

$$d\lambda_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2} |x|^2} dx_1 \cdots dx_n; \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n. \quad (2.1.4)$$

Then the random variable

$$\omega \rightarrow (\langle \omega, \xi_1 \rangle, \langle \omega, \xi_2 \rangle, \ldots, \langle \omega, \xi_n \rangle) \quad (2.1.5)$$

has distribution $\lambda_n$. Equivalently,

$$E[f(\langle \cdot, \xi_1 \rangle, \ldots, \langle \cdot, \xi_n \rangle)] = \int_{\mathbb{R}^n} f(x) d\lambda_n(x) \quad \text{for all} \quad f \in L^1(\lambda_n). \quad (2.1.6)$$

**Proof** It suffices to prove this for $f \in C_0^\infty(\mathbb{R}^n)$; the general case then follows by taking the limit in $L^1(\lambda_n)$. If $f \in C_0^\infty(\mathbb{R}^n)$, then $f$ is the inverse Fourier transform of its Fourier transform $\hat{f}$:
\[ f(x) = (2\pi)^{-\frac{n}{2}} \int \hat{f}(y)e^{i(x,y)}dy \]

where
\[ \hat{f}(y) = (2\pi)^{-\frac{n}{2}} \int f(x)e^{-i(x,y)}dx, \]

where \((x, y)\) denotes the usual inner product in \(\mathbb{R}^d\). Then (2.1.3) gives
\[
E[f(\langle \cdot, \xi_1 \rangle, \ldots, \langle \cdot, \xi_n \rangle)] = (2\pi)^{-\frac{n}{2}} \int \hat{f}(y)e^{-\frac{1}{2}y^2}dy
\]
\[
= (2\pi)^{-\frac{n}{2}} \int \left( \int f(x)e^{-i(x,y)}dx \right)e^{-\frac{1}{2}|y|^2}dy
\]
\[
= (2\pi)^{-n} \int f(x) \left( \int e^{-i(x,y)}e^{-\frac{1}{2}|y|^2}dy \right)dx
\]
\[
= (2\pi)^{-\frac{n}{2}} \int f(x)e^{-\frac{1}{2}|x|^2}dx
\]
\[
= \int f(x)d\lambda_n(x),
\]

where we have used the well-known formula
\[
\int e^{i\alpha t - \beta t^2}dt = \left(\frac{\pi}{\beta}\right)^{\frac{1}{2}}e^{-\frac{\alpha^2}{4\beta}}. \quad (2.1.7)
\]

(See Exercise 2.4.) \hfill \Box

For an alternative proof of Lemma 2.1.2, see Exercise 2.5.

Remark  Note that (2.1.6) applies in particular to polynomials
\[
(x_1, \ldots, x_n) = \sum_{|\alpha|\leq N} c_\alpha x^\alpha,
\]

where the sum is taken over all \(n\)-dimensional multi-indices \(\alpha = (\alpha_1, \ldots, \alpha_n)\) and \(x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n}\). Let \(\mathcal{P}\) denote the family of all functions \(p : \mathcal{S}'(\mathbb{R}^d) \to \mathbb{R}\) of the form
\[
p(\omega) = f(\langle \omega, \xi_1 \rangle, \ldots, \langle \omega, \xi_n \rangle)
\]
for some polynomial \( f \). We call such functions \( p \) **stochastic polynomials**. Similarly, we let \( \mathcal{E} \) denote the family of all linear combinations of functions \( f : \mathcal{S}'(\mathbb{R}^d) \to \mathbb{R} \) of the form

\[
f(\omega) = \exp[i(\omega, \phi)] \quad \text{where} \quad \phi \in \mathcal{S}(\mathbb{R}^d).
\]

Such functions are called **stochastic exponentials**. The following result is useful.

**Theorem 2.1.3.** \( \mathcal{P} \) and \( \mathcal{E} \) are dense in \( L^p(\mu_1) \), for all \( p \in [1, \infty) \).

**Proof** See Theorem 1.9, p. 7, in Hida et al. (1993).

**Definition 2.1.4.** The **1-dimensional (d-parameter) smoothed white noise** is the map

\[
w : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \to \mathbb{R}
\]

given by

\[
w(\phi) = w(\phi, \omega) = \langle \omega, \phi \rangle; \quad \omega \in \mathcal{S}'(\mathbb{R}^d), \phi \in \mathcal{S}(\mathbb{R}^d).
\] (2.1.8)

**Remark** In Section 2.3 we will define the singular white noise \( W(x, \omega) \). We may regard \( w(\phi) \) as obtained by smoothing \( W(x, \omega) \) by \( \phi \).

Using Lemma 2.1.2 it is not difficult to prove that if \( \phi \in L^2(\mathbb{R}^d) \) and we choose \( \phi_n \in \mathcal{S}(\mathbb{R}^d) \) such that \( \phi_n \to \phi \) in \( L^2(\mathbb{R}^d) \), then

\[
\langle \omega, \phi \rangle := \lim_{n \to \infty} \langle \omega, \phi_n \rangle \quad \text{exists in} \quad L^2(\mu_1)
\] (2.1.9)

and is independent of the choice of \( \{\phi_n\} \) (Exercise 2.6). In particular, if we define

\[
\tilde{B}(x) := \tilde{B}(x_1, \ldots, x_d, \omega) = \langle \omega, \chi_{[0,x_i]} \chi \cdots \chi_{[0,x_d]} \rangle; x = (x_1, \ldots, x_d) \in \mathbb{R}^d,
\] (2.1.10)

where \([0,x_i]\) is interpreted as \([x_i,0]\) if \( x_i < 0 \), then \( \tilde{B}(x, \omega) \) has an \( x \)-continuous version \( B(x, \omega) \), which becomes a \( d \)-parameter Brownian motion.

By a **\( d \)-parameter Brownian motion** we mean a family \( \{X(x, \cdot)\}_{x \in \mathbb{R}^d} \) of random variables on a probability space \((\Omega, \mathcal{F}, P)\) such that

\[
X(0, \cdot) = 0 \quad \text{almost surely with respect to} \quad P,
\] (2.1.11)

\( \{X(x, \omega)\} \) is a Gaussian stochastic process (i.e., \( Y = (X(x^{(1)}, \cdot), \ldots, X(x^{(n)}, \cdot)) \) has a multinormal distribution with mean zero for all
2.1 White Noise

$x^{(1)}, \ldots, x^{(n)} \in \mathbb{R}^d$ and all $n \in \mathbb{N}$) and, further, for all 
$x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d_+$, that $X(x, \cdot) X(y, \cdot)$ have 
the covariance $\prod_{i=1}^d x_i \wedge y_i$. For general $x, y \in \mathbb{R}^d$ the covariance is 
$\prod_{i=1}^d \int_{\mathbb{R}} \theta_{x_i}(s) \theta_{y_i}(s) ds$, where $\theta_{x_i}(t_1, \ldots, t_d) = \theta_{x_1}(t_1) \cdots \theta_{x_d}(t_d)$, with 
(2.1.12)

$$
\theta_{x_j}(s) = \begin{cases} 
1 & \text{if } 0 < s \leq x_j \\
-1 & \text{if } x_j < s \leq 0 \\
0 & \text{otherwise}
\end{cases}
$$

We also require that 
$X(x, \omega)$ has continuous paths, i.e., that $x \to X(x, \omega)$ 
is continuous for almost all $\omega$ with respect to $P$. (2.1.13)

We have to verify that $\tilde{B}(x, \omega)$ defined by (2.1.10) satisfies (2.1.11) and 
(2.1.12) and that $\tilde{B}$ has a continuous version. Property (2.1.11) is evident. 
To prove (2.1.12), we choose $x^{(1)}, \ldots, x^{(n)} \in \mathbb{R}^d_+, c_1, \ldots, c_n \in \mathbb{R}$ and put 
$$
\chi^{(j)}(t) = \chi_{[0, x^{(j)}_1]} \times \cdots \times [0, x^{(j)}_d](t) ; t \in \mathbb{R}^d
$$

where $x^{(j)} = (x^{(j)}_1, \ldots, x^{(j)}_d)$, and compute 
$$
E\left[ \exp \left( i \sum_{j=1}^n c_j \tilde{B}(x^{(j)}) \right) \right] = E\left[ \exp \left( i \langle \cdot, \sum_{j=1}^n c_j \chi^{(j)} \rangle \right) \right] 
= \exp \left( -\frac{1}{2} \left\| \sum_{j=1}^nc_j \chi^{(j)} \right\|^2 \right) 
= \exp \left( -\frac{1}{2} \int_{\mathbb{R}^d} \left( \sum_{j=1}^n c_j \chi^{(j)}(t) \right)^2 dt \right) 
= \exp \left( -\frac{1}{2} \sum_{i,j=1}^n c_i c_j \int_{\mathbb{R}^d} \chi^{(i)}(t) \chi^{(j)}(t) dt \right) 
= \exp \left( -\frac{1}{2} c^T V c \right),
$$

where $c = (c_1, \ldots, c_n)$ and $V = [V_{ij}] \in \mathbb{R}^{n \times n}$ is the symmetric non-negative 
definite matrix with entries 
$$
V_{ij} = \int_{\mathbb{R}^d} \chi^{(i)}(y) \chi^{(j)}(y) dy = (\chi^{(i)}, \chi^{(j)}).$$
This proves that \( Y = (\tilde{B}(x^{(1)}), \ldots, \tilde{B}(x^{(n)})) \) is Gaussian with mean zero and covariance matrix \( V = [V_{ij}] \). For \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) let \( \chi_x(t) = \chi_{[0,x_1]} \times \cdots \times [0,x_d](t); t \in \mathbb{R}^d \). Then for \( y = (y_1, \ldots, y_d) \in \mathbb{R}^d \), we have

\[
D^2 := \| \chi_x - \chi_y \|^2 = \prod_{i=1}^d \| \chi_{[0,x_i]} - \chi_{[0,y_i]} \|^2 = \prod_{i=1}^d |x_i - y_i|.
\]

Hence by (2.1.6)

\[
E[|\tilde{B}(x) - \tilde{B}(y)|^2] = E[(\chi_x - \chi_y)^2] = D^2 E\left[ \left( \frac{\chi_x - \chi_y}{D} \right)^2 \right] = D^2 \int_{\mathbb{R}} t^2 d\lambda_1(t) = D^2,
\]

which proves that \( \tilde{B} \) satisfies (2.1.12).

Finally, using Kolmogorov’s continuity theorem (see, e.g., Stroock and Varadhan (1979), Theorem 2.1.6) we obtain that \( \tilde{B}(x) \) has a continuous version \( B(x) \), which then becomes a \( d \)-parameter Brownian motion. See Exercise 2.7.

We remark that for \( d = 1 \) we get the classical (1-parameter) Brownian motion \( B(t) \) if we restrict ourselves to \( t \geq 0 \). For \( d = 2 \) we get what is often called the Brownian sheet.

With this definition of Brownian motion it is natural to define the \( d \)-parameter Wiener–Itô integral of \( \phi \in L^2(\mathbb{R}^d) \) by

\[
\int_{\mathbb{R}^d} \phi(x) dB(x, \omega) := \langle \omega, \phi \rangle; \quad \omega \in \mathcal{S}'(\mathbb{R}^d).
\] (2.1.14)

We see that by appealing to the Bochner–Minlos theorem we have obtained not only a simple description of white noise, but also an easy construction of Brownian motion. The relation between these two fundamental concepts can also be expressed as follows.

Using integration by parts for Wiener–Itô integrals (Appendix B), we get

\[
\int_{\mathbb{R}^d} \phi(x) dB(x) = (-1)^d \int_{\mathbb{R}^d} \frac{\partial^d \phi}{\partial x_1 \cdots \partial x_d}(x) B(x) dx.
\] (2.1.15)

Hence

\[
w(\phi) = \int_{\mathbb{R}^d} \phi(x) dB(x) = \left( (-1)^d \frac{\partial^d \phi}{\partial x_1 \cdots \partial x_d}, B \right) = \left( \phi, \frac{\partial^d B}{\partial x_1 \cdots \partial x_d} \right),
\] (2.1.16)
where \((\cdot, \cdot)\) denotes the usual inner product in \(L^2(\mathbb{R}^d)\). In other words, in the sense of distributions we have, for almost all \(\omega\),
\[
w = \frac{\partial^d B}{\partial x_1 \cdots \partial x_d}, \tag{2.1.17}
\]

We will give other formulations of this connection between Brownian motion and white noise in Section 2.5.

Using \(w(\phi, \omega)\) we can construct a stochastic process, called the *smoothed white noise process* \(W_\phi(x, \omega)\), as follows: Set
\[
W_\phi(x, \omega) := w(\phi_x, \omega), \quad x \in \mathbb{R}^d, \quad \omega \in \mathcal{S}'(\mathbb{R}^d), \tag{2.1.18}
\]
where
\[
\phi_x(y) = \phi(y - x) \tag{2.1.19}
\]
is the \(x\)-shift of \(\phi\); \(x, y \in \mathbb{R}^d\).

Note that \(\{W_\phi(x, \cdot)\}_{x \in \mathbb{R}^d}\) has the following three properties:

If \(\text{supp} \phi_{x_1} \cap \text{supp} \phi_{x_2} = \emptyset\), then \(W_\phi(x_1, \cdot)\) and \(W_\phi(x_2, \cdot)\) are independent. \(\tag{2.1.20}\)

\(\{W_\phi(x, \cdot)\}_{x \in \mathbb{R}^d}\) is a stationary process, i.e., for all \(n \in \mathbb{N}\) and for all \(x^{(1)}, \ldots, x^{(n)}\) and \(h \in \mathbb{R}^d\), the joint distribution of
\[
(W_\phi(x^{(1)} + h, \cdot), \ldots, W_\phi(x^{(n)} + h, \cdot)) \tag{2.1.21}
\]
is independent of \(h\).

For each \(x \in \mathbb{R}^d\), the random variable \(W_\phi(x, \cdot)\) is normally distributed with mean 0 and variance \(\|\phi\|^2\). \(\tag{2.1.22}\)

So \(\{W_\phi(x, \omega)\}_{x \in \mathbb{R}^d}\) is indeed a mathematical model for what one usually intuitively thinks of as white noise. In explicit applications the test function or “window” \(\phi\) can be chosen such that the diameter of \(\text{supp} \phi\) is the maximal distance within which \(W_\phi(x_1, \cdot)\) and \(W_\phi(x_2, \cdot)\) might be correlated.

Figure 2.1 shows computer simulations of the 2-parameter white noise process \(W_\phi(x, \omega)\) where \(\phi(y) = \chi_{[0,h] \times [0,h]}(y); y \in \mathbb{R}^2\) for \(h = 1/50\) (left) and for \(h = 1/20\) (right).

![Two sample paths of white noise](image-url)
2.1.2 The (Smoothed) White Noise Vector

We now proceed to define the multidimensional case. If \( m \) is a natural number, we define

\[
S := \prod_{i=1}^{m} S(\mathbb{R}^d), \quad S' := \prod_{i=1}^{m} S'(\mathbb{R}^d), \quad B := \prod_{i=1}^{m} B(S'(\mathbb{R}^d))
\]  

(2.1.23)

and equip \( S' \) with the product measure

\[
\mu_m = \mu_1 \times \mu_1 \times \cdots \times \mu_1,
\]

(2.1.24)

where \( \mu_1 \) is the 1-dimensional white noise probability measure. It is then easy to see that we have the following property:

\[
\int_{S'} e^{i\langle \omega, \phi \rangle} d\mu_m(\omega) = e^{-\frac{1}{2} \| \phi \|^2} \quad \text{for all } \phi \in S.
\]

(2.1.25)

Here \( \langle \omega, \phi \rangle = \langle \omega_1, \phi_1 \rangle + \cdots + \langle \omega_m, \phi_m \rangle \) is the action of \( \omega = (\omega_1, \ldots, \omega_m) \in S' \) on \( \phi = (\phi_1, \ldots, \phi_m) \in S \), where \( \langle \omega_k, \phi_k \rangle \) is the action of \( \omega_k \in S'(\mathbb{R}^d) \) on \( \phi_k \in S(\mathbb{R}^d) \); \( k = 1, 2, \ldots, m \).

Furthermore,

\[
\| \phi \| = \| \phi \|_K = \left( \sum_{k=1}^{m} \| \phi_k \|^2 \right)^{\frac{1}{2}} = \left( \sum_{k=1}^{m} \int_{\mathbb{R}^d} \phi_k^2(x) dx \right)^{\frac{1}{2}}
\]

(2.1.26)

is the norm of \( \phi \) in the Hilbert space \( K \) defined as the orthogonal sum of \( m \) identical copies of \( L^2(\mathbb{R}^d) \), viz. \( K = \bigoplus_{k=1}^{m} L^2(\mathbb{R}^d) \).

We will call the triplet \((S', B, \mu_m)\) the \( d \)-parameter multidimensional white noise probability space. The parameter \( m \) is called the white noise dimension. The \( m \)-dimensional smoothed white noise

\[
w : S \times S' \to \mathbb{R}^m
\]

is then defined by

\[
w(\phi) = w(\phi, \omega) = (\langle \omega_1, \phi_1 \rangle, \ldots, \langle \omega_m, \phi_m \rangle) \in \mathbb{R}^m
\]

(2.1.27)

if \( \omega = (\omega_1, \ldots, \omega_m) \in S', \phi = (\phi_1, \ldots, \phi_m) \in S \). If the value of \( m \) is clear from the context, we sometimes write \( \mu \) for \( \mu_m \).

As in the 1-dimensional case, we now proceed to define \( m \)-dimensional Brownian motion \( B(x) = B(x, \omega) = (B_1(x, \omega), \ldots, B_m(x, \omega)) \); \( x \in \mathbb{R}^d, \omega \in S' \) as the \( x \)-continuous version of the process

\[
\tilde{B}(x, \omega) = (\langle \omega_1, \chi_{[0,x_1]} \times \cdots \times [0,x_d] \rangle, \ldots, \langle \omega_m, \chi_{[0,x_1]} \times \cdots \times [0,x_d] \rangle).
\]

(2.1.28)
From this we see that $B(x)$ consists of $m$ independent copies of 1-dimensional Brownian motion. Combining (2.1.27) and (2.1.14) we get

$$w(\phi) = \left( \int \phi_1(x) dB_1(x), \ldots, \int \phi_m(x) dB_m(x) \right). \tag{2.1.29}$$

Using $w(\phi, \omega)$, we can construct $m$-dimensional smoothed white noise process $W_\phi(x, \omega)$ as follows:

$$W_\phi(x, \omega) := w(\phi_x, \omega) \tag{2.1.30}$$

for $\phi = (\phi_1, \ldots, \phi_m) \in \mathcal{S}$, $\omega = (\omega_1, \ldots, \omega_m) \in \mathcal{S}'$, where

$$\phi_x(y) = (\phi_1(y-x), \ldots, \phi_m(y-x)); \quad x, y \in \mathbb{R}^d. \tag{2.1.31}$$

### 2.2 The Wiener–Itô Chaos Expansion

There are (at least) two ways of constructing the classical Wiener–Itô chaos expansion:

(A) by Hermite polynomials,
(B) by multiple Itô integrals.

Both approaches are important, and it is useful to know them both and to know the relationship between them. For us the first construction will play the major role. We will therefore introduce this method in detail first, then sketch the other construction, and finally compare the two.

#### 2.2.1 Chaos Expansion in Terms of Hermite Polynomials

The Hermite polynomials $h_n(x)$ are defined by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}); \quad n = 0, 1, 2, \ldots \tag{2.2.1}$$

Thus the first Hermite polynomials are

$$h_0(x) = 1, \quad h_1(x) = x, \quad h_2(x) = x^2 - 1, \quad h_3(x) = x^3 - 3x,$$

$$h_4(x) = x^4 - 6x^2 + 3, \quad h_5(x) = x^5 - 10x^3 + 15x, \ldots.$$

The Hermite functions $\xi_n(x)$ are defined by

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} h_{n-1}(\sqrt{2}x); \quad n = 1, 2, \ldots \tag{2.2.2}$$
The most important properties of \( h_n \) and \( \xi_n \) are given in Appendix C. Some properties we will often use follow.

\[
\xi_n \in \mathcal{S}(\mathbb{R}) \quad \text{for all } n. \tag{2.2.3}
\]

The collection \( \{\xi_n\}_{n=1}^{\infty} \) constitutes an orthonormal basis for \( L^2(\mathbb{R}) \).

\[
\sup_{x \in \mathbb{R}} |\xi_n(x)| = O(n^{-\frac{1}{2}}). \tag{2.2.5}
\]

The statement (2.2.3) follows from the fact that \( h_n \) is a polynomial of degree \( n \). Proofs of statements (2.2.4) and (2.2.5) can be found in Hille and Phillips (1957), Chapter 21.

We will use these functions to define an orthogonal basis for \( L^2(\mu_m) \), where \( \mu_m = \mu_1 \times \cdots \times \mu_1 \) as before. Since the 1-dimensional case is simpler and also the case we will use most, we first do the construction in this case.

**Case 1** \((m = 1)\). In the following, we let \( \delta = (\delta_1, \ldots, \delta_d) \) denote \( d \)-dimensional multi-indices with \( \delta_1, \ldots, \delta_d \in \mathbb{N} \). By (2.2.4) it follows that the family of tensor products

\[
\xi_{\delta} := \xi_{(\delta_1, \ldots, \delta_d)} := \xi_{\delta_1} \otimes \cdots \otimes \xi_{\delta_d}; \quad \delta \in \mathbb{N}^d \tag{2.2.6}
\]

forms an orthonormal basis for \( L^2(\mathbb{R}^d) \). Let \( \delta^{(j)} = (\delta_1^{(j)}, \delta_2^{(j)}, \ldots, \delta_d^{(j)}) \) be the \( j \)th multi-index number in some fixed ordering of all \( d \)-dimensional multi-indices \( \delta = (\delta_1, \ldots, \delta_d) \in \mathbb{N}^d \). We can, and will, assume that this ordering has the property that

\[
i < j \Rightarrow \delta_1^{(i)} + \delta_2^{(i)} + \cdots + \delta_d^{(i)} \leq \delta_1^{(j)} + \delta_2^{(j)} + \cdots + \delta_d^{(j)}, \tag{2.2.7}
\]

i.e., that the \( \{\delta^{(j)}\}_{j=1}^{\infty} \) occur in increasing order.

Now define

\[
\eta_j := \xi_{\delta^{(j)}} = \xi_{\delta_1^{(j)}} \otimes \cdots \otimes \xi_{\delta_d^{(j)}}; \quad j = 1, 2, \ldots \tag{2.2.8}
\]

We will need to consider multi-indices of arbitrary length. To simplify the notation, we regard multi-indices as elements of the space \((\mathbb{N}_0^d)_c\) of all sequences \( \alpha = (\alpha_1, \alpha_2, \ldots) \) with elements \( \alpha_i \in \mathbb{N}_0 \) and with compact support, i.e., with only finitely many \( \alpha_i \neq 0 \). We write

\[
\mathcal{J} = (\mathbb{N}_0^d)_c. \tag{2.2.9}
\]

**Definition 2.2.1** \((m = 1)\). Let \( \alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{J} \). Then we define

\[
H_\alpha(\omega) = H^{(1)}_\alpha(\omega) := \prod_{i=1}^{\infty} h_{\alpha_i}((\omega, \eta_i)); \quad \omega \in \mathcal{S}'(\mathbb{R}^d). \tag{2.2.10}
\]
Case 2 ($m > 1$). In this case we have to proceed one step further from (2.2.8) to obtain an orthonormal basis for $\mathcal{K} = \bigoplus_{k=1}^m L^2(\mathbb{R}^d)$. Define the following elements of $\mathcal{K}$:

$$e^{(1)} = (\eta_1, 0, \ldots, 0)$$
$$e^{(2)} = (0, \eta_1, \ldots, 0)$$
$$\vdots$$
$$e^{(m)} = (0, 0, \ldots, \eta_1).$$

Then repeat with $\eta_1$ replaced by $\eta_2$:

$$e^{(m+1)} = (\eta_2, 0, \ldots, 0)$$
$$\vdots$$
$$e^{(2m)} = (0, 0, \ldots, \eta_2),$$

and so on.

In short, for every $k \in \mathbb{N}$ there are unique numbers $i \in \{1, \ldots, m\}$ and $j \in \mathbb{N}$ such that $k = i + (j - 1)m$. Then we have

$$e^{(k)} = e^{(i+(j-1)m)} = (0, 0, \ldots, \eta_j, \ldots, 0) = \eta_j e^{(i)} \in \mathcal{K}, \quad (2.2.11)$$

where $e^{(i)}$ is the multi-index with 1 on entry number $i$ and 0 otherwise.

**Definition 2.2.2** ($m > 1$). For $\alpha \in \mathcal{J}$ define

$$H_\alpha(\omega) = H_\alpha^{(m)}(\omega) = \prod_{k=1}^\infty h_{\alpha_k}(\langle \omega, e^{(k)} \rangle). \quad (2.2.12)$$

Here $\omega = (\omega_1, \ldots, \omega_m) \in \mathcal{S}'$ and

$$\langle \omega, e^{(k)} \rangle = \langle \omega_1, e^{(k)}_1 \rangle + \cdots + \langle \omega_m, e^{(k)}_m \rangle = \langle \omega_i, \eta_j \rangle \text{ if } k = i + (j - 1)m. \quad (2.2.13)$$

Therefore we can also write

$$H_\alpha^{(m)}(\omega) = \prod_{\substack{k=1 \atop k = i+(j-1)m}}^\infty h_{\alpha_k}(\langle \omega_i, \eta_j \rangle); \quad \omega \in \mathcal{S}'. \quad (2.2.14)$$

For example, if $\alpha = e^{(k)}$ with $k = i + (j - 1)m$, we get

$$H_{e^{(k)}} = \langle \omega, e^{(k)} \rangle = \langle \omega_i, \eta_j \rangle. \quad (2.2.15)$$
There is an alternative description of the family \( \{ H_{\alpha}^{(m)} \} \) that is natural from a tensor product point of view:

For \( \Gamma = (\gamma^{(1)}, \ldots, \gamma^{(m)}) \in J^m = J \times \cdots \times J \), \( \omega = (\omega_1, \ldots, \omega_m) \in \mathcal{S}' \) define

\[
H_{\Gamma}^{(m)}(\omega) = \prod_{i=1}^{m} H_{\gamma^{(i)}}^{(1)}(\omega_i),
\]

where each \( H_{\gamma^{(i)}}^{(1)}(\omega_i) \) is as in (2.2.10). Then we see that

\[
H_{\Gamma}^{(m)}(\omega) = \prod_{i=1}^{m} \prod_{j=1}^{\infty} h_{\gamma^{(i)}}(\omega, \eta_j) = H_{\alpha}^{(m)}(\omega),
\]

where \( \alpha = (\alpha_1, \alpha_2, \ldots) \in J \) is related to \( \Gamma = (\gamma^{(i)}) \in J^m \) by

\[
\alpha_k = \gamma^{(i)}_j \quad \text{if} \quad i + (j-1)m = k.
\]

**Theorem 2.2.3.** For any \( m \geq 1 \) the family \( \{ H_{\alpha} \}_{\alpha \in J} = \{ H_{\Gamma} \}_{\Gamma \in J^m} \) constitutes an orthogonal basis for \( L^2(\mu_m) \). Moreover, if \( \alpha = (\alpha_1, \alpha_2, \ldots) \in J \), we have the norm expression

\[
\|H_{\alpha}\|_{L^2(\mu_m)}^2 = \alpha_1! \alpha_2! \cdots.
\]

**Proof** First consider the case where we have \( m = 1 \). Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) be two multi-indices. Then using Lemma 2.1.2, where we have \( E = E_{\mu_1} \), we get the expression

\[
E[H_{\alpha}H_{\beta}] = E\left[ \prod_{i=1}^{n} h_{\alpha_i}(\omega, \eta_i)h_{\beta_i}(\omega, \eta_i) \right] = \int \prod_{i=1}^{n} h_{\alpha_i}(x_i)h_{\beta_i}(x_i)d\lambda_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} \int h_{\alpha_i}(x_i)h_{\beta_i}(x_i)d\lambda_1(x_i).
\]

From the well-known orthogonality relations for Hermite polynomials (see, e.g., (C.10) in Appendix C), we have that

\[
\int h_j(x)h_k(x)e^{-\frac{1}{2}x^2} dx = \delta_{j,k} \sqrt{2\pi k}!.
\]

We therefore obtain (2.2.18) and that \( H_{\alpha} \) and \( H_{\beta} \) are orthogonal if \( \alpha \neq \beta \). To prove completeness of the family \( \{ H_{\alpha} \} \), we note that by Theorem 2.1.3
any \( g \in L^2(\mu_1) \) can be approximated in \( L^2(\mu_1) \) by stochastic polynomials of the form

\[
p_n(\omega) = f_n(\langle \omega, \eta_1 \rangle, \ldots, \langle \omega, \eta_n \rangle).
\]

Now the polynomial \( f_n(x_1, \ldots, x_n) \) can be written as a linear combination of products of Hermite polynomials \( h_{\alpha}(x_1)h_{\alpha_2}(x_2) \cdots h_{\alpha_n}(x_n) \). Then, of course, \( p_n(\omega) \) is the corresponding linear combination of functions \( H_{\alpha}(\omega) \) where \( \alpha = (\alpha_1, \ldots, \alpha_n) \).

The general case \( m \geq 1 \) follows from the above case using the tensor product structure. For completeness, we give the details.

With \( H_{\alpha} = H_{\alpha}^{(m)} \) defined as in Definition 2.2.2, we get, with \( \mu = \mu_m, E = E_\mu, \alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \):

\[
E[H_\alpha H_\beta] = E \left[ \prod_{k=1}^{n} h_{\alpha_k}(\langle \omega, e^{(k)} \rangle) h_{\beta_k}(\langle \omega, e^{(k)} \rangle) \right]
\]

\[
= E \left[ \prod_{k=1}^{n} h_{\alpha_k}(\langle \omega_{i(k)}, \eta_{j(k)} \rangle) h_{\beta_k}(\langle \omega_{i(k)}, \eta_{j(k)} \rangle) \right]
\]

\[
= E \left[ \prod_{k=1}^{n} h_{\alpha_k}(\langle \omega_1, \eta_j(k) \rangle) \cdot h_{\beta_k}(\langle \omega_1, \eta_j(k) \rangle) \right]
\]

\[
= \prod_{u=1}^{m} E_{\mu_u} \left[ \prod_{k=1}^{n} h_{\alpha_k}(\langle \omega_u, \eta_j(k) \rangle) h_{\beta_k}(\langle \omega_u, \eta_j(k) \rangle) \right]
\]

\[
= \prod_{u=1}^{m} E_{\mu_u} \left[ \prod_{v=1}^{n} \left\{ h_{\alpha_k}(\langle \omega_u, \eta_v \rangle) \cdot h_{\beta_k}(\langle \omega_u, \eta_v \rangle) \right\}_{k=1}^{m} = u + (v-1)m \right]
\]

\[
= \prod_{u=1}^{m} \prod_{v=1}^{n} \int_{\mathbb{R}} \left\{ h_{\alpha_k}(x_u) h_{\beta_k}(x_u) \right\}_{k=1}^{m} = u + (v-1)m d\lambda_1(x_u)
\]

\[
= \prod_{u=1}^{m} \prod_{v=1}^{n} \left\{ \delta_{\alpha_k, \beta_k} \alpha_k! \right\}_{k=1}^{m} = u + (v-1)m
\]

\[
= \prod_{u=1}^{m} \prod_{k=1}^{n} \delta_{\alpha_k, \beta_k} \alpha_k! = \begin{cases} 
\alpha! & \text{if } \alpha = \beta \\
0 & \text{if } \alpha \neq \beta.
\end{cases}
\]

We conclude that \( \{ H_\alpha \}_\alpha \) is an orthogonal family in \( L^2(\mu) \) and that (2.2.18) holds.
Finally, since the span of \( \{ H^{(1)}_{\alpha} \}_{\alpha \in \mathcal{J}} \) is dense in \( L^2(\mu_1) \), it follows by (2.2.17) that the span of \( \{ H^{(m)}_{\alpha} \}_{\alpha \in \mathcal{J}} \) is dense in \( L^2(\mu_m) \). This completes the proof. \( \Box \)

From now on we fix the parameter dimension \( d \geq 1 \), the white noise dimension \( m \geq 1 \) and we fix a state space dimension \( N \geq 1 \). Let

\[
L^2(\mu_m) = \bigoplus_{k=1}^{N} L^2(\mu_m). \tag{2.2.20}
\]

Applying Theorem 2.2.3 to each component of \( L^2(\mu_m) \), we get

**Theorem 2.2.4 (Wiener–Itô chaos expansion theorem).** Every \( f \in L^2(\mu_m) \) has a unique representation

\[
f(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha(\omega) \tag{2.2.21}
\]

where \( c_\alpha \in \mathbb{R}^N \) for all \( \alpha \).

Moreover, we have the isometry

\[
\|f\|_{L^2(\mu_m)}^2 = \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2, \tag{2.2.22}
\]

where \( c_\alpha^2 = |c_\alpha|^2 = (c_\alpha, c_\alpha) \) denotes the inner product in \( \mathbb{R}^N \).

**Remark** The major part of this book will be based on this construction. It must be admitted that the definitions behind (2.2.21) are rather complicated. Nevertheless the expression is notationally simple and quite easy to apply as long as we can avoid the underlying structure.

**Exercise 2.2.5** \((N = 1, m = 1)\)

i) The 1-dimensional smoothed white noise \( w(\phi, \omega) \) defined in (2.1.8) has the expansion

\[
w(\phi, \omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha(\omega)
\]

where \( c_\alpha = |c_\alpha|^2 = (c_\alpha, c_\alpha) \) denotes the inner product in \( \mathbb{R}^N \).

where \( \epsilon^{(j)} = (0, 0, \ldots, 1, \ldots) \) with 1 on entry number \( j \), 0 otherwise. The convergence is in \( L^2(\mu) \).

In other words,

\[
w(\phi, \omega) = \sum_{\alpha} c_\alpha H_\alpha(\omega)
\]
with

\[ c_\alpha = \begin{cases} 
(\phi, \eta_j) & \text{if } \alpha = \epsilon^j \\
0 & \text{otherwise.}
\end{cases} \quad (2.2.23) \]

ii) The 1-dimensional, d-parameter Brownian motion \( B(x, \omega) \) is defined by (2.1.10):

\[ B(x, \omega) = \langle \omega, \psi \rangle, \]

where

\[ \psi(y) = \chi_{[0,x_1] \times \ldots \times [0,x_d]}(y). \]

Proceeding as above, we write

\[ \psi(y) = \sum_{j=1}^{\infty} (\psi, \eta_j) \eta_j(y) = \sum_{j=1}^{\infty} \int_0^x \eta_j(u) du \eta_j(y), \]

where we have used the multi-index notation

\[ \int_0^x \eta_j(u) du = \int_0^{x_1} \eta_j(u_1, \ldots, u_d) du_1 \ldots du_d = \prod_{k=1}^d \int_0^{x_k} \xi_{\beta_k}(t_k) dt_k \]

when \( x = (x_1, \ldots, x_d) \) (see (2.2.8)). Therefore,

\[ B(x, \omega) = \left\langle \omega, \sum_{j=1}^{\infty} \int_0^x \eta_j(u) du \eta_j \right\rangle = \sum_{j=1}^{\infty} \int_0^x \eta_j(u) du \langle \omega, \eta_j \rangle, \]

so \( B(x, \omega) \) has the expansion

\[ B(x, \omega) = \sum_{j=1}^{\infty} \int_0^x \eta_j(u) du H_{\epsilon^{(j)}}(\omega). \quad (2.2.24) \]

**Example 2.2.6 (N = m > 1).**

i) Next consider \( m \)-dimensional smoothed white noise defined by (2.1.27):

\[ w(\phi, \omega) = (\langle \omega_1, \phi_1 \rangle, \ldots, \langle \omega_m, \phi_m \rangle), \]

where \( \omega = (\omega_1, \ldots, \omega_m) \in S', \phi = (\phi_1, \ldots, \phi_m) \in S. \)
Using the same procedure as in the previous example, we get

\[
\mathbf{w}(\phi, \omega) = \left( \langle \omega_1, \sum_{j=1}^{\infty} (\phi_1, \eta_j) \eta_j \rangle, \ldots, \langle \omega_m, \sum_{j=1}^{\infty} (\phi_m, \eta_j) \eta_j \rangle \right)
= \left( \sum_{j=1}^{\infty} (\phi_1, \eta_j) \langle \omega_1, \eta_j \rangle, \ldots, \sum_{j=1}^{\infty} (\phi_m, \eta_j) \langle \omega_m, \eta_j \rangle \right).
\]

Since by (2.2.15)

\[
\langle \omega_i, \eta_j \rangle = H_{\epsilon(i+(j-1)m)}(\omega),
\]
we conclude that the \(i\)th component, \(1 \leq i \leq m\), of \(\mathbf{w}(\phi, \omega)\), \(w_i(\phi, \omega)\), can be written

\[
w_i(\phi, \omega) = \sum_{j=1}^{\infty} (\phi_i, \eta_j) \langle \omega_i, \eta_j \rangle = \sum_{j=1}^{\infty} (\phi_i, \eta_j) H_{\epsilon(i+(j-1)m)}(\omega).
\]

(2.2.25)

Thus

\[
w_i(\phi, \omega) = (\phi_i, \eta_1) H_{\epsilon(i)} + (\phi_i, \eta_2) H_{\epsilon(i+m)} + \cdots.
\]

Note that the expansions of \(\{w_i\}_{i=1}^{m}\) involve disjoint families of \(\{H_{\epsilon(k)}\}\).

ii) A similar expansion can be found for \(m\)-dimensional \(d\)-parameter Brownian motion

\[
\mathbf{B}(x) = \mathbf{B}(x, \omega) = (B_1(x, \omega), \ldots, B_m(x, \omega))
\]
defined by (see (2.1.28))

\[
\mathbf{B}(x, \omega) = (\langle \omega_1, \psi \rangle, \ldots, \langle \omega_m, \psi \rangle); \ (\omega_1, \ldots, \omega_m) \in S',
\]

where

\[
\psi(y) = \chi_{[0,x_1] \times \cdots \times [0,x_d]}(y); \ y \in \mathbb{R}^d.
\]

So from (2.2.24) and (2.2.25) we get

\[
\mathbf{B}(x, \omega) = \left( \sum_{j=1}^{\infty} \int_0^x \eta_j(u) du \langle \omega_1, \eta_j \rangle, \ldots, \sum_{j=1}^{\infty} \int_0^x \eta_j(u) du \langle \omega_m, \eta_j \rangle \right)
= \left( \sum_{j=1}^{\infty} \int_0^x \eta_j(u) du H_{\epsilon(i+(j-1)m)}(\omega), \ldots, \sum_{j=1}^{\infty} \int_0^x \eta_j(u) du H_{\epsilon(jm)}(\omega) \right).
\]
Hence the $i$th component, $B_i(x)$, has expansion

$$B_i(x) = \sum_{j=1}^{\infty} \int_{0}^{x} \eta_j(u) du H_{\epsilon(i+(j-1)m)}$$

$$= \int_{0}^{x} \eta_1(u) du H_{\epsilon(i)} + \int_{0}^{x} \eta_2(u) du H_{\epsilon(i+m)} + \int_{0}^{x} \eta_3(u) du H_{\epsilon(i+2m)} + \cdots$$

(2.2.26)

Again we note that the expansions of $\{B_i(x)\}_{i=1}^{m}$ involve disjoint families of $\{H_{\epsilon(k)}\}$.

Note that for white noise and Brownian motion it is natural to have $N = m$. In general, however, one considers functions of white noise or Brownian motion, and in this case $N$ and $m$ need not be related. See Exercise 2.8.

### 2.2.2 Chaos Expansion in Terms of Multiple Itô Integrals

The chaos expansion (2.2.21)–(2.2.22) has an alternative formulation in terms of iterated Itô integrals. Although this formulation will not play a central role in our presentation, we give a brief review of it here, because it makes it easier for the reader to relate the material of the previous sections of this chapter to other literature of related content. Moreover, we will need this version in Section 2.5.

For convenience of notation we set $N = m = d = 1$ for the rest of this section. For the definition and basic properties of (1-parameter) Itô integrals, the reader is referred to Appendix B. For more information, see, e.g., Chung and Williams (1990), Karatzas and Shreve (1991), or Øksendal (2003). If $\psi(t_1, \ldots, t_n)$ is a symmetric function in its $n$ variables $t_1, \ldots, t_n$, then we define its $n$-tuple Itô integral by the formula $(n \geq 1)$

$$\int_{\mathbb{R}^n} \psi dB^{\otimes n} := n! \int_{-\infty}^{t_n} \int_{-\infty}^{t_{n-1}} \cdots \int_{-\infty}^{t_2} \psi(t_1, t_2, \ldots, t_n) dB(t_1) dB(t_2) \cdots dB(t_n),$$

(2.2.27)

where the integral on the right consists of $n$ iterated Itô integrals (note that in each step the corresponding integrand is adapted because of the limits of the preceding integrals). Applying the Itô isometry $n$ times we see that this iterated integral exists iff $\psi \in L^2(\mathbb{R}^n)$, then we have

$$E \left[ \left( \int_{\mathbb{R}^n} \psi dB^{\otimes n} \right)^2 \right] = n! \int_{\mathbb{R}^n} \psi(t_1, \ldots, t_n)^2 dt_1 \cdots dt_n = n! \| \psi \|^2.$$  

(2.2.28)
For \( n = 0 \) we adopt the convention that \( \int_{\mathbb{R}^0} \psi dB^0 = \psi = \|\psi\|_{L^2(\mathbb{R}^0)} \) when \( \psi \) is constant. Let \( \alpha = (\alpha_1, \ldots, \alpha_k) \) be a multi-index, let \( n = |\alpha| \) and let \( \xi_1, \xi_2, \ldots \) be the Hermite functions defined in (2.2.2). Then by a fundamental result in Itô (1951), we have

\[
\int_{\mathbb{R}^n} \xi_1^{\hat{\otimes} \alpha_1} \cdots \xi_k^{\hat{\otimes} \alpha_k} dB^{\otimes n} = \prod_{j=1}^k h_{\alpha_j}(\langle \omega, \xi_j \rangle).
\]

(2.2.29)

Here \( \hat{\otimes} \) denotes the symmetrized tensor product. So, for example, if \( f, g : \mathbb{R} \to \mathbb{R} \), then

\[
(f \otimes g)(x_1, x_2) = f(x_1)g(x_2); \ (x_1, x_2) \in \mathbb{R}^2
\]

and

\[
(f \hat{\otimes} g)(x_1, x_2) = \frac{1}{2} [f \otimes g + g \otimes f](x_1, x_2); \ (x_1, x_2) \in \mathbb{R}^2,
\]

and similarly for higher dimensions and for symmetric tensor powers.

Therefore, by comparison with Definition 2.2.1 we can reformulate (2.2.29) as

\[
\int_{\mathbb{R}^n} \xi^{\hat{\otimes} \alpha} dB^{\otimes n} = H_\alpha(\omega),
\]

(2.2.30)

where we have used multi-index notation

\[
\xi^{\hat{\otimes} \alpha} = \xi_1^{\hat{\otimes} \alpha_1} \cdots \xi_k^{\hat{\otimes} \alpha_k}.
\]

(2.2.31)

Now assume \( f \in L^2(\mu) \) has the Wiener–Itô chaos expansion (2.2.21)

\[
f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega).
\]

We rewrite this as

\[
f(\omega) = \sum_{n=0}^\infty \sum_{|\alpha|=n} c_\alpha H_\alpha(\omega) = \sum_{n=0}^\infty \sum_{|\alpha|=n} c_\alpha \int_{\mathbb{R}^n} \xi^{\hat{\otimes} \alpha} dB^{\otimes n}
\]

\[
= \sum_{n=0}^\infty \int_{\mathbb{R}^n} \sum_{|\alpha|=n} c_\alpha \xi^{\hat{\otimes} \alpha} dB^{\otimes n}.
\]

Hence

\[
f(\omega) = \sum_{n=0}^\infty \int_{\mathbb{R}^n} f_n dB^{\otimes n},
\]

(2.2.32)

with

\[
f_n = \sum_{|\alpha|=n} c_\alpha \xi^{\hat{\otimes} \alpha} \in \hat{L}^2(\mathbb{R}^n),
\]

(2.2.33)

where \( \hat{L}^2(\mathbb{R}^n) \) denotes the symmetric functions in \( L^2(\mathbb{R}^n) \).
More generally, from (2.2.22) and (2.2.28) we have
\[ \| f \|_{L^2(\mu)}^2 = \sum_{n=0}^{\infty} n! \| f_n \|_{L^2(\mathbb{R}^n)}^2. \] (2.2.34)

We summarize this as follows.

**Theorem 2.2.7 (The Wiener–Itô chaos expansion theorem II).** If \( f \in L^2(\mu) \), then there exists a unique sequence of (deterministic) functions \( f_n \in \hat{L}^2(\mathbb{R}^n) \) such that
\[ f(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n dB^\otimes n. \] (2.2.35)

Moreover, we have the isometry
\[ \| f \|_{L^2(\mu)}^2 = \sum_{n=0}^{\infty} n! \| f_n \|_{L^2(\mathbb{R}^n)}^2. \] (2.2.36)

This result extends to arbitrary parameter dimension \( d \). See Itô (1951).

### 2.3 The Hida Stochastic Test Functions and Stochastic Distributions. The Kondratiev Spaces \((S)^{m_1;m_2}_\rho;N(S)^{m_1;m_2}_{-\rho}\)

As we saw in the previous section, the growth condition
\[ \sum_{\alpha} \alpha!c_\alpha^2 < \infty \] (2.3.1)

assures that
\[ f(\omega) := \sum_{\alpha} c_\alpha H_\alpha(\omega) \in L^2(\mu). \]

In the following we will replace condition (2.3.1) by various other conditions. We thus obtain a family of (generalized) function spaces that relates to \( L^2(\mu) \) in a natural way. At the same time these spaces form an environment of stochastic test function spaces and stochastic distribution spaces, in a way that is analogous to the spaces \( S(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset S'(\mathbb{R}^d) \). These spaces provide a favorable setting for the study of stochastic (ordinary and partial) differential equations. They were originally constructed on spaces of sequences by Kondratiev (1978), and later extended by him and others. See Kondratiev et al. (1994) and the references therein.
Let us first recall the characterizations of $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ in terms of Fourier coefficients: As in (2.2.7) we let \( \{\delta^{(j)}\}_{j=1}^{\infty} = \{(\delta^{(j)}_1, \ldots, \delta^{(j)}_d)\}_{j=1}^{\infty} \) be a fixed ordering of all \( d \)-dimensional multi-indices \( \delta = (\delta_1, \ldots, \delta_d) \in \mathbb{N}^d \) satisfying (2.2.7). In general, if \( \alpha = (\alpha_1, \ldots, \alpha_j, \ldots) \in J, \beta = (\beta_1, \ldots, \beta_j, \ldots) \in (\mathbb{N}^N)_c \) are two finite sequences, we will use the notation
\[
\alpha^\beta = \alpha_1^{\beta_1} \alpha_2^{\beta_2} \ldots \alpha_j^{\beta_j} \ldots \text{ where } \alpha_0^0 = 1.
\] (2.3.2)

**Theorem 2.3.1** Reed and Simon (1980), Theorem V. 13–14.

a) Let \( \phi \in L^2(\mathbb{R}^d) \), so that
\[
\phi = \sum_{j=1}^{\infty} a_j \eta_j,
\] (2.3.3)
where \( a_j = (\phi, \eta_j); j = 1, 2, \ldots \), are the Fourier coefficients of \( \phi \) with respect to \( \{\eta_j\}_{j=1}^{\infty} \), with \( \eta_j \) as in (2.2.8). Then \( \phi \in S(\mathbb{R}^d) \) if and only if
\[
\sum_{j=1}^{\infty} a_j^2 (\delta^{(j)})^\gamma < \infty
\] (2.3.4)
for all \( d \)-dimensional multi-indices \( \gamma = (\gamma_1, \ldots, \gamma_d) \).

b) The space \( S'(\mathbb{R}^d) \) can be identified with the space of all formal expansions
\[
T = \sum_{j=1}^{\infty} b_j \eta_j
\] (2.3.5)
such that
\[
\sum_{j=1}^{\infty} b_j^2 (\delta^{(j)})^{-\theta} < \infty
\] (2.3.6)
for some \( d \)-dimensional multi-index \( \theta = (\theta_1, \ldots, \theta_d) \).

If (2.3.6) holds, then the action of \( T \in S'(\mathbb{R}^d) \) given by (2.3.5) on \( \phi \in S(\mathbb{R}^d) \) given by (2.3.2) reads
\[
\langle T, \phi \rangle = \sum_{j=1}^{\infty} a_j b_j.
\] (2.3.7)

We now formulate a stochastic analogue of Theorem 2.3.1. The following quantity is crucial:

If \( \gamma = (\gamma_1, \ldots, \gamma_j, \ldots) \in (\mathbb{R}^N)_c \) (i.e., only finitely many of the real numbers \( \gamma_j \) are nonzero), we write
\[
(2\mathbb{N})^\gamma := \prod_j (2j)^{\gamma_j}.
\] (2.3.8)
As before, \(d\) is the parameter dimension, \(m\) is the dimension of the white noise vector, \(\mu_m = \mu_1 \times \cdots \times \mu_1\) as in (2.1.24), and \(N\) is the state space dimension.

**Definition 2.3.2. The Kondratiev spaces of stochastic test function and stochastic distributions.**

**a)** The stochastic test function spaces

Let \(N\) be a natural number. For \(0 \leq \rho \leq 1\), let

\[(S)^N = (S)^{m;N}\]

consist of those

\[f = \sum_\alpha c_\alpha H_\alpha \in L^2(\mu_m) = \bigoplus_{k=1}^N L^2(\mu_m) \quad \text{with} \quad c_\alpha \in \mathbb{R}^N\]

such that

\[\|f\|^2_{\rho,k} := \sum_\alpha c_\alpha^2 (\alpha!)^{1+\rho}(2N)^{k\alpha} < \infty \quad \text{for all} \quad k \in \mathbb{N}\]  \hspace{1cm} (2.3.9)

where

\[c_\alpha^2 = |c_\alpha|^2 = \sum_{k=1}^N (c_\alpha^{(k)})^2 \quad \text{if} \quad c_\alpha = (c_\alpha^{(1)}, \ldots, c_\alpha^{(N)}) \in \mathbb{R}^N.\]

**b)** The stochastic distribution spaces

For \(0 \leq \rho \leq 1\), let

\[(S)^{-N} = (S)^{m;N}\]

consist of all formal expansions

\[F = \sum_\alpha b_\alpha H_\alpha \quad \text{with} \quad b_\alpha \in \mathbb{R}^N\]

such that

\[\|F\|^2_{-\rho,-q} := \sum_\alpha b_\alpha^2 (\alpha!)^{1-\rho}(2N)^{-q\alpha} < \infty \quad \text{for some} \quad q \in \mathbb{N}.\]  \hspace{1cm} (2.3.10)

The family of seminorms \(\|f\|_{\rho,k}; \quad k \in \mathbb{N}\) gives rise to a topology on \((S)^N\); and we can regard \((S)^{-N}\) as the dual of \((S)^N\) by the action

\[\langle F, f \rangle = \sum_\alpha (b_\alpha, c_\alpha)\alpha!\]  \hspace{1cm} (2.3.11)

if

\[F = \sum_\alpha b_\alpha H_\alpha \in (S)^{-N}; \quad f = \sum_\alpha c_\alpha H_\alpha \in (S)^N\]
and \((b_{\alpha}, c_{\alpha})\) is the usual inner product in \(\mathbb{R}^N\). Note that this action is well defined since
\[
\sum_{\alpha} |(b_{\alpha}, c_{\alpha})| \alpha! = \sum_{\alpha} |(b_{\alpha}, c_{\alpha})|(\alpha!)^{\frac{1-\rho}{2}} (\alpha!)^{\frac{1+\rho}{2}} (2N)^{-\frac{q_{\alpha}}{2}} (2N)^{\frac{q_{\alpha}}{2}} \\
\leq \left( \sum_{\alpha} b_{\alpha}^2 (\alpha!)^{1-\rho} (2N)^{-q_{\alpha}} \right)^{\frac{1}{2}} \left( \sum_{\alpha} c_{\alpha}^2 (\alpha!)^{1+\rho} (2N)^{q_{\alpha}} \right)^{\frac{1}{2}} < \infty
\]
for \(q\) large enough. When the value of \(m\) is clear from the context we simply write \((S)^N_\rho\), \((S)^N_{-\rho}\) instead of \((S)^m_N\), \((S)^{m:N}_{-\rho}\), respectively. If \(N = 1\), we write \((S)^1_\rho\), \((S)^{-1}_{-\rho}\) instead of \((S)^1\), \((S)^{-1}_{-\rho}\), respectively.

**Remarks**

a) Note that for general \(\rho \in [0, 1]\) we have
\[
(S)^N_1 \subset (S)^N_\rho \subset (S)^N_0 \subset \mathbb{L}^2(\mu_m) \subset (S)^N_{-0} \subset (S)^N_{-1}. \tag{2.3.12}
\]
From (2.3.11) we see that if \(F = (F_1, \ldots, F_N)\) and \(G = (G_1, \ldots, G_N)\) both belong to \(\mathbb{L}^2(\mu_m)\), then the action of \(F\) on \(G\) is given by
\[
\langle F, G \rangle = E \left[ \sum_{i=1}^N F_i G_i \right]. \tag{2.3.13}
\]

b) In some cases it is useful to consider various generalizations of the spaces \((S)^{m:N}_\rho\), \((S)^{m:N}_{-\rho}\). For example, the coefficients \(c_{\alpha}, b_{\alpha}\) may not be constants, but may depend on some random parameter \(\hat{\omega}\) that is independent of our white noise. In these cases we assume that \(b_{\alpha}(\hat{\omega}), c_{\alpha}(\hat{\omega}) \in L^2(\nu)\), where \(\nu\) is the probability measure for \(\hat{\omega}\). Then the definitions above apply, with the modification that in (2.3.9) we replace \(c_{\alpha}^2\) by \(\|c_{\alpha}\|_{L^2(\nu)}^2\) and in (2.3.11) we interpret \((b_{\alpha}, c_{\alpha})\) as
\[
E_\nu [b_{\alpha} c_{\alpha}] = \int_{\hat{\Omega}} b_{\alpha}(\hat{\omega}) c_{\alpha}(\hat{\omega}) d\nu(\hat{\omega}) = (b_{\alpha}, c_{\alpha})_{L^2(\nu)}. \tag{2.3.14}
\]

Another useful generalization (where the \(c_{\alpha}\) are elements of Sobolev spaces) is discussed in Våge (1996a).

c) The quantity \((2N)^\alpha\) in (2.3.8) will be applied frequently, so it is useful to have some estimates of it.

First note that if \(\alpha = \epsilon^{(k)}\), we get
\[
(2N)^\epsilon^{(k)} = 2k. \tag{2.3.15}
\]

Next we state the following result from Zhang (1992).
**Proposition 2.3.3 Zhang (1992).** We have that
\[
\sum_{\alpha \in J} (2N)^{-q\alpha} < \infty \tag{2.3.16}
\]
if and only if \(q > 1\).

**Proof** First assume \(q > 1\). If \(\alpha = (\alpha_1, \alpha_2, \ldots) \in J\), define
\[
\text{Index } \alpha = \max\{j; \alpha_j \neq 0\}.
\]
Consider
\[
a_n := \sum_{\text{Index } \alpha = n} (2N)^{-q\alpha} = \sum_{\alpha_1 \cdots \alpha_{n-1} \geq 0, \alpha_n \geq 1} \prod_{j=1}^{\infty} (2j)^{-q\alpha_j}
\]
This gives
\[
\frac{a_n}{a_{n+1}} - 1 = \frac{(2n+2)^q - 1}{(2n)^q} - 1 = \left(1 + \frac{1}{n}\right)^q - (2n)^{-q} - 1. \tag{2.3.17}
\]
In particular,
\[
\frac{a_n}{a_{n+1}} - 1 \geq \frac{q}{n} - (2n)^{-q}.
\]
Hence
\[
\liminf_{n \to \infty} n \left(\frac{a_n}{a_{n+1}} - 1\right) \geq q > 1
\]
and, therefore, by Abel’s criterion for convergence,
\[
\sum_{\alpha} (2N)^{-q\alpha} = \sum_{n=0} a_n < \infty,
\]
as claimed. Conversely, if \(q = 1\), then, by (2.3.17) above,
\[
\frac{a_n}{a_{n+1}} = 1 + \frac{1}{2n}, \text{ so } \lim_{n \to \infty} n \left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n \to \infty} n \cdot \frac{1}{2n} = \frac{1}{2} < 1.
\]
Hence \(\sum_{n=0}^\infty a_n = \infty\) by Abel’s criterion. \(\square\)
The following useful result relates the sum in (2.3.16) to sums of the type appearing in Theorem 2.3.1. It was pointed out to us by Y. Hu (private communication).

**Lemma 2.3.4.** Let \( \delta^{(j)} = (\delta_1^{(j)}, \delta_2^{(j)}, \ldots, \delta_d^{(j)}) \) be as in (2.2.7). For all \( j \in \mathbb{N} \) and all \( d \in \mathbb{N} \) we have

\[
\frac{j}{d} \leq \delta_1^{(j)} \cdot \delta_2^{(j)} \cdots \delta_d^{(j)} \leq jd.
\]

**Proof** The case \( d = 1 \) is trivial, so we fix \( d \geq 2 \). Since \( \delta_k^{(j)} \leq j \) (by (2.2.7)), the second inequality is immediate. To prove the first inequality, we fix \( j \) and set

\[
M = \delta_1^{(j)} + \delta_2^{(j)} + \cdots + \delta_d^{(j)}.
\]

Note that \( M \geq d \). Consider the minimization problem

\[
\inf \left\{ f(x_1,\ldots,x_d) \mid x_i \in [1, \infty) \text{ for all } i; \sum_{i=1}^{d} x_i = M \right\},
\]

where \( f(x) = x_1 x_2 \cdots x_d \). Clearly a minimum exists. Using the Lagrange multiplier method we see that the only candidate for a minimum point when \( x_i > 1 \) for all \( i \) is \((x_1,x_2,\ldots,x_d) = (M/d,\ldots,M/d)\), which gives the value

\[
f\left(\frac{M}{d},\ldots,\frac{M}{d}\right) = \left(\frac{M}{d}\right)^d.
\]

If one or several \( x_i \)'s have the value 1, then the minimization problem can be reduced to the case when \( d \) and \( M \) are replaced by \( d-1 \) and \( M-1 \), respectively. Since

\[
\left(\frac{M-1}{d-1}\right)^{d-1} \leq \left(\frac{M}{d}\right)^d,
\]

we conclude by induction on \( d \) that

\[
x_1 \cdots x_d \geq M - d + 1 \text{ for all } (x_1,\ldots,x_d) \in [1, \infty)^d \text{ with } \sum_{i=1}^{d} x_i = M.
\]

(2.3.18)

To finish the proof of the lemma we now compare \( M \) and \( j \):

Since \( \delta_1^{(j)} + \cdots + \delta_d^{(j)} = M \) and the sequence \( \{(\delta_1^{(i)},\ldots,\delta_d^{(i)})\}_{i=1}^{\infty} \) is increasing (see (2.2.7)), we know that

\[
\delta_1^{(i)} + \cdots + \delta_d^{(i)} \leq M \text{ for all } i < j.
\]

Now (by a known result in combinatorics) the total number of multi-indices \((\delta_1,\ldots,\delta_d) \in \mathbb{N}^d\) such that \( \delta_1 + \delta_2 + \cdots + \delta_d \leq M \) is equal to

\[
\sum_{n=d}^{M} \binom{n-1}{d-1} = \binom{M}{d}.
\]
Therefore
\[ j \leq \binom{M}{d} = \frac{M(M-1)\cdots(M-d+1)}{d!} \leq (M-d+1)^d \]
or
\[ M - d + 1 \geq j^{\frac{1}{d}}. \]

Combined with (2.3.18) this gives
\[ \delta_1^{(j)} \delta_2^{(j)} \cdots \delta_d^{(j)} \geq j^{\frac{1}{d}}. \]
\[ \square \]

As a consequence of this, we obtain the following alternative characterization of the spaces \((S)^N_\rho, (S)^N_{-\rho}\). This characterization has often been used as a definition of the Kondratiev spaces (see, e.g., Holden et al. (1995a), and the references therein). As usual we let \((\delta_1^{(j)}, \ldots, \delta_d^{(j)})\) be as in (2.2.7).

In this connection the following notation is convenient:

With \((\delta_1^{(j)}, \ldots, \delta_d^{(j)})\) as in (2.2.7), let \(\Delta = (\Delta_1, \ldots, \Delta_k, \ldots) \in \mathbb{N}^\mathbb{N}\) be the sequence defined by
\[ \Delta_j = 2^d \delta_1^{(j)} \delta_2^{(j)} \cdots \delta_d^{(j)}; \quad j = 1, 2, \ldots. \] (2.3.19)

Then if \(\alpha = (\alpha_1, \ldots, \alpha_j, \ldots) \in (\mathbb{R}^N)_c\), we define
\[ \Delta^\alpha = \Delta_1^{\alpha_1} \Delta_2^{\alpha_2} \cdots \Delta_j^{\alpha_j} \cdots = \prod_{j=1}^{\infty} (2^d \delta_1^{(j)} \cdots \delta_d^{(j)})^{\alpha_j}, \] (2.3.20)
in accordance with the general multi-index notation (2.3.2).

**Corollary 2.3.5.** Let \(0 \leq \rho \leq 1\). Then we have
a) \( f = \sum_\alpha c_\alpha H_\alpha \) (with \(c_\alpha \in \mathbb{R}^N\) for all \(\alpha\)) belongs to \((S)^N_\rho\) if and only if
\[ \sum_\alpha c_\alpha^2 (\alpha!)^{1+\rho} \Delta^{k\alpha} < \infty \quad \text{for all} \quad k \in \mathbb{N}. \] (2.3.21)
b) The formal expansion \( F = \sum_\alpha b_\alpha H_\alpha \) (with \(b_\alpha \in \mathbb{R}^N\) for all \(\alpha\)) belongs to \((S)^N_{-\rho}\) if and only if
\[ \sum_\alpha b_\alpha^2 (\alpha!)^{1-\rho} \Delta^{-q\alpha} < \infty \quad \text{for some} \quad q \in \mathbb{N}. \] (2.3.22)
Proof By the second inequality of Lemma 2.3.4 we see that
\[
\Delta^{k\alpha} = \prod_{j=1}^{\infty} (2^d \delta_1^{(j)} \ldots \delta_d^{(j)})^{k\alpha_j} \leq \prod_{j=1}^{\infty} (2^d j^d)^{k\alpha_j} = \prod_{j=1}^{\infty} (2^d)^{dk\alpha_j} \tag{2.3.23}
\]
for all \(\alpha \in \mathcal{J}\) and all \(k \in \mathbb{N}\).

From the first inequality of Lemma 2.3.4 we get
\[
\prod_{j=1}^{\infty} (2^d \delta_1^{(j)} \ldots \delta_d^{(j)})^{k\alpha_j} \geq \prod_{j=1}^{\infty} (2^{d-1} j^d)^{k\alpha_j} \geq \prod_{j=1}^{\infty} (2^d)^{dk\alpha_j} \geq \prod_{j=1}^{\infty} (2^d)^{d\alpha_j} \tag{2.3.24}
\]
for all \(\alpha \in \mathcal{J}\) and all \(k \in \mathbb{N}\). These two inequalities show the equivalence of the criterion in a) with (2.3.9) and the equivalence of the criterion in b) with (2.3.10).

\[\square\]

Example 2.3.6. If \(\phi \in \mathcal{S}(\mathbb{R}^d)\), then
\[
w(\phi, \omega) \in (\mathcal{S})_1.
\]

Indeed, by (2.2.23) we have
\[
w(\phi, \omega) = \sum_{j=1}^{\infty} (\phi, \eta_j) H_{e(j)}(\omega) = \sum_{\alpha} c_{\alpha} H_{\alpha},
\]
so
\[
\sum_{\alpha} c_{\alpha}^2 (\alpha!)^2 \Delta^{k\alpha} = \sum_{j=1}^{\infty} (\phi, \eta_j)^2 (2^d \delta_1^{(j)} \ldots \delta_d^{(j)})^k
\]
\[
= 2^{dk} \sum_{j=1}^{\infty} (\phi, \eta_j)^2 (\delta_1^{(j)} \ldots \delta_d^{(j)})^k < \infty
\]
by Theorem 2.3.1a. Hence \(w(\phi, \omega) \in (\mathcal{S})_1\) by Corollary 2.3.5.

\[\square\]

Note that with our notation \((\mathcal{S})_0\) and \((\mathcal{S})_{-0}\) are different spaces. In fact, they coincide with the Hida spaces \((\mathcal{S})\) and \((\mathcal{S})^*\), respectively, which we describe below.

Remark The definition of stochastic test function and distribution spaces used in Kondratiev et al. (1994), and Kondratiev et al. (1995a), does not coincide with ours. However, the two definitions are equivalent, as we will now show.

Let us first recall Kondratiev’s definition. Let \(\phi \in L^2(\mu_1)\) be given by
\[
\phi = \sum_{\alpha} c_{\alpha} H_{\alpha}(\omega).
\]
For \( p \in \mathbb{R} \) define

\[
\mathcal{K}_p := \{ \phi \in L^2(\mu_1) : \|\phi\|^2_p < +\infty \}
\]

where

\[
\|\phi\|^2_p = \sum_{n=0}^{\infty} (n!)^2 \sum_{|\alpha| = n} c^2_{\alpha}(2N)^{\alpha p} < +\infty.
\]

The Kondratiev test function space \((\mathcal{K})_1\) is defined as

\[
(\mathcal{K})_1 = \bigcap_p \mathcal{K}_p,
\]

the projective limit of \(\mathcal{K}_p\).

The Kondratiev distribution space \((\mathcal{K})_{-1}\) is the inductive limit of \(\mathcal{K}_{-p}\), the dual of \(\mathcal{K}_p\).

According to our definition,

\[
\|\phi\|_{1,p}^2 = \sum_{\alpha} c^2_{\alpha}(\alpha!)^2(2N)^{\alpha p} = \sum_{n=0}^{\infty} \sum_{|\alpha| = n} (\alpha!)^2 c^2_{\alpha}(2N)^{\alpha p}.
\]

Obviously \(\alpha! \leq n!\) if \(|\alpha| = n\). Therefore

\[
\|\phi\|_1 \leq \|\phi\|_p.
\]

Hence

\[
(\mathcal{K})_1 \subset (\mathcal{S})_1.
\]

On the other hand, if \(\alpha_1 + \alpha_2 + \cdots + \alpha_m = n, \alpha_i \geq 1\) we have

\[
n! = \alpha_1!(\alpha_1 + 1)(\alpha_1 + 2) \cdots (\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + 1) \cdots (\alpha_1 + \cdots + \alpha_m)
\leq \alpha_1!\alpha_1(\alpha_2 + 1)!\alpha_2(\alpha_3 + 1)! \cdots \alpha_m(\alpha_m + 1)!
\leq \prod_{j=1}^{m} (\alpha_j(\alpha_j + 1))\alpha_j!
= \left( \prod_{j=1}^{m} \alpha_j! \right) \prod_{j=1}^{m} 2\alpha_j^2
\leq \alpha! \prod_{j=1}^{m} (2j)^{2\alpha_j}
\text{ (since } 2\alpha_j^2 \leq (2j)^{2\alpha_j} = \alpha!(2N)^{2\alpha}.\text{)}
\]

Hence

\[
\|\phi\|^2_p = \sum_{n=0}^{\infty} (n!)^2 \sum_{|\alpha| = n} c^2_{\alpha}(2N)^{\alpha p} \leq \sum_{n=0}^{\infty} \sum_{|\alpha| = n} (\alpha!)^2(2N)^{4\alpha} c^2_{\alpha}(2N)^{\alpha p}
= \sum_{\alpha} (\alpha!)^2 c^2_{\alpha}(2N)^{\alpha(p+4)} = \|\phi\|^2_{1,p+4}.
\]

This shows \((\mathcal{S})_1 \subset (\mathcal{K})_1\), and hence \((\mathcal{S})_1 = (\mathcal{K})_1\).
2.3.1 The Hida Test Function Space \((\mathcal{S})\) and the Hida Distribution Space \((\mathcal{S})^*\)

There is an extensive literature on these spaces. See Hida et al. (1993), and the references therein. According to the characterization in Zhang (1992), we can describe these spaces, generalized to arbitrary dimension \(m\), as follows:

**Proposition 2.3.7** Zhang (1992). a) The Hida test function space \((\mathcal{S})^N\) consists of those

\[
f = \sum_{\alpha} c_{\alpha} H_{\alpha} \in L^2(\mu_m) \quad \text{with} \quad c_{\alpha} \in \mathbb{R}^N,
\]

such that

\[
\sup_{\alpha} \{c_{\alpha}^2 \alpha!(2N)^k\} < \infty \quad \text{for all} \quad k < \infty. \tag{2.3.25}
\]

b) The Hida distribution space \((\mathcal{S})^{*,N}\) consists of all formal expansions

\[
F = \sum_{\alpha} b_{\alpha} H_{\alpha} \quad \text{with} \quad b_{\alpha} \in \mathbb{R}^N
\]

such that

\[
\sup_{\alpha} \{b_{\alpha}^2 \alpha!(2N)^{-q}\} < \infty \quad \text{for some} \quad q < \infty. \tag{2.3.26}
\]

Hence, after comparison with Definition 2.3.2, we see that

\[(\mathcal{S})^N = (\mathcal{S})^{m;N}_0 \quad \text{and} \quad (\mathcal{S})^{*,N} = (\mathcal{S})^{m;N}_0. \tag{2.3.27}\]

If \(N = 1\), we write

\[(\mathcal{S})^1 = (\mathcal{S}) \quad \text{and} \quad (\mathcal{S})^{*,1} = (\mathcal{S})^*.\]

**Corollary 2.3.8.** For \(N = 1\) and \(p \in (1, \infty)\) we have

\[(\mathcal{S}) \subset L^p(\mu_m) \subset (\mathcal{S})^*. \tag{2.3.28}\]

**Proof** We give a proof in the case \(m = d = 1\). Since \(L^p(\mu) \supset L^p(\mu)\) for \(p' > p\) and the dual of \(L^p(\mu)\) is \(L^q(\mu)\) with \(1/p + 1/q = 1\) if \(1 < p < \infty\), it suffices to prove that

\[(\mathcal{S}) \subset L^p(\mu) \quad \text{for all} \quad p \in (1, \infty).\]

To this end choose

\[
f = \sum_{\alpha} c_{\alpha} H_{\alpha} \in (\mathcal{S}).
\]
Then
\[
\|f\|_{L^p(\mu)} \leq \sum_{\alpha} |c_{\alpha}| \|H_{\alpha}\|_{L^p(\mu)}.
\]

If \( \alpha = (\alpha_1, \ldots, \alpha_k) \), we have \( H_{\alpha}(\omega) = h_{\alpha_1}(\langle \omega, \xi_1 \rangle) h_{\alpha_2}(\langle \omega, \xi_2 \rangle) \cdots h_{\alpha_k}(\langle \omega, \xi_k \rangle) \) for \( \omega \in S'(\mathbb{R}) \). Hence, by independence,
\[
\|H\|_{L^p(\mu)} = E[|H_{\alpha}|^p] = \prod_{j=1}^k E[h_{\alpha_j}^p(\langle \omega, \xi_j \rangle)] = \sum_{\alpha} |c_{\alpha}| \|H_{\alpha}\|_{L^p(\mu)}.
\tag{2.3.29}
\]

Note that by (2.2.29), we have
\[
h_{\alpha_j}(\langle \omega, \xi_j \rangle) = \int_{\mathbb{R}^n} \xi_j^{\hat{\alpha}_j}(x) dB^{\hat{\alpha}_j}(x).
\tag{2.3.30}
\]

By the Carlen–Kree estimates in Carlen and Kree (1991), we have, in general, for \( p \geq 1, n \in \mathbb{N}, \phi \in L^2(\mathbb{R}) \),
\[
\left\| \int_{\mathbb{R}^n} \phi \hat{n}(x) dB^{\hat{n}}(x) \right\|_{L^p(\mu)} \leq \sqrt{n!}(\theta_p \sqrt{e})^n (2\pi n)^{-\frac{1}{2}} \|\phi\|^n,
\tag{2.3.31}
\]
where
\[
\theta_p = 1 + \sqrt{1 + \frac{1}{p}}.
\tag{2.3.32}
\]

Applied to (2.3.29)–(2.3.30), this gives
\[
\|H\|_{L^p(\mu)} \leq \prod_{j=1}^k \sqrt{\alpha_j!(\theta_p \sqrt{e})^\alpha_j} (2\pi \alpha_j)^{-\frac{1}{2}} \leq \sqrt{\alpha!(\theta_p \sqrt{e})^\alpha}.
\]

Hence, by (2.3.25),
\[
\|f\|_{L^p(\mu)} \leq \sum_{\alpha} |c_{\alpha}| \sqrt{\alpha!(\theta_p \sqrt{e})^\alpha}
\leq \sum_{\alpha} |c_{\alpha}| \sqrt{\alpha!(2N)^{\alpha}}(\theta_p \sqrt{e})^\alpha (2N)^{-k\alpha}
\leq \sup_{\alpha} \left\{ |c_{\alpha}| \sqrt{\alpha!(2N)^{\alpha}} \right\} \sum_{\alpha} (\theta_p \sqrt{e})^\alpha (2N)^{-k\alpha}
\leq \infty
\]
for \( k \) large enough. \qed
2.3.2 Singular White Noise

One of the many useful properties of $(\mathcal{S})^*$ is that it contains the singular or pointwise white noise.

Definition 2.3.9. a) The 1-dimensional (d-parameter) singular white noise process is defined by the formal expansion

$$W(x) = W(x, \omega) = \sum_{k=1}^{\infty} \eta_k(x) H_{\epsilon(k)}(\omega); \ x \in \mathbb{R}^d,$$  \hspace{1cm} (2.3.33)

where $\{\eta_k\}_{k=1}^{\infty}$ is the basis of $L^2(\mathbb{R}^d)$ defined in (2.2.8) while $H_\alpha = H_\alpha^{(1)}$ is defined by (2.2.10).

b) The $m$-dimensional (d-parameter) singular white noise process is defined by

$$W(x) = W(x, \omega) = (W_1(x, \omega), \ldots, W_m(x, \omega)),$$

where the $i$th component $W_i(x)$, of $W(x)$, has expansion

$$W_i(x) = \sum_{j=1}^{\infty} \eta_j(x) H_{\epsilon_{i+j-1}m}$$

$$= \eta_1(x) H_{\epsilon^{(i)}} + \eta_2(x) H_{\epsilon_{i+m}} + \eta_3(x) H_{\epsilon_{i+2m}} + \cdots.$$  \hspace{1cm} (2.3.34)

(Compare with the expansion (2.2.25) we have for smoothed $m$-dimensional white noise.)

Proposition 2.3.10.

$$W(x, \omega) \in (\mathcal{S})^{*,m} \text{ for each } x \in \mathbb{R}^d.$$  

Proof (i) $m = 1$. We must show that the expansion (2.3.34) satisfies condition (2.3.10) for $\rho = 0$, i.e.,

$$\sum_{k=1}^{\infty} \eta_k^2(x)(2k)^{-q} < \infty$$  \hspace{1cm} (2.3.35)

for some $q \in \mathbb{N}$. By (2.2.5) and (2.2.8) we have $|\eta_k^2(x)| \leq C$ for all $k = 1, 2, \ldots, x \in \mathbb{R}^d$ for a constant $C$.

Therefore, by Proposition 2.3.3, the series in (2.3.35) converges for all $q > 1$.

(ii) $m > 1$. The proof in this case is similar to the above, replacing $\eta_k$ by $e^{(k)}$. \hfill \square
Remark Using (2.2.11) we may rewrite (2.3.34) as
\[
W(x, \omega) = \sum_{i=1, \ldots, m}^\infty \sum_{j=1, 2, \ldots} e^{i+(j-1)m}(x)H_{(i+m+j-1)m}(\omega)
\]
(2.3.36)

By comparing the expansion (2.3.33) for singular white noise \( W(x) \) with the expansion (2.2.24) for Brownian motion \( B(x) \), we see that
\[
W_i(x) = \partial^d \partial x_1 \cdots \partial x_d B_i(x) \quad \text{in } (S)^*; \quad \text{for } 1 \leq i \leq m = N, d \geq 1 \quad (2.3.37)
\]
In particular,
\[
W(t) = \frac{d}{dt} B(t) \quad \text{in } (S)^* \quad (d = m = N = 1) \quad (2.3.38)
\]

See Exercise 2.30. See also (2.5.27).

Thus we may say that \( m \)-dimensional singular white noise \( W(x, \omega) \) consists of \( m \) independent copies of 1-dimensional singular white noise. Here “independence” is interpreted in the sense that if we truncate the summations over \( j \) to a finite number of terms, then the components are independent when they are regarded as random variables in \( L^2(\mu_m) = L^2(\mu_1 \times \cdots \times \mu_1) \).

In spite of Proposition 2.3.10 and the fact that also many other important Brownian functionals belong to \((S)^* \) (see Hida et al. (1993)), the space \((S)^* \) turns out to be too small for the purpose of solving stochastic ordinary and partial differential equations. We will return to this in Chapters 3 and 4, where we will give examples of such equations with no solution in \((S)^* \) but a unique solution in \((S)_{-1} \).

### 2.4 The Wick Product

The Wick product was introduced in Wick (1950) as a tool to renormalize certain infinite quantities in quantum field theory. In stochastic analysis the Wick product was first introduced by Hida and Ikeda (1965). A systematic, general account of the traditions of both mathematical physics and probability theory regarding this subject was given in Dobrushin and Minlos (1977). In Meyer and Yan (1989), this kind of construction was extended to cover Wick products of Hida distributions. We should point out that this (stochastic) Wick product does not in general coincide with the Wick product in physics, as defined, e.g., in Simon (1974). See also the survey in Gjessing et al. (1993).
Today the Wick product is also important in the study of stochastic (ordinary and partial) differential equations. In general, one can say that the use of this product corresponds to – and extends naturally – the use of Itô integrals. We now explain this in more detail.

The (stochastic) Wick product can be defined in the following way:

**Definition 2.4.1.** The Wick product $F \diamond G$ of two elements

$$F = \sum_{\alpha} a_{\alpha} H_{\alpha}, \quad G = \sum_{\alpha} b_{\alpha} H_{\alpha} \in (S)_{-1}^{m;N} \quad \text{with} \quad a_{\alpha}, b_{\alpha} \in \mathbb{R}^{N} \tag{2.4.1}$$

is defined by

$$F \diamond G = \sum_{\alpha, \beta} (a_{\alpha}, b_{\beta}) H_{\alpha + \beta}. \tag{2.4.2}$$

With this definition the Wick product can be described in a very simple manner. What is not obvious from the construction, however, is that $F \diamond G$ in fact does not depend on our particular choice of base elements for $L^{2}(\mu)$. It is possible to give a direct proof of this, but the details are tedious. A sketch of a proof is given in Appendix D.

In the $L^{2}(\mu)$ case the basis independence of the Wick product can also be seen from the following formulation of Wick multiplication in terms of multiple Itô integrals (see Theorem 2.2.7).

**Proposition 2.4.2.** Let $N = m = d = 1$. Assume that $f, g \in L^{2}(\mu)$ have the following representation in terms of multiple Itô integrals:

$$f(\omega) = \sum_{i=0}^{\infty} \int_{\mathbb{R}^{i}} f_{i} dB^{\otimes i}, \quad g(\omega) = \sum_{j=0}^{\infty} \int_{\mathbb{R}^{j}} g_{j} dB^{\otimes j}. \tag{2.2.30}$$

Suppose $f \circ g \in L^{2}(\mu)$. Then

$$(f \circ g)(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^{n}} \sum_{i+j=n} f_{i} \hat{\otimes} g_{j} dB^{\otimes n}. \tag{2.4.3}$$

**Proof** By (2.4.2) we have

$$(f \circ g)(\omega) = \sum_{\alpha, \beta} a_{\alpha} b_{\beta} H_{\alpha + \beta}(\omega)$$

and by (2.2.30) we have

$$H_{\alpha + \beta}(\omega) = \int_{\mathbb{R}^{|\alpha + \beta|}} \xi_{\alpha} \hat{\otimes} \xi_{\beta} dB^{\otimes |\alpha + \beta|}.$$
Combining this with \((2.2.33)\), we get
\[
(f \circ g)(\omega) = \sum_{n=0}^{\infty} \sum_{|\alpha + \beta| = n} a_{\alpha}b_{\beta} \left( \int_{\mathbb{R}^n} \xi_\alpha \hat{\otimes} \xi_\beta dB^\otimes n \right)
\]
\[
= \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} \left( \sum_{i+j=n} \sum_{|\alpha| = i} a_{\alpha} \xi_\alpha \hat{\otimes} \sum_{|\beta| = j} b_{\beta} \xi_\beta \right) dB^\otimes n
\]
\[
= \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} \left( \sum_{i+j=n} f_i \hat{\otimes} g_j \right) dB^\otimes n,
\]

as claimed. \(\square\)

**Example 2.4.3.** Let \(0 \leq t_0 < t_1 < \infty\) and assume that \(h(\omega) \in L^2(\mu_1)\) is \(\mathcal{F}_{t_0}\)-measurable. Then
\[
h \circ (B(t_1) - B(t_0)) = h \cdot (B(t_1) - B(t_0)). \tag{2.4.4}
\]

**Proof** If
\[
h(\omega) = \sum_{i=0}^{\infty} \int \chi_{[t_0, t_1]}(x) dB^\otimes i(x),
\]
then each of the functions \(h_i(x)\) must satisfy
\[
h_i(x) = 0 \text{ almost surely outside } \{x; x_j \leq t_0 \text{ for } j = 1, 2, \ldots, n\}.
\]

Therefore the symmetrized tensor product of \(\chi_{[t_0, t_1]}(s)\) and \(h_i(x_1, \ldots, x_n)\) is given by (with \(x_{n+1} = s\))
\[
(\chi_{[t_0, t_1]} \hat{\otimes} h_i)(x_1, \ldots, x_{n+1}) = \frac{h(y)}{n+1} \chi_{[t_0, t_1]}(\max_{j} \{x_j\}),
\]
where \(y = (y_1, y_2, \ldots, y_n)\) is an arbitrary permutation of the remaining \(x_i\) when \(\tilde{y} := x_{\tilde{j}} := \max_{1 \leq i \leq n+1} \{x_i\}\) is removed. (For almost all \((x_1, \ldots, x_{n+1})\) there is a unique such \(\tilde{j}\).)

Since
\[
B(t_1) - B(t_0) = \int \chi_{[t_0, t_1]}(s) dB(s),
\]
we get, by \((2.4.3)\),
\[
h \circ (B(t_1) - B(t_0))
\]
\[
= \sum_{n=0}^{\infty} \int_{\mathbb{R}^{n+1}} (\chi_{[t_0, t_1]} \hat{\otimes} h_i)(x_1, \ldots, x_{n+1}) dB^\otimes (n+1)(x_1, \ldots, x_{n+1})
\]
\[\sum_{n=0}^{\infty} (n+1)! \int_{t_0}^{t_1} \int_{x_0}^{x_n} \cdots \int_{x_0}^{x_1} \frac{1}{n+1} \chi_{[t_0,t_1]}(y) h dB(x_1) \cdots dB(x_{n+1})\]

\[= \sum_{n=0}^{\infty} n! \int_{t_0}^{t_1} \left[ \int_{0}^{x_2} \int_{0}^{x_3} \cdots \int_{0}^{x_n} h(x_1, \ldots, x_n) dB(x_1) dB(x_2) \cdots \right] dB(x_{n+1})\]

\[= \sum_{n=0}^{\infty} n! \left( \int_{0}^{t_1} h(x_1, \ldots, x_n) dB(x_1) dB(x_2) \cdots \right) \left( \int_{t_0}^{t_1} dB(x_{n+1}) \right)\]

\[= \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} h(x) dB \bigotimes^n (x) \cdot (B(t_1) - B(t_0)) = h \cdot (B(t_1) - B(t_0)).\]

\[\square\]

An important property of the spaces \((S)_{-1}, (S)_1\) and \((S)^*, (S)\) is that they are closed under Wick products.

**Lemma 2.4.4.**

a) \(F, G \in (S)_{m:N} \Rightarrow F \circ G \in (S)_{m-1}^{m+1}\);

b) \(f, g \in (S)_1^{m:N} \Rightarrow f \circ g \in (S)_1^{m+1}\);

c) \(F, G \in (S)^{*,N} \Rightarrow F \circ G \in (S)^{*,1};\)

d) \(f, g \in (S)^N \Rightarrow f \circ g \in (S).\)

**Proof** We may assume \(N = 1\).

a) Take \(F = \sum_\alpha a_\alpha H_\alpha, G = \sum_\beta b_\beta H_\beta \in (S)_{-1}\). This means that there exist \(q_1\) such that

\[\sum_\alpha a_\alpha^2 (2N)^{-q_1} < \infty \quad \text{and} \quad \sum_\beta b_\beta^2 (2N)^{-q_1} < \infty. \quad (2.4.5)\]

We note that \(F \circ G = \sum_{\alpha, \beta} a_\alpha b_\beta H_{\alpha + \beta} = \sum_\gamma (\sum_{\alpha + \beta = \gamma} a_\alpha b_\beta) H_\gamma\) and then set \(c_\gamma = \sum_{\alpha + \beta = \gamma} a_\alpha b_\beta\). With \(q = q_1 + k\) we have

\[\sum_\gamma (2N)^{-q} c_\gamma^2 = \sum_\gamma (2N)^{-k\gamma} (2N)^{-q_1} \left( \sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right)^2 \]

\[\leq \sum_\gamma (2N)^{-k\gamma} (2N)^{-q_1} \left( \sum_{\alpha + \beta = \gamma} a_\alpha^2 \right) \left( \sum_{\alpha + \beta = \gamma} b_\beta^2 \right) \]

\[= \sum_\gamma (2N)^{-k\gamma} \left( \sum_{\alpha + \beta = \gamma} a_\alpha^2 (2N)^{-q_1} \right) \left( \sum_{\alpha + \beta = \gamma} b_\beta^2 (2N)^{-q_1} \right) \]

\[\leq \left( \sum_\gamma (2N)^{-k\gamma} \right) \left( \sum_{\alpha} a_\alpha^2 (2N)^{-q_1} \right) \left( \sum_{\beta} b_\beta^2 (2N)^{-q_1} \right) < \infty \quad (2.4.6)\]

for \(k > 1\), by Proposition 2.3.3. The proofs of b), c) and d) are similar. \(\square\)
The following basic algebraic properties of the Wick product follow directly from the definition.

**Lemma 2.4.5.**

a) *(Commutative law)* \( F, G \in (S)_{-1}^{m,N} \Rightarrow F \odot G = G \odot F. \)

b) *(Associative law)* \( F, G, H \in (S)_{-1}^{m,1} \Rightarrow F \odot (G \odot H) = (F \odot G) \odot H. \)

c) *(Distributive law)* \( F, A, B \in (S)_{-1}^{m,N} \Rightarrow F \odot (A + B) = F \odot A + F \odot B. \)

The Wick powers \( F^{\odot k}; k = 0, 1, 2, \ldots \) of \( F \in (S)_{-1} \) are defined inductively as follows:

\[
\begin{cases}
F^{\odot 0} = 1 \\
F^{\odot k} = F \odot F^{\odot (k-1)} & \text{for } k = 1, 2, \ldots,
\end{cases}
\]  

(2.4.7)

More generally, if \( p(x) = \sum_{n=0}^{N} a_n x^n; \ a_n \in \mathbb{R}, \ x \in \mathbb{R}, \) is a polynomial, then we define its Wick version \( p^\circ : (S)_{-1} \rightarrow (S)_{-1} \)

by

\[
p^\circ(F) = \sum_{n=0}^{N} a_n F^{\odot n} \quad \text{for } F \in (S)_{-1}.
\]  

(2.4.8)

Later we will extend this construction to more general functions than polynomials (see Section 2.6).

### 2.4.1 Some Examples and Counterexamples

For simplicity, we will assume that we have \( N = m = d = 1 \) in this paragraph. If \( F, G \in L^p(\mu) \) for \( p > 1 \), then it also makes sense to consider the ordinary (pointwise) product

\[
(F \cdot G)(\omega) = F(\omega) \cdot G(\omega).
\]

How does this product compare to the Wick product \( (F \odot G)(\omega) \)? This is a difficult question in general. Let us first consider some illustrating examples:

**Example 2.4.6.** Suppose at least one of \( F \) and \( G \) is deterministic, e.g., that \( F = a_0 \in \mathbb{R} \). Then

\[
F \odot G = F \cdot G.
\]
Hence the Wick product coincides with the ordinary product in the deterministic case. In particular, if \( F = 0 \), then \( F \odot G = 0 \).

**Example 2.4.7.** Suppose \( F, G \in L^2(\mu) \) are both Gaussian, i.e.,

\[
F(\omega) = a_0 + \sum_{k=1}^{\infty} a_k H_{\epsilon(k)}(\omega), \quad G(\omega) = b_0 + \sum_{l=1}^{\infty} b_l H_{\epsilon_l}(\omega),
\]

where

\[
\sum_{k=1}^{\infty} a_k^2 < \infty, \quad \sum_{l=1}^{\infty} b_l^2 < \infty.
\]

Then we have

\[
(F \odot G)(\omega) = a_0 b_0 + \sum_{k, l=1}^{\infty} a_k b_l H_{\epsilon(k) + \epsilon_l}(\omega).
\]

Now

\[
h_{\epsilon(k) + \epsilon_l} = \begin{cases} h_{\epsilon(k)} h_{\epsilon_l} & \text{for } k \neq l \\ h_{\epsilon(k)}^2 - 1 & \text{for } k = l. \end{cases}
\]

Hence

\[
(F \odot G)(\omega) = F(\omega) \cdot G(\omega) - \sum_{k=1}^{\infty} a_k b_k.
\]

This result can be restated in terms of Itô integrals, as follows: We may write

\[
F(\omega) = a_0 + \int_{\mathbb{R}} f(t) dB(t),
\]

where \( f(t) = \sum_{k=1}^{\infty} a_k \xi_k(t) \in L^2(\mathbb{R}) \), and, similarly,

\[
G(\omega) = b_0 + \int_{\mathbb{R}} g(t) dB(t),
\]

with \( g(t) = \sum_{k=1}^{\infty} b_k \xi_k(t) \in L^2(\mathbb{R}) \). Then (2.4.10) states that

\[
\left( \int_{\mathbb{R}} f(t) dB(t) \right) \odot \left( \int_{\mathbb{R}} g(t) dB(t) \right) = \left( \int_{\mathbb{R}} f(t) dB(t) \right) \cdot \left( \int_{\mathbb{R}} g(t) dB(t) \right) - \int_{\mathbb{R}} f(t) g(t) dt.
\]
In particular, choosing \( f = g = \chi_{[0,t]} \) we obtain

\[
B(t)^{\odot 2} = B(t)^2 - t. \tag{2.4.14}
\]

Note, in particular, that \( B(t)^{\odot 2} \) is not positive (but see Example 2.6.15). Similarly, for the smoothed white noise we obtain

\[
w(\phi) \odot w(\psi) = w(\phi) \cdot w(\psi) - (\phi, \psi) \tag{2.4.15}
\]

for \( \phi, \psi \in L^2(\mathbb{R}^d) \) with \( (\phi, \psi) = \int_{\mathbb{R}^d} \phi(x)\psi(x)dx \). (See Exercise 2.9.)

Note that if \( \psi = \phi \) and \( \|\phi\| = 1 \), this can be written

\[
w(\phi)^{\odot 2} = h_2(w(\phi)); \quad \|\phi\| = 1. \tag{2.4.16}
\]

This suggests the general formula

\[
w(\phi)^{\odot n} = h_n(w(\phi)); \quad \|\phi\| = 1. \tag{2.4.17}
\]

To prove (2.4.17) we use that the Wick product is independent of the choice of basis elements of \( L^2(\mathbb{R}^d) \). (See Appendix D.) In this case, where \( w(\phi) \) and its Wick powers all belong to \( L^2(\mu) \), the basis independence follows from Proposition 2.4.2. Therefore, we may assume that \( \phi = \eta_1 \), and then

\[
w(\phi)^{\odot n} = h_1((\omega, \eta_1))^{\odot n} = H_{\epsilon_1}^{\odot n}(\omega) = H_{n\epsilon_1}(\omega) = h_n((\omega, \eta_1)) = h_n(w(\phi)).
\]

**Example 2.4.8 Gjessing (1993).** The \( L^p(\mu) \) spaces are not closed under Wick products.

For example, choose \( \phi \in S(\mathbb{R}^d) \) with \( \|\phi\|_{L^2} = 1 \), put \( \theta(\omega) = \langle \omega, \phi \rangle \) and define

\[
X(\omega) = \begin{cases} 
1 & \text{if } \langle \omega, \phi \rangle \geq 0 \\
0 & \text{if } \langle \omega, \phi \rangle < 0.
\end{cases}
\]

Then

\[
X \in L^\infty(\mu) \quad \text{but} \quad X^{\odot 2} \notin L^2(\mu).
\]

**Example 2.4.9 Gjessing (1993).** Independence of \( X \) and \( Y \) is not enough to ensure that

\[
X \odot Y = X \cdot Y.
\]

To see this, let \( X, \theta \) be as in the previous example. Then \( Y = \theta^{\odot 2} = \theta^2 - 1 \) is independent of \( X \), but \( X \odot Y \) and \( X \cdot Y \) are not equal. In fact, they do not even have the same second moments. See, however, Propositions 8.2 and 8.3 in Benth and Potthoff (1996).
Example 2.4.10 Gjessing (1993). The Wick product is not local, i.e., the value of \((X \diamond Y)(\omega_0)\) is not (in general) determined by the values of \(X\) and \(Y\) in a neighborhood \(V\) of \(\omega_0\) in \(S'(\mathbb{R}^d)\).

2.5 Wick Multiplication and Hitsuda/Skorohod Integration

In this section we put \(N = m = d = 1\) for simplicity. One of the most striking features of the Wick product is its relation to Hitsuda/Skorohod integration. In short, this relation can be expressed as

\[
\int_{\mathbb{R}} Y(t) \delta B(t) = \int_{\mathbb{R}} Y(t) \diamond W(t) dt. \tag{2.5.1}
\]

Here the left-hand side denotes the Hitsuda/Skorohod integral of the stochastic process \(Y(t) = Y(t, \omega)\) (which coincides with the Itô integral if \(Y(t)\) is adapted; see Appendix B), while the right-hand side is to be interpreted as an \((S)^*\)-valued (Pettis) integral. Strictly speaking the right-hand side of (2.5.1) represents a generalization of the Hitsuda/Skorohod integral. For simplicity we will call this generalization the Skorohod integral.

The relation (2.5.1) explains why the Wick product is so natural and important in stochastic calculus. It is also the key to the fact that Itô calculus (with Itô’s formula, etc.) with ordinary multiplication is equivalent to ordinary calculus with Wick multiplication. To illustrate the content of this statement, consider the example with \(Y(t) = B(t) \cdot \chi_{[0,T]}(t)\) in (2.5.1): Then the left hand side becomes, by Itô’s formula,

\[
\int_0^T B(t) dB(t) = \frac{1}{2} B^2(T) - \frac{1}{2} T \quad (\text{assuming } B(0) = 0), \tag{2.5.2}
\]

while (formal) Wick calculation makes the right hand side equal to

\[
\int_0^T B(t) \diamond W(t) dt = \int_0^T B(t) \diamond B'(t) dt = \frac{1}{2} B(T)^{\diamond 2}, \tag{2.5.3}
\]

which is equal to (2.5.2) by virtue of (2.4.14).

This computation will be made rigorous later (Example 2.5.11), and we will illustrate applications of this principle in Chapters 3 and 4.

Various versions of (2.5.1) have been proved by several authors. A version involving the operator \(\partial_t^\diamond\) is proved in Hida et al. (1993), see Theorem 8.7 and subsequent sections. In Lindstrøm et al. (1992), a formula of the type (2.5.1) is proved, but under stronger conditions than necessary. In Benth (1993), the
result was extended to be valid under the sole condition that the left hand side exists. The proof we present here is based on Benth (1993). First we recall the definition of the Skorohod integral:

Let \( Y(t) = Y(t, \omega) \) be a stochastic process such that

\[
E[Y(t)^2] < \infty \quad \text{for all } t. \tag{2.5.4}
\]

Then, by Theorem 2.2.7, \( Y(t) \) has a Wiener-Itô chaos expansion

\[
Y(t) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(s_1, \ldots, s_n, t) dB^{\otimes n}(s_1, \ldots, s_n), \tag{2.5.5}
\]

where \( f_n(\cdot, t) \in \hat{L}^2(\mathbb{R}^n) \) for \( n = 0, 1, 2, \ldots \) and for each \( t \). Let

\[
\hat{f}_n(s_1, \ldots, s_n, s_{n+1})
\]

be the symmetrization of \( f_n(s_1, \ldots, s_{n+1}) \) wrt the \( n+1 \) variables \( s_1, \ldots, s_n, s_{n+1} \).

**Definition 2.5.1.** Assume that

\[
\sum_{n=0}^{\infty} (n+1)! \| \hat{f}_n \|_{L^2(\mathbb{R}^{n+1})}^2 < \infty. \tag{2.5.6}
\]

Then we define the Skorohod integral of \( Y(t) \), denoted by

\[
\int_{\mathbb{R}} Y(t) dB(t),
\]

by

\[
\int_{\mathbb{R}} Y(t) dB(t) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^{n+1}} \hat{f}_n(s_1, \ldots, s_{n+1}) dB^{\otimes(n+1)}(s_1, \ldots, s_{n+1}). \tag{2.5.7}
\]

By (2.5.6) and (2.5.7) the Skorohod integral belongs to \( L^2(\mu) \) and

\[
\left\| \int_{\mathbb{R}} Y(t) dB(t) \right\|_{L^2(\mu)}^2 = \sum_{n=0}^{\infty} (n+1)! \| \hat{f}_n \|_{L^2(\mathbb{R}^{n+1})}^2. \tag{2.5.8}
\]

Note that we do not require that the process be adapted. In fact, the Skorohod integral may be regarded as an extension of the Itô integral to non-adapted (anticipating) integrands. This was proved in Nualart and Zakai (1986). See also Theorem 8.5 in Hida et al. (1993), and the references there. For completeness we include a proof here.
First we need a result (of independent interest) about how to characterize adaptedness of a process in terms of the coefficients of its chaos expansion.

**Lemma 2.5.2.** Suppose \( Y(t) \) is a stochastic process with \( E[Y^2(t)] < \infty \) for all \( t \) and with the multiple Itô integral expansion

\[
Y(t) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(x, t) dB^{\otimes n}(x) \quad \text{with} \quad f_n(\cdot, t) \in \hat{L}^2(\mathbb{R}^n) \quad \text{for all} \quad n.
\]

(2.5.9)

Then \( Y(t) \) is \( \mathcal{F}_t \)-adapted if and only if

\[
\text{supp} f_n(\cdot, t) \subset \{ x \in \mathbb{R}^n_+; \; x_i \leq t \quad \text{for} \quad i = 1, 2, \ldots, n \},
\]

(2.5.10)

for all \( n \).

Here support is interpreted as essential support with respect to Lebesgue measure:

\[
\text{supp} f_n(\cdot, t) = \bigcap \{ F; \; F \text{ closed}, \; f_n(x, t) = 0 \quad \text{for a.e.} \quad x \notin F \}.
\]

**Proof** We first observe that for all \( n \) and all \( f \in \hat{L}^2(\mathbb{R}^n) \), we have

\[
E \left[ \int_{\mathbb{R}^n} f(x) dB^{\otimes n}(x) \bigg| \mathcal{F}_t \right]
= E \left[ n! \int_{-\infty}^{t_n} \int_{-\infty}^{t_2} \ldots \int_{-\infty}^{t_1} f(t_1, \ldots, t_n) dB(t_1) dB(t_2) \ldots dB(t_n) \bigg| \mathcal{F}_t \right]
= n! \int_{0}^{t} \int_{0}^{t_n} \int_{0}^{t_2} \ldots \int_{0}^{t_1} f(t_1, \ldots, t_n) dB(t_1) dB(t_2) \ldots dB(t_n)
= \int_{\mathbb{R}^n} f(x) \chi_{[0,t]^n}(x) dB^{\otimes n}(x).
\]

Therefore we get

\[
Y(t) \text{ is } \mathcal{F}_t\text{-adapted}
\]

\[
\Leftrightarrow \quad E \left[ Y(t) \big| \mathcal{F}_t \right] = Y_t \quad \text{for all} \quad t
\]

\[
\Leftrightarrow \quad \sum_{n=0}^{\infty} E \left[ \int_{\mathbb{R}^n} f_n(x, t) dB^{\otimes n}(x) \bigg| \mathcal{F}_t \right] = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(x, t) dB^{\otimes n}(x)
\]

\[
\Leftrightarrow \quad \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(x, t) \chi_{[0,t]^n}(x) dB^{\otimes n}(x) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(x, t) dB^{\otimes n}(x)
\]

\[
\Leftrightarrow \quad f_n(x, t) \chi_{[0,t]^n}(x) = f_n(x, t) \quad \text{for all} \quad t \text{ and almost all} \quad x,
\]

by the uniqueness of the expansion. \( \square \)
The corresponding characterization for Hermite chaos expansions is

**Lemma 2.5.3.** Suppose \( Y(t) \) is a stochastic process with \( E[Y^2(t)] < \infty \) for all \( t \) and with the Hermite chaos expansion

\[
Y(t) = \sum_{\alpha} c_{\alpha}(t) H_{\alpha}(\omega). \tag{2.5.11}
\]

Then \( Y(t) \) is \( \mathcal{F}_t \)-adapted if and only if

\[
\text{supp} \left\{ \sum_{|\alpha|=n} c_{\alpha}(t) \xi^{\hat{\alpha}}(x) \right\} \subset \{ x \in \mathbb{R}^n; x_i \leq t \ \text{for} \ i = 1, \ldots, n \} \tag{2.5.12}
\]

for all \( n \).

**Proof** This follows from Lemma 2.5.2 and (2.2.33).

**Proposition 2.5.4.** Suppose \( Y(t) \) is an \( \mathcal{F}_t \)-adapted stochastic process such that

\[
\int_{\mathbb{R}} E[Y^2(t)]dt < \infty.
\]

Then \( Y(t) \) is both Skorohod-integrable and Itô integrable, and the two integrals coincide:

\[
\int_{\mathbb{R}} Y(t) \delta B(t) = \int_{\mathbb{R}} Y(t) dB(t). \tag{2.5.13}
\]

**Proof** Suppose \( Y(t) \) has the expansion

\[
Y(t) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(x,t) dB^{\otimes n}(x); f_n(\cdot,t) \in \hat{L}^2(\mathbb{R}^n) \ \text{for all} \ n.
\]

Since \( Y(t) \) is adapted we know that \( f_n(x_1,x_2,\ldots,x_n,t) = 0 \) if \( \max_{1 \leq i \leq n} \{x_i\} > t \), a.e.

Therefore, the symmetrization \( \hat{f}_n(x_1,\ldots,x_n,t) \) of \( f_n(x_1,\ldots,x_n,t) \) satisfies (with \( x_{n+1} = t \))

\[
\hat{f}_n(x_1,\ldots,x_n,x_{n+1}) = \frac{1}{n+1} f(y_1,\ldots,y_n, \max_{1 \leq i \leq n+1} \{x_i\}),
\]

where \( (y_1,\ldots,y_n) \) is an arbitrary permutation of the remaining \( x_j \) when the maximum value \( x_j := \max_{1 \leq i \leq n+1} \{x_i\} \) is removed. This maximum is obtained for a unique \( j \), for almost all \( x \in \mathbb{R}^{n+1} \) with respect to Lebesgue measure.
Hence the Itô integral of $Y(t)$ is

$$
\int_Y Y(t)dB(t)
= \sum_{n=0}^{\infty} \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} f_n(x_1, \ldots, x_n, t)dB^{\otimes n}(x) \right) dB(t)
= \sum_{n=0}^{\infty} n! \int_{\mathbb{R}} \left( \int_{-\infty}^{t} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} f_n(x_1, \ldots, x_n, t)dB(x_1) \cdots dB(x_n) \right) dB(t)
= \sum_{n=0}^{\infty} n!(n+1) \int_{\mathbb{R}^{n+1}} \hat{f}_n(x_1, \ldots, x_n, x_{n+1})dB^{\otimes (n+1)}(x_1, \ldots, x_n, x_{n+1})
= \int_{\mathbb{R}} Y(t)\delta B(t),
$$

as claimed. \hfill \Box

We now proceed to consider integrals with values in $(\mathcal{S})^*$. 

**Definition 2.5.5.** A function $Z(t) : \mathbb{R} \to (\mathcal{S})^*$ (also called an $(\mathcal{S})^*$-valued process) is called $(\mathcal{S})^*$-integrable if

$$\langle Z(t), f \rangle \in L^1(\mathbb{R}, dt) \quad \text{for all} \quad f \in (\mathcal{S}). \quad (2.5.14)$$

Then the $(\mathcal{S})^*$-integral of $Z(t)$, denoted by $\int_{\mathbb{R}} Z(t)dt$, is the (unique) $(\mathcal{S})^*$-element such that

$$\left\langle \int_{\mathbb{R}} Z(t)dt, f \right\rangle = \int_{\mathbb{R}} \langle Z(t), f \rangle dt; \quad f \in (\mathcal{S}). \quad (2.5.15)$$

**Remark** It is a consequence of Proposition 8.1 in Hida et al. (1993) that (2.5.15) defines $\int_{\mathbb{R}} Z(t)dt$ as an element in $(\mathcal{S})^*$.

**Lemma 2.5.6.** Assume that $Z(t) \in (\mathcal{S})^*$ has the chaos expansion

$$Z(t) = \sum_{\alpha} c_{\alpha}(t)H_{\alpha}, \quad (2.5.16)$$

where

$$\sum_{\alpha} \alpha!\|c_{\alpha}\|^2_{L^1(\mathbb{R})}(2N)^{-p\alpha} < \infty \quad \text{for some} \quad p < \infty. \quad (2.5.17)$$
Then $Z(t)$ is $(S)^*$-integrable and

$$
\int \mathbb{R} Z(t)dt = \sum_{\alpha} \int \mathbb{R} c_\alpha(t)dt H_\alpha. \quad (2.5.18)
$$

**Proof** Let $f = \sum_{\alpha} a_\alpha H_\alpha \in (S)$. Then by (2.5.17)

$$
\int \mathbb{R} \big| \langle Z(t), f \rangle \big| dt = \int \mathbb{R} \bigg| \sum_{\alpha} \alpha! a_\alpha c_\alpha(t) \bigg| dt \leq \sum_{\alpha} \alpha! |a_\alpha| \|c_\alpha\|_{L^1(\mathbb{R})}
$$

$$
= \sum_{\alpha} \sqrt{\alpha!} |a_\alpha|(2N)^{\frac{\alpha p}{2}} \sqrt{\alpha!} \|c_\alpha\|_{L^1(\mathbb{R})}(2N)^{\frac{-\alpha p}{2}}
$$

$$
\leq \left( \sum_{\alpha} \alpha! a_\alpha^2 (2N)^{\alpha p} \right)^{\frac{1}{2}} \left( \sum_{\alpha} \alpha! \|c_\alpha\|^2_{L^1(\mathbb{R})}(2N)^{-\alpha p} \right)^{\frac{1}{2}} < \infty.
$$

Hence $Z(t)$ is $(S)^*$-integrable and

$$
\left\langle \int \mathbb{R} Z(t)dt, f \right\rangle = \int \mathbb{R} \langle Z(t), f \rangle dt = \int \mathbb{R} \sum_{\alpha} \alpha! a_\alpha c_\alpha(t) dt
$$

$$
= \sum_{\alpha} \alpha! a_\alpha \int \mathbb{R} c_\alpha(t) dt = \left\langle \sum_{\alpha} \int \mathbb{R} c_\alpha(t) dt H_\alpha, f \right\rangle,
$$

which proves (2.5.18). \qed

**Lemma 2.5.7.** Suppose

$$
Y(t) = \sum_{\alpha} c_\alpha(t)H_\alpha \in (S)^* \quad \text{for all } t \in \mathbb{R},
$$

and that there exists $q < \infty$ such that

$$
K := \sup_{\alpha} \{ \alpha! \|c_\alpha\|^2_{L^1(\mathbb{R})}(2N)^{-q\alpha} \} < \infty. \quad (2.5.19)
$$

Choose $\phi \in \mathcal{S}(\mathbb{R})$. Then $Y(t) \circ W(t)$ and $Y(t) \circ W_\phi(t)$ (with $W_\phi(t)$ as in (2.1.15)) are both $(S)^*$-integrable and

$$
\int \mathbb{R} Y(t) \circ W(t) dt = \sum_{\alpha,k} \int \mathbb{R} c_\alpha(t) \xi_k(t) dt H_{\alpha + \epsilon(k)}. \quad (2.5.20)
$$

and

$$
\int \mathbb{R} Y(t) \circ W_\phi(t) dt = \sum_{\alpha,k} \int \mathbb{R} c_\alpha(t) (\phi_t(\cdot), \xi_k) dt H_{\alpha + \epsilon(k)}. \quad (2.5.21)
$$
Proof We prove (2.5.20), the proof of (2.5.21) being similar. Since

\[ Y(t) \circ W(t) = \sum_{\alpha,k} c_\alpha(t) \xi_k(t) H_{\alpha + \epsilon(k)} = \sum_{\beta} \sum_{\alpha,k \in \beta} c_\alpha(t) \xi_k(t) H_\beta, \]

the result follows from Lemma 2.5.6 if we can verify that

\[ M(p) := \sum_{\beta} \beta! \left\| \sum_{\alpha,k \in \beta} c_\alpha(t) \xi_k(t) \right\|_{L^1(\mathbb{R})}^2 (2N)^{-p\beta} < \infty \]

for some \( p < \infty \).

By (2.2.5) we have, for some constant \( C < \infty \),

\[ \int_{\mathbb{R}} |c_\alpha(t)||\xi_k(t)| dt \leq C \int_{\mathbb{R}} |c_\alpha(t)| dt = C \|c_\alpha\|_{L^1(\mathbb{R})}. \]

Note that for each \( \beta, \alpha \) there is at most one \( k \) such that \( \alpha + \epsilon(k) = \beta \). Therefore

\[
\left\| \sum_{\alpha,k \in \beta} c_\alpha(t) \xi_k(t) \right\|_{L^1(\mathbb{R})}^2 \leq \left[ \sum_{\alpha,k \in \beta} \|c_\alpha\|_{L^1(\mathbb{R})} \right]^2 \\
\leq C^2 \left[ \sum_{\alpha,k \in \beta} \|c_\alpha\|_{L^1(\mathbb{R})} \right]^2 \\
\leq C^2 (l(\beta))^2 \sum_{\alpha,k \in \beta, \alpha + \epsilon(k) = \beta} \|c_\alpha\|^2_{L^1(\mathbb{R})},
\]

where \( l(\beta) \) is the number of nonzero elements of \( \beta \), i.e., \( l(\beta) \) is the length of \( \beta \). We conclude that

\[
M(2q) \leq C^2 \sum_{\alpha,k} ((\alpha + \epsilon(k))! (l(\alpha + \epsilon(k)))^2 \|c_\alpha\|^2_{L^1(\mathbb{R})} (2N)^{-2q(\alpha + \epsilon(k))} \\
\leq C^2 K \sum_{\alpha,k} (\alpha + \epsilon(k))! \frac{1}{\alpha!} (l(\alpha + \epsilon(k)))^2 (2N)^{-q\alpha} (2N)^{-2q\epsilon(k)} \\
\leq C^2 K \sum_{\alpha,k} (|\alpha| + 1)^3 2^{-|\alpha|} k^{-2q} \epsilon(k) < \infty \quad \text{for} \quad q > \frac{1}{2}.
\]

\[ \square \]

Corollary 2.5.8. Let \( Y(t) = \sum_\alpha c_\alpha(t) H_\alpha \) be a stochastic process such that \( \int_a^b E[Y_t^2] dt < \infty \) for some \( a, b \in \mathbb{R}, a < b \). Then \( Y(t) \circ W(t) \) is \((S)^*\)-integrable over \([a, b]\) and
\[
\int_a^b Y(t) \diamond W(t) dt = \sum_{\alpha, k} \int_a^b c_\alpha(t) \xi_k(t) dt H_{\alpha + \epsilon(k)}. \quad (2.5.22)
\]

**Proof** We have

\[
\sum_{\alpha} \alpha! \int_a^b c_\alpha^2(t) dt = \int_a^b E[Y(t)^2] dt < \infty,
\]

hence (2.5.19) holds, so by Lemma 2.5.7 the corollary follows.

We are now ready to prove the main result of this section. \(\Box\)

**Theorem 2.5.9.** Assume that \(Y(t) = \sum_{\alpha} c_\alpha(t) H_\alpha\) is a Skorohod-integrable stochastic process. Let \(a, b \in \mathbb{R}, a < b\). Then \(Y(t) \diamond W(t)\) is \((S)^*\)-integrable over \([a, b]\) and we have

\[
\int_a^b Y(t) \delta B(t) = \int_a^b Y(t) \diamond W(t) dt. \quad (2.5.23)
\]

**Proof** By the preceding corollary and by replacing \(c_\alpha(t)\) by \(c_\alpha(t) \chi_{[a,b]}(t)\), we see that it suffices to verify that

\[
\int \mathbb{R} Y(t) \delta B(t) = \sum_{\alpha, k} (c_\alpha, \xi_k) H_{\alpha + \epsilon(k)}, \quad (2.5.24)
\]

where \((c_\alpha, \xi_k) = \int_{\mathbb{R}} c_\alpha(t) \xi_k(t) dt\). This will be done by computing the left-hand side explicitly: Let

\[
Y(t) = \sum_{n=0}^\infty \int_{\mathbb{R}^n} c_n(t) \xi_{\alpha}(u_1, \ldots, u_n) dB^{\otimes n}(u_1, \ldots, u_n).
\]

Then by (2.2.33) we have

\[
Y(t) = \sum_{n=0}^\infty \int_{\mathbb{R}^n} c_n(t) \xi_{\alpha}(u_1, \ldots, u_n) dB^{\otimes n}(u_1, \ldots, u_n)
\]

\[
= \sum_{n=0}^\infty \sum_{|\alpha|=n} \sum_{k=1}^\infty (c_\alpha, \xi_k(t)) \xi_{\alpha}(u_1, \ldots, u_n) dB^{\otimes n}(u_1, \ldots, u_n).
\]

Now the symmetrization of

\[
\xi_k(u_0) \xi_{\alpha}(u_1, \ldots, u_n) = \xi_k(u_0)(\xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_j})(u_1, \ldots, u_n),
\]

where \(\alpha = (\alpha_1, \ldots, \alpha_j)\), as a function of \(u_0, \ldots, u_n\) is simply

\[
\xi_{\alpha + \epsilon(k)} = \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_k+1} \otimes \cdots \otimes \xi_{\alpha_j}. \quad (2.5.25)
\]
Therefore the Skorohod integral of $Y(s)$ becomes, by (2.5.7) and (2.2.30),

$$
\int Y(t)\delta B(t) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^{n+1}} \sum_{|\alpha|=n} \sum_{k=1}^{\infty} (c_{\alpha}, \xi_k) \xi^{(\alpha+\epsilon(k))} dB^{(n+1)}
$$

$$
= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \sum_{k=1}^{\infty} (c_{\alpha}, \xi_k) H_{\alpha+\epsilon(k)} = \sum_{\alpha,k} (c_{\alpha}, \xi_k) H_{\alpha+\epsilon(k)},
$$
as claimed.

To illustrate the contents of these results, we consider some simple examples.

**Example 2.5.10.** It is immediate from the definition that

$$
\int_0^t 1 \delta B(s) = B(t)
$$

(assuming, as before, $B(0) = 0$), so from Theorem 2.5.9 we have

$$
B(t) = \int_0^t W(s)ds.
$$

(2.5.26)

In other words, we have proved that as elements of $(S)^*$ we have

$$
\frac{dB(t)}{dt} = W(t),
$$

(2.5.27)

where differentiation is in $(S)^*$ (compare with (2.1.17)).

More generally, if we choose $Y(t)$ to be deterministic, $Y(t, \omega) = \psi(t) \in L^2(\mathbb{R})$, then by Theorem 2.5.9 and Proposition 2.5.4

$$
\int \psi(t)dB(t) = \int \psi(t)\delta B(t) = \int \psi(t) \circ W(t)dt.
$$

(2.5.28)

**Example 2.5.11.** Let us apply Theorem 2.5.9 and Corollary 2.5.8 to compute the Skorohod integral

$$
\int_0^t B(s)\delta B(s) = \int_0^t B(s) \circ W(s)ds.
$$

From Example 2.2.5 we know that

$$
B(s) = \sum_{j=1}^{\infty} \int_0^s \xi_j(r)dr H_{\epsilon(j)}(\omega),
$$
which substituted in (2.5.22) gives

\[ \int_0^t B(s) \delta B(s) = \sum_{j,k} \int_0^t \int_0^s \xi_j(r) dr \xi_k(s) ds H_{\epsilon(j)+\epsilon(k)}. \]

Integration by parts gives

\[ \int_0^t \int_0^s \xi_j(r) dr \xi_k(s) ds = \int_0^t \xi_j(r) dr \int_0^t \xi_k(s) ds - \int_0^t \int_0^s \xi_k(s) ds \xi_j(r) dr. \]

Hence, by the symmetry of \( j \) and \( k \),

\[ \int_0^t B(s) \delta B(s) = \frac{1}{2} \sum_{j,k} \int_0^t \xi_j(r) dr \int_0^t \xi_k(s) ds H_{\epsilon(j)+\epsilon(k)}. \]

By the Wick product definition (2.4.2) this is equal to \( \frac{1}{2} B(t) \diamond B(t) \). Hence we obtain, using (2.4.14), the familiar formula

\[ \int_0^t B(s) dB(s) = \int_0^t B(s) \delta B(s) = \frac{1}{2} B(t)^2 = \frac{1}{2} B^2(t) - \frac{1}{2} t. \]

We can more easily obtain this formula if we use (2.5.27) and work in \((S)^*\). Then

\[ \int_0^t B(s) \delta B(s) = \int_0^t B(s) \diamond W(s) ds = \int_0^t B(s) \diamond B'(s) ds = \int_0^t \frac{1}{2} B(s)^\diamond^2 = \frac{1}{2} B(t)^\diamond^2 = \frac{1}{2} B^2(t) - \frac{1}{2} t. \]

**Corollary 2.5.12.** Suppose that \( Y(t) = Y(t, \omega) \) is Skorohod-integrable, that \( h(\omega) \in (S)^* \) does not depend on \( t \) and that \( h \diamond Y(t) \) is Skorohod-integrable. Then for \( a < b \) we have

\[ \int_a^b h \diamond Y(t) \delta B(t) = h \diamond \int_a^b Y(t) \delta B(t). \] \hspace{1cm} (2.5.29)

**Proof** By Theorem 2.5.9 we have

\[ \int_a^b h \diamond Y(t) \delta B(t) = \int_a^b h \diamond Y(t) \diamond W(t) dt = h \diamond \int_a^b Y(t) \delta B(t). \]
Example 2.5.13. Choose \( Y(t) = \chi_{[a,b]}(t)h(\omega) \), with \( h \in L^2(\mu_1), a < b \). Then the Skorohod integral becomes

\[
\int_a^b h(\omega) \delta B_s(\omega) = \int_a^b h(\omega) \circ W(s) ds = h(\omega) \circ (B(b) - B(a)). \quad (2.5.30)
\]

Finally, we state and prove a smoothed version of Theorem 2.5.9.

**Theorem 2.5.14.** Let \( Y(t) = Y(t, \omega) \) be a stochastic process such that

\[
\int_{\mathbb{R}} E[Y^2(t)] dt < \infty. \quad (2.5.31)
\]

Choose \( \phi \in S(\mathbb{R}) \) and let

\[
(\phi \ast Y)(t, \omega) = \int_{\mathbb{R}} \phi(t - s) Y(s, \omega) ds \quad (2.5.32)
\]

be the convolution of \( \phi \) and \( Y(\cdot, \omega) \), for almost all \( \omega \). Suppose \( (\phi \ast Y)(t) \) is Skorohod-integrable. Then \( Y(t) \circ W_\phi(t) \) is \((S)^*\)-integrable, and we have

\[
\int_{\mathbb{R}} (\phi \ast Y)(t) \delta B(t) = \int_{\mathbb{R}} Y(t) \circ W_\phi(t) dt. \quad (2.5.33)
\]

**Proof** Suppose \( Y(s) \) has the expansion

\[
Y(s) = \sum_\alpha c_\alpha(s) H_\alpha.
\]

Then by (2.5.31) and Lemma 2.5.7 \( Y(t) \circ W_\phi(t) \) is \((S)^*\)-integrable. Applying Theorem 2.5.9 with \( Y(t) \) replaced by \( (\phi \ast Y)(t) \), we get, by (2.1.18),

\[
\int_{\mathbb{R}} (\phi \ast Y)(t) \delta B(t) = \int_{\mathbb{R}} (\phi \ast Y)(t) \circ W(t) dt \\
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \phi(t - s) Y(s) ds \right) \circ W(t) dt \\
= \int_{\mathbb{R}} Y(s) \circ \int_{\mathbb{R}} \phi(t - s) W(t) dt ds = \int_{\mathbb{R}} Y(s) \circ W_\phi(s) ds.
\]

\( \square \)
2.6 The Hermite Transform

Since the Wick product satisfies all the ordinary algebraic rules for multiplication, one can carry out calculations in much the same way as with usual products. Problems arise, however, when limit operations are involved. To handle these situations it is convenient to apply a transformation, called the Hermite transform or the $H$-transform, which converts Wick products into ordinary (complex) products and convergence in $(S)_{-1}$ into bounded, pointwise convergence in a certain neighborhood of 0 in $\mathbb{C}^N$. This transform, which first appeared in Lindstrøm et al. (1991), has been applied by the authors in many different connections. We will see several of these applications later. We first give the definition and some of its basic properties.

**Definition 2.6.1.** Let $F = \sum_\alpha b_\alpha H_\alpha \in (S)^N_{-1}$ with $b_\alpha \in \mathbb{R}^N$ as in Definition 2.3.2. Then the Hermite transform of $F$, denoted by $\mathcal{H}F$ or $\tilde{F}$, is defined by

$$\mathcal{H}F(z) = \tilde{F}(z) = \sum_\alpha b_\alpha z^\alpha \in \mathbb{C}^N$$

(when convergent),

(2.6.1)

where $z = (z_1, z_2, \ldots) \in \mathbb{C}^N$ (the set of all sequences of complex numbers) and

$$z^\alpha = z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_n^{\alpha_n} \ldots$$

(2.6.2)

if $\alpha = (\alpha_1, \alpha_2, \ldots) \in J$, where $z_j^0 = 1$.

**Example 2.6.2** ($N = m = 1$).

i) The 1-dimensional smoothed white noise $w(\phi)$ has chaos expansion (see (2.2.23))

$$w(\phi, \omega) = \sum_{j=1}^n (\phi, \eta_j) H_{\epsilon(j)}(\omega),$$

(2.6.3)

and therefore the Hermite transform $\tilde{w}(\phi)$ of $w(\phi)$ is

$$\tilde{w}(\phi)(z) = \sum_{j=1}^\infty (\phi, \eta_j)z_j,$$

(2.6.4)

which is convergent for all $z = (z_1, z_2, \ldots) \in (\mathbb{C}^N)_c$.

ii) The 1-dimensional ($d$-parameter) Brownian motion $B(x)$ has chaos expansion (see (2.2.24))

$$B(x, \omega) = \sum_{j=1}^\infty \int_0^x \eta_j(u)duH_{\epsilon(j)}(\omega),$$

(2.6.5)
and therefore
\[ \tilde{B}(x)(z) = \sum_{j=1}^{\infty} \int_{0}^{x} \eta_j(u) \, du \, z_j; \quad z = (z_1, z_2, \ldots) \in (\mathbb{C}^{N})_c, \quad (2.6.6) \]

where \((\mathbb{C}^{N})_c\) is the set of all finite sequences in \(\mathbb{C}^{N}\).

iii) The 1-dimensional singular white noise \(W(x, \omega)\) has the expansion (see (2.2.23))
\[ W(x, \omega) = \sum_{j=1}^{\infty} \eta_j(x) H_{\epsilon(j)}(\omega), \quad (2.6.7) \]
and therefore
\[ \tilde{W}(x)(z) = \sum_{j=1}^{\infty} \eta_j(x) z_j; \quad z = (z_1, z_2, \ldots) \in (\mathbb{C}^{N})_c. \quad (2.6.8) \]

**Example 2.6.3\((N = m > 1)\).**

i) The \(m\)-dimensional smoothed white noise \(w(\phi)\) has chaos expansion (see (2.2.25))
\[ w(\phi, \omega) = (w_1(\phi, \omega), \ldots, w_m(\phi, \omega)), \]
with
\[ w_i(\phi, \omega) = w(\phi_i, \omega_i) = \sum_{j=1}^{\infty} (\phi_i, \eta_j) H_{\epsilon_i + (j-1)m}(\omega); \quad 1 \leq i \leq m. \quad (2.6.9) \]
Hence the Hermite transform of coordinate \(w_i(\phi, \omega)\) of \(w(\phi, \omega)\) is, for \(z \in (\mathbb{C}^{N})_c\),
\[ \tilde{w}_i(\phi)(z) = \sum_{j=1}^{\infty} (\phi_i, \eta_j) z_{(j-1)i + m} \]
\[ = (\phi_i, \eta_1) z_i + (\phi_i, \eta_2) z_{i+m} + (\phi_i, \eta_3) z_{i+2m} + \cdots; \quad 1 \leq i \leq m. \quad (2.6.10) \]
Note that different components of \(w\) involve disjoint families of \(z_k\)-variables when we take the \(\mathcal{H}\)-transform.

ii) For the \(m\)-dimensional \(d\)-parameter Brownian motion
\[ \mathbf{B}(x, \omega) = (B_1(x, \omega), \ldots, B_m(x, \omega)) \]
we have (see (2.2.26))

\[ B_i(x, \omega) = \sum_{j=1}^{\infty} \int_{0}^{x} \eta_j(u) du H_{\epsilon_i(j-1)m} \]  \hspace{1cm} (2.6.11)

and hence

\[ \tilde{B}_i(x)(z) = \sum_{j=1}^{\infty} \int_{0}^{x} \eta_j(u) du z_{i(j-1)m}; \quad 1 \leq i \leq m. \]  \hspace{1cm} (2.6.12)

iii) The \( m \)-dimensional singular white noise

\[ W(x, \omega) = (W_1(x, \omega), \ldots, W_m(x, \omega)) \]

has expansion (see (2.3.34))

\[ W_i(x) = \sum_{j=1}^{\infty} \eta_j(x) H_{\epsilon_i(j-1)m}; \quad 1 \leq i \leq m, \]  \hspace{1cm} (2.6.13)

and therefore

\[ \tilde{W}_i(x)(z) = \sum_{j=1}^{\infty} \eta_j(x) z_{i(j-1)m}; \quad 1 \leq i \leq m, \quad z \in (\mathbb{C}^N)_c. \]  \hspace{1cm} (2.6.14)

Note that if \( F = \sum_{\alpha} b_\alpha H_\alpha \in (S)^N_{-\rho} \) for \( \rho < 1 \), then \( \mathcal{H}F(z_1, z_2, \ldots) \) converges for all finite sequences \( (z_1, z_2, \ldots) \) of complex numbers. To see this we write

\[ \sum_{\alpha} |b_\alpha| z^\alpha \leq \sum_{\alpha} |b_\alpha| (\alpha!)^{(1-\rho)} (2N)^{-\alpha q} \frac{\alpha q}{2} \]

\[ \leq \left( \sum_{\alpha} |b_\alpha|^2 (\alpha!)^{1-\rho} (2N)^{-\alpha q} \right)^{\frac{1}{2}} \left( \sum_{\alpha} |z^\alpha|^2 (\alpha!)^{\rho-1} (2N)^{\alpha q} \right)^{\frac{1}{2}} \]  \hspace{1cm} (2.6.15)

Now if \( z = (z_1, \ldots, z_n) \) with \( |z_j| \leq M \), then

\[ \sum_{\alpha} |z^\alpha|^2 (\alpha!)^{\rho-1} (2N)^{\alpha q} \leq \sum_{\alpha} M^{2|\alpha|} (\alpha!)^{\rho-1} 2^q |\alpha|^q |\alpha| < \infty \]  \hspace{1cm} (2.6.16)

for all \( q < \infty \). If \( q \) is large enough, then by Proposition 2.3.3, the expression (2.6.15) is finite.

If \( F \in (S)^N_{-1} \), however, we can only obtain convergence of \( \mathcal{H}F(z_1, z_2, \ldots) \) in a neighborhood of the origin. We have
\[
\sum_{\alpha} |b_\alpha z^{\alpha}| \leq \left( \sum_{\alpha} b_\alpha^2 (2N)^{-\alpha q} \right)^{1/2} \left( \sum_{\alpha} |z^{\alpha}|^2 (2N)^{\alpha q} \right)^{1/2}
\]  
(2.6.17)

where the first factor on the right hand side converges for \( q \) large enough. For such a value of \( q \) we have convergence of the second factor if \( z \in \mathbb{C}^N \) with
\[
|z_j| < (2j)^{-q} \quad \text{for all } j.
\]

**Definition 2.6.4.** For \( 0 < R, q < \infty \), define the infinite-dimensional neighborhoods \( \mathbb{K}_q(R) \) of 0 in \( \mathbb{C}^N \) by
\[
\mathbb{K}_q(R) = \left\{ (\zeta_1, \zeta_2, \ldots) \in \mathbb{C}^N; \sum_{\alpha \neq 0} |\zeta^{\alpha}|^2 (2N)^{\alpha q} < R^2 \right\}.
\]  
(2.6.18)

Note that
\[
q \leq Q, r \leq R \Rightarrow \mathbb{K}_Q(r) \subset \mathbb{K}_q(R).
\]  
(2.6.19)

For any \( q < \infty, \delta > 0 \) and natural number \( k \), there exists \( \epsilon > 0 \) such that
\[
z = (z_1, \ldots, z_k) \in \mathbb{C}^k \quad \text{and} \quad |z_i| < \epsilon; \quad 1 \leq i \leq k \Rightarrow z \in \mathbb{K}_q(\delta).
\]  
(2.6.20)

The conclusions above can be stated as follows:

**Proposition 2.6.5.**  
\textbf{a)} If \( F \in (S)_{-\rho}^N \) for some \( \rho \in [-1, 1) \), then the Hermite transform \( (\mathcal{H}F)(z) \) converges for all \( z \in (\mathbb{C}^N)_c \).
\textbf{b)} If \( F \in (S)_{-1}^N \), then there exists \( q < \infty \) such that \( (\mathcal{H}F)(z) \) converges for all \( z \in \mathbb{K}_q(R) \) for all \( R < \infty \).

One of the reasons why the Hermite transform is so useful, is the following result, which is an immediate consequence of Definition 2.4.1 and Definition 2.6.1.

**Proposition 2.6.6.** If \( F, G \in (S)_{-1}^N \), then
\[
\mathcal{H}(F \odot G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z)
\]  
(2.6.21)

for all \( z \) such that \( \mathcal{H}F(z) \) and \( \mathcal{H}G(z) \) exist. The product on the right hand side of (2.6.21) is the complex bilinear product between two elements of \( \mathbb{C}^N \) defined by
\[
(\zeta_1, \ldots, \zeta_N) \cdot (w_1, \ldots, w_N) = \sum_{i=1}^N \zeta_i w_i; \quad \zeta_i, w_i \in \mathbb{C}.
\]

Note that there is no complex conjugation in this definition.
Example 2.6.7. Referring to Examples 2.6.2 and 2.6.3, we get the following Hermite transforms:

(i) \( \mathcal{H}(w^2(\phi))(z) = \sum_{j,k=1}^{\infty} (\phi, \eta_j)(\phi, \eta_k)z_jz_k \)

(ii) \( \mathcal{H}(B^2(x))(z) = \sum_{j,k=1}^{\infty} \left( \int_0^x \eta_j(u)du \right) \left( \int_0^x \eta_k(u)du \right)z_jz_k \)

(iii) \( \mathcal{H}(W^3(x))(z) = \sum_{i,j,k=1}^{\infty} \eta_i(x)\eta_j(x)\eta_k(x)z_iz_jz_k \)

(iv) \( \mathcal{H}(W_1(x) \circ W_2(x))(z) = \left( \sum_{j=1}^{\infty} \eta_j(x)z_{2j-1} \right) \cdot \left( \sum_{k=1}^{\infty} \eta_k(x)z_{2k} \right) = \sum_{j,k=1}^{\infty} \eta_j(x)\eta_k(x)z_{2j-1}z_{2k}; \ z = (z_1, z_2, \ldots) \)

The Characterization Theorem for \( (S)_N^{-1} \)

Proposition 2.6.5 states that the \( \mathcal{H} \)-transform of any \( F \in (S)_N^{-1} \) is a \( \mathbb{C}^N \)-valued analytic function on \( \mathbb{K}_q(R) \) for all \( R < \infty \), if \( q < \infty \) is large enough. It is natural to ask if the converse is true: Is every \( \mathbb{C}^N \)-valued analytic function \( g \) on \( \mathbb{K}_q(R) \) (for some \( R < \infty, q < \infty \)) the \( \mathcal{H} \)-transform of some element in \( (S)_N^{-1} \)? The answer is yes, if we add the condition that \( g \) be bounded on some \( \mathbb{K}_q(R) \) (see Theorem 2.6.11).

To prove this, we first establish some auxiliary results. We say that a formal power series in infinitely many complex variables \( z_1, z_2, \ldots \)

\[
g(z) = \sum_{\alpha} a_{\alpha}z^\alpha; \ a_{\alpha} \in \mathbb{C}^N, z = (z_1, z_2, \ldots)
\]

is convergent at \( z \) if

\[
\sum_{\alpha} |a_{\alpha}|z^\alpha < \infty. \tag{2.6.22}
\]

If this holds, the series has a well-defined sum that we denote by \( g(z) \).

Proposition 2.6.8. Let \( g(z) = \sum_{\alpha} a_{\alpha}z^\alpha, \ a_{\alpha} \in \mathbb{C}^N, z = (z_1, z_2, \ldots) \) be a formal power series in infinitely many variables. Suppose there exist \( q < \infty, M < \infty \) and \( \delta > 0 \) such that \( g(z) \) is convergent for \( z \in \mathbb{K}_q(\delta) \) and \( |g(z)| \leq M \) for all \( z \in \mathbb{K}_q(\delta) \).
Then
\[ \sum_{\alpha} |a_\alpha z^\alpha| \leq M \, A(q) \quad \text{for all} \quad z \in \mathbb{K}_{3q}(\delta), \]
where, by Proposition 2.3.3,
\[ A(q) := \sum_{\alpha} (2N)^{-q \alpha} < \infty \quad \text{for} \quad q > 1. \]

To prove this proposition we need the two lemmas below.

**Lemma 2.6.9.** Suppose \( f(z) = \sum_{k=0}^\infty a_k z^k \) is an analytic function in one complex variable \( z \) such that
\[ \sup_{|z| \leq r} |f(z)| \leq M. \tag{2.6.23} \]
Then \( |a_k z^k| \leq M \) for all \( k \) and all \( z \) with \( |z| \leq r \).

**Proof** By the Cauchy formula
\[ f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|\zeta| = r} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta, \quad \text{for} \quad |z| < r, \tag{2.6.24} \]
and we have
\[ |f^{(k)}(0)| \leq k! r^{-k} M. \tag{2.6.25} \]
Hence
\[ |a_k z^k| = \frac{|f^{(k)}(0)| z^k}{k!} \leq M \left| \frac{z^k}{r} \right| \leq M, \quad \text{for} \quad |z| \leq r. \tag{2.6.26} \]
\[ \square \]

**Lemma 2.6.10.** Let \( g(z) = \sum_{\alpha} a_\alpha z^\alpha \) be an analytic function in \( n \) complex variables such that there exists \( M < \infty \) and \( c_1, \ldots, c_n > 0, \delta > 0 \) such that
\[ |g(z)| \leq M, \tag{2.6.27} \]
when \( z \in \mathbb{K} := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n; \ c_1|z_1|^2 + \cdots + c_n|z_n|^2 \leq \delta^2 \} \). Then \( |a_\alpha z^\alpha| \leq M \) for \( z \in \mathbb{K} \), for all \( \alpha \).

**Proof** Use the previous lemma and induction. For example, for \( n = 2 \) the proof is the following: Write \( g(z_1, z_2) = \sum_{k=0}^\infty A_k(z_2) z_1^k \). Fix \( z_2 \) such that \( c_2 z_2^2 \leq \delta^2 \), and let
\[ f(z_1) = \sum_{k=0}^\infty A_k(z_2) z_1^k, \quad \text{for} \quad (z_1, z_2) \in \mathbb{K}. \tag{2.6.28} \]
2.6 The Hermite Transform

By the previous lemma we have \(|A_k(z_2)z_1^k| \leq M\) for \((z_1, z_2) \in K\). Applying the same lemma to

\[ f(z_2) = A_k(z_2) = \sum_{l=0}^{\infty} a_{kl} z_2^l, \quad (2.6.29) \]

we get that \(|a_{kl} z_2^l| \leq \frac{M}{|z_1|^k}\) or \(|a_{kl} z_1^k z_2^l| \leq M\), as claimed.

\[ \square \]

**Proof of Proposition 2.6.8**

We may assume \(N = 1\). Without loss of generality we can assume that \(q\) is so large that \(\sum_{\alpha} (2N)^{-q_{\alpha}} < \infty\), and we put \(Q = 3q\). Then by Lemma 2.6.10 we have

\[ |a_{\alpha} w^\alpha| \leq M \text{ for all } w \in K_q(\delta). \quad (2.6.30) \]

Choose \(z \in K_{3q}(\delta)\). Then if \(w_j = (2j)^q z_j\),

\[ \sum_{\alpha} |w^\alpha|^2 (2N)^{q_{\alpha}} = \sum_{\alpha} (2N)^{3q_{\alpha}} |w^\alpha|^2 < \delta, \quad (2.6.32) \]

so \(w \in K_q(\delta)\). Therefore

\[ \sum_{\alpha} |a_{\alpha}||z^\alpha| \leq \left( \sum_{\alpha} |a_{\alpha}|^2 |z^\alpha|^2 (2N)^{q_{\alpha}} \right)^{\frac{1}{2}} \left( \sum_{\alpha} (2N)^{-q_{\alpha}} \right)^{\frac{1}{2}} \]

\[ \leq \left( \sum_{\alpha} |a_{\alpha}|^2 |w^\alpha|^2 (2N)^{-q_{\alpha}} \right)^{\frac{1}{2}} \left( \sum_{\alpha} (2N)^{-q_{\alpha}} \right)^{\frac{1}{2}} \]

\[ \leq M \sum_{\alpha} (2N)^{-q_{\alpha}}. \quad (2.6.33) \]

\[ \square \]

**Theorem 2.6.11 (Characterization theorem for \((S)^{N-1}\)).**

a) If \(F(\omega) = \sum_{\alpha} a_{\alpha} H_{\alpha}(\omega) \in (S)^{N-1}\), where \(a_{\alpha} \in \mathbb{R}^n\), then there is \(q < \infty, M_q < \infty\) such that

\[ |\tilde{F}(z)| \leq \sum_{\alpha} |a_{\alpha}||z^\alpha| \leq M_q \left( \sum_{\alpha} (2N)^{q_{\alpha}} |z^\alpha|^2 \right)^{\frac{1}{2}} \text{ for all } z \in (\mathbb{C}^N)_c. \]

\[ (2.6.34) \]

In particular, \(\tilde{F}\) is a bounded analytic function on \(K_q(R)\) for all \(R < \infty\).

b) Conversely, suppose \(g(z) = \sum_{\alpha} b_{\alpha} z^\alpha\) is a given power series of \(z \in (\mathbb{C}^N)_c\) with \(b_{\alpha} \in \mathbb{R}^N\) such that there exists \(q < \infty\) and \(\delta > 0\), such that \(g(z)\) is absolutely convergent when \(z \in K_q(\delta)\) and

\[ \sup_{z \in K_q(\delta)} |g(z)| < \infty. \]

\[ (2.6.35) \]
Then there exists a unique $G \in (S)_{-1}^N$ such that $\tilde{G} = g$, namely

$$G(\omega) = \sum_\alpha b_\alpha H_\alpha(\omega).$$

(2.6.36)

c) Let $F = \sum_\alpha c_\alpha H_\alpha(\omega) \in (S)_{-1,-q}$. Then we have

$$\sup_{z \in K_q(R)} |\mathcal{H}F(z)| \leq R||F||_{-1,-q} \text{ for all } R > 0$$

d) Suppose there exist $q > 1, \delta > 0$ such that

$$M_q(\delta) := \sup_{z \in K_q(\delta)} \left| \sum_\alpha c_\alpha z^\alpha \right| < \infty$$

Then there exists $r \geq q$ such that

$$S_r := \sup_\alpha |c_\alpha|(2N)^{-r\alpha} \leq M_q(\delta)A(q),$$

where

$$A(q) := \sum_\alpha (2N)^{-q\alpha} \text{ (see Proposition 2.6.8)},$$

and such that $F := \sum_\alpha c_\alpha H_\alpha$ satisfies

$$||F||_{-1,-2r} \leq A(q) \sup_{z \in K_q(\delta)} |\mathcal{H}F(z)|$$

e) For all $R > 0, q > 1$ there exist $r \geq q$ such that

$$\sup_{z \in K_q(R)} |\mathcal{H}F(z)| \leq R||F||_{-1,-q} \leq R||F||_{-1,-2r} \leq RA(q) \sup_{z \in K_q(R)} |\mathcal{H}F(z)|$$

Proof  a) We have

$$|\tilde{F}(z)| \leq \sum_\alpha |a_\alpha||z^\alpha| \leq \left( \sum_\alpha |a_\alpha|^2 (2N)^{-q\alpha} \right)^{\frac{1}{2}} \left( \sum_\alpha |z^\alpha|^2 (2N)^{q\alpha} \right)^{\frac{1}{2}}.$$

(2.6.37)

Since $F \in (S)_{-1}^N$, we see that $M_q^2 := \sum_\alpha |a_\alpha|^2 (2N)^{-q\alpha} < \infty$ if $q$ is large enough.

b) Conversely, assume that (2.6.35) holds. For $r < \infty$ and $k$ a natural number, choose $\zeta = \zeta^{(r,k)} = (\zeta_1, \zeta_2, \ldots, \zeta_k)$ with

$$\zeta_j = (2j)^{-r}; \ 1 \leq j \leq k.$$  (2.6.38)
Then
\[ \sum_{\alpha} |\zeta^\alpha|^2 (2N)^{-r\alpha} \leq \sum_{\alpha} (2N)^{-r\alpha} < \delta^2 \] (2.6.39)
if \( r \) is large enough, say \( r \geq q_1 \). Hence
\[ \zeta \in K_r(\delta) \text{ for } r \geq q_1. \]

By Proposition 2.6.8 we have
\[ \sum_{\alpha} |b_{\alpha}| z_{\alpha} | \leq MA(q) \text{ for } z \in K_{3q}(\delta), \]
where \( M = \sup \{ |g(z)| ; z \in K_q(\delta) \} \). Hence, if \( r \geq \max(3q, q_1) \), we get
\[ \sum_{\text{Index } \alpha \leq k} |b_{\alpha}| (2N)^{-r\alpha} = \sum_{\text{Index } \alpha \leq k} |b_{\alpha}| |\zeta^\alpha| \]
\[ \leq \sum_{\text{Index } \alpha \leq k} |b_{\alpha}| |z^\alpha| \]
\[ \leq MA(q), \text{ for } z \in K_{3q}(\delta), \]
(2.6.40)
where Index \( \alpha \) is the position of the last nonzero element in the sequence \( (\alpha_1, \alpha_2, \ldots) \).

Now let \( k \to \infty \) to deduce that
\[ K := \sup_{\alpha} |b_{\alpha}| (2N)^{-r\alpha} < \infty. \] (2.6.41)

This gives
\[ \sum_{\alpha} |b_{\alpha}|^2 (2N)^{-2r\alpha} \leq K \sum_{\alpha} |b_{\alpha}| (2N)^{-r\alpha} < \infty, \] (2.6.42)
and hence \( G := \sum_{\alpha} b_{\alpha} H_{\alpha} \in (S)_{-1} \) as claimed.

c) If \( z \in K_q(R) \) we have
\[ |\mathcal{H}F(z)| = \left| \sum_{\alpha} c_{\alpha} z^\alpha \right| \]
\[ \leq \sum_{\alpha} |c_{\alpha}| (2N)^{-\frac{3}{2} \alpha} |z^\alpha| (2N)^{\frac{3}{2} \alpha} \]
\[ \leq \left( \sum_{\alpha} |z^\alpha|^2 (2N)^{q\alpha} \right)^{\frac{1}{2}} \left( \sum_{\alpha} |c_{\alpha}|^2 (2N)^{-q\alpha} \right)^{\frac{1}{2}} \]
\[ \leq R||F||_{-1,-q} \]
d) Suppose $M_q(\delta) < \infty$. Then it follows as in (2.6.41) that there exists $r \geq q$ such that $S_r \leq M_q(\delta)A(q)$ and
\[
\left\| \sum_{\alpha} c_{\alpha} H_{\alpha} \right\|_{-1,-2r}^2 = \sum_{\alpha} |c_{\alpha}|^2 (2N)^{-2r\alpha} \\
\leq S_r \sum_{\alpha} |c_{\alpha}| (2N)^{-r\alpha} \\
\leq S_r M_q(\delta)A(q) \leq A^2(q) M_q^2(\delta)
\]
e) This is a synthesis of c) and d).

From this we deduce the following useful result:

Theorem 2.6.12. Kondratiev et al. (1994), Theorem 12 (Analytic functions operate on $\mathcal{H}$-transforms). Suppose $g = \mathcal{H}X$ for some $X \in (S)_{-1}^N$, and let $M \in \mathbb{N}$. Let $f$ be a $C^M$-valued analytic function on a neighborhood $U$ of $\zeta_0 := g(0)$ in $\mathbb{C}^N$ such that the Taylor expansion of $f$ around $\zeta_0$ has real coefficients. Then there exists a unique $Y \in (S)_{-1}^M$ such that
\[
\mathcal{H}Y = f \circ g.
\]

Proof Let $r > 0$ be such that
\[
\{ \zeta \in \mathbb{C}^N; |\zeta - \zeta_0| < r \} \subset U.
\]

Then choose $q < \infty$ such that $g(z)$ is a bounded analytic function on $\mathbb{K}_q(1)$ and such that
\[
|g(z) - \zeta_0| < \frac{r}{2} \quad \text{for} \quad z \in \mathbb{K}_q(1).
\]
(This is possible by the estimate (2.6.37)). Then $f \circ g$ is a bounded analytic function on $\mathbb{K}_q(1)$, so the result follows from Theorem 2.6.11.

Definition 2.6.13 (Generalized expectation). Let $X = \sum_{\alpha} c_{\alpha} H_{\alpha} \in (S)_{-1}^N$. Then the vector $c_0 = \tilde{X}(0) \in \mathbb{R}^N$ is called the generalized expectation of $X$ and is denoted by $E[X]$. In the case when $X = F \in L^p(\mu)$ for some $p > 1$ then the generalized expectation of $F$ coincides with the usual expectation
\[
E[F] = \int_{S'} F(\omega) d\mu(\omega).
\]

To see this we note that for $N = 1$ the action of $F \in L^p(\mu)$ on $f \in L^p(\mu)^* = L^q(\mu)$ (where $1/p + 1/q = 1$) is given by
\[
\langle F, f \rangle = E[Ff],
\]
so that, in particular,

\[ E[F] = \langle F, 1 \rangle. \]

On the other hand, if \( F = \sum_\alpha c_\alpha H_\alpha \), then by (2.3.11)

\[ \langle F, 1 \rangle = c_0 = \tilde{F}(0). \]

So

\[ E[F] = c_0 = \tilde{F}(0) \tag{2.6.44} \]

for \( F \in L^p(\mu), p > 1 \), as claimed.

In fact, (2.6.43) also holds if \( F \in L^1(\mu) \cap (\mathcal{S})_{-1} \). (See Exercise 2.10.)

Note that from this definition we have

\[ E[X \circ Y] = (E[X], E[Y]) \text{ for all } X, Y \in (\mathcal{S})^N_1, \tag{2.6.45} \]

where \((\cdot, \cdot)\) denotes the inner product in \( \mathbb{R}^N \), and, in particular,

\[ E[X \circ Y] = E[X]E[Y]; X, Y \in (\mathcal{S})_{-1}. \tag{2.6.46} \]

Thanks to Theorem 2.6.12 we can construct the Wick versions \( f^\circ \) of analytic functions \( f \) as follows:

**Definition 2.6.14 (Wick versions of analytic functions).** Let \( X \in (\mathcal{S})^N_1 \) and let \( f : U \to \mathbb{C}^M \) be an analytic function, where \( U \) is a neighborhood of \( \zeta_0 := E[X] \). Assume that the Taylor series of \( f \) around \( \zeta_0 \) has coefficients in \( \mathbb{R}^M \). Then the Wick version \( f^\circ(X) \) of \( f \) applied to \( X \) is defined by

\[ f^\circ(X) = \mathcal{H}^{-1}(f \circ \tilde{X}) \in (\mathcal{S})^M_{-1}. \tag{2.6.47} \]

In other words, if \( f \) has the power series expansion

\[ f(z) = \sum a_\alpha (z - \zeta_0)^\alpha \text{ with } a_\alpha \in \mathbb{R}^M, \]

then

\[ f^\circ(X) = \sum a_\alpha (X - \zeta_0)^\circ_\alpha \in (\mathcal{S})^M_{-1}. \tag{2.6.48} \]

**Example 2.6.15.** If the function \( f : \mathbb{C}^N \to \mathbb{C}^M \) is entire, i.e., analytic in the whole space \( \mathbb{C}^N \), then \( f^\circ(X) \) is defined for all \( X \in (\mathcal{S})^N_{-1} \). For example,

i) *The Wick exponential* of \( X \in (\mathcal{S})_{-1} \) is defined by

\[ \exp^\circ X = \sum_{n=0}^{\infty} \frac{1}{n!} X^{\circ n}. \tag{2.6.49} \]
Using the Hermite transform we see that the Wick exponential has the same algebraic properties as the usual exponential. For example,

$$\exp^\diamond [X + Y] = \exp^\diamond [X] \diamond \exp^\diamond [Y]; \; X, Y \in (S)_{-1}. \quad (2.6.50)$$

ii) The analytic logarithm, \( f(z) = \log z \), is well-defined in any simply connected domain \( U \subset \mathbb{C} \) not containing the origin. If we require that \( 1 \in U \), then we can choose the branch of \( f(z) = \log z \) with \( f(1) = 0 \). For any \( X \in (S)_{-1} \) with \( E[X] \neq 0 \), choose a simply connected \( U \subset \mathbb{C} \setminus \{0\} \) such that \( \{1, E[X]\} \subset U \) and define the Wick-logarithm of \( X \), \( \log^\diamond X \), by

$$\log^\diamond X = H^{-1}(\log(\tilde{X}(z)) \in (S)_{-1}). \quad (2.6.51)$$

If \( E[X] \neq 0 \), we have

$$\exp^\diamond (\log^\diamond (X)) = X. \quad (2.6.52)$$

For all \( X \in (S)_{-1} \), we have

$$\log^\diamond (\exp^\diamond X) = X. \quad (2.6.53)$$

Moreover, if \( E[X] \neq 0 \) and \( E[Y] \neq 0 \), then

$$\log^\diamond (X \diamond Y) = \log^\diamond X + \log^\diamond Y. \quad (2.6.54)$$

iii) Similarly, if \( E[X] \neq 0 \), we can define the Wick inverse \( X^\diamond (-1) \in (S)_{-1} \), having the property that

$$X \diamond X^{\diamond (-1)} = 1. \quad \text{(2.6.56)}$$

Moreover, if \( E[X] \neq 0 \), we can define the Wick powers \( X^{\diamond r} \in (S)_{-1} \) for all real numbers \( r \).

**Remark** Note that, with the generalized expectation \( E[Y] \) defined for \( Y \in (S)_{-1} \) as in Definition 2.6.13, we have

$$E[\exp^\diamond [X]] = \exp[E[X]]; \; X \in (S)_{-1}, \quad (2.6.55)$$

simply because

$$E[\exp^\diamond [X]] = \mathcal{H}(\exp^\diamond [X])(0) = \exp[\mathcal{H}(X)(0)] = \exp[E[X]].$$

**Positive Noise**

An important special case of the Wick exponential is obtained by choosing \( X \) to be smoothed white noise \( w(\phi) \). Since \( w(\phi, \cdot) \in L^2(\mu) \), the usual exponential function \( \exp \) can also be applied to \( w(\phi, \omega) \) for almost all \( \omega \), and the relation between these two quantities is given by the following result.
Lemma 2.6.16.

\[ \exp^\diamond [w(\phi, \omega)] = \exp \left[ w(\phi, \omega) - \frac{1}{2} \|\phi\|^2 \right]; \phi \in L^2(\mathbb{R}^d) \] (2.6.56)

where \( \|\phi\| = \|\phi\|_{L^2(\mathbb{R}^d)} \).

Proof By basis independence, which in this \( L^2(\mu) \)-case follows from Proposition 2.4.2 (see Appendix D for the general case), we may assume that \( \phi = c \eta_1 \), in which case we get

\[ \exp^\diamond [w(\phi)] = \sum_{n=0}^{\infty} \frac{1}{n!} w(\phi)^n = \sum_{n=0}^{\infty} \frac{1}{n!} c^n \langle \omega, \eta_1 \rangle^n \]

\[ = \sum_{n=0}^{\infty} \frac{c^n}{n!} H_{n_1}^\omega(\omega) = \sum_{n=0}^{\infty} \frac{c^n}{n!} H_{n_1}(\omega) \]

\[ = \exp \left[ w(\phi) - \frac{1}{2} \|\phi\|^2 \right], \]

where we have used the generating property of the Hermite polynomials (see Appendix C).

In particular, (2.6.55) shows that \( \exp^\diamond w(\phi) \) is positive for all \( \phi \in L^2(\mu) \) and all \( \omega \). Moreover, if

\[ W_{\phi}(x, \omega) := w(\phi_x, \omega); \ x \in \mathbb{R}^d \]

is the smoothed white noise process defined in (2.1.15), then the process

\[ K_{\phi}(x, \omega) := \exp^{\diamond}[W_{\phi}(x, \omega)] \]

has the following three properties (compare with (2.1.20)–(2.1.22)):

If \( \text{supp} \phi_{x_1} \cap \text{supp} \phi_{x_2} = \emptyset \), then \( K_{\phi}(x_1, \cdot) \) and \( K_{\phi}(x_2, \cdot) \) are independent. (2.6.58)

\( \{K_{\phi}(x, \cdot)\}_{x \in \mathbb{R}^d} \) is a stationary process. (2.6.59)

For each \( x \in \mathbb{R}^d \) the random variable \( K_{\phi}(x, \cdot) > 0 \) has a lognormal distribution (i.e., \( \log K_{\phi}(x, \cdot) \) has a normal distribution) and \( E[K_{\phi}(x, \cdot)] = 1 \), \( \Var[K_{\phi}(x, \cdot)] = \exp[\|\phi\|^2] - 1 \). (2.6.60)

Properties (2.6.57) and (2.6.58) follow directly from the corresponding properties (2.1.20) and (2.1.21) for \( W_{\phi}(x, \cdot) \). The first parts of (2.6.59)
follow from (2.6.56) and the fact that \( E[W_\phi(x,\cdot)^k] = 0 \) for all \( k \geq 1 \). The last part of (2.6.59) is left as an exercise for the reader (Exercise 2.11).

These three properties make \( K_\phi(x,\omega) \) a good mathematical model for many cases of “positive noise” occurring in various applications. In particular, the function \( K_\phi(x,\omega) \) is suitable as a model for the \textit{stochastic permeability} of a heterogeneous, isotropic rock. See (1.1.5) and Section 4.6. We shall call \( K_\phi(x,\cdot) \) the \textit{smoothed positive noise process}. Similarly, we call

\[
K(x,\cdot) = \exp^\diamond [W(x,\cdot)] \in (\mathcal{S})^*
\tag{2.6.61}
\]

the \textit{singular positive noise process}. Computer simulations of the 1-parameter (i.e., \( d = 1 \)) positive noise process \( K_\phi(x,\omega) \) for a given \( \phi \) are shown in Figure 2.2.

Computer simulations of the 2-parameter (i.e., \( d = 2 \)) positive noise process \( K_\phi(x,\omega) \) where \( \phi(y) = \epsilon \chi_{[0,h] \times [0,h]}(y); \ y \in \mathbb{R}^2 \) are shown on Figure 2.3.

\section*{The Positive Noise Matrix}

When the (deterministic) medium is anisotropic, the non-negative permeability function \( k(x) \) in Darcy’s law (1.1.5) must be replaced by a permeability matrix \( K(x) = [K_{ij}(x)] \in \mathbb{R}^{d \times d} \). The interpretation of the \((i,j)\)th element, \( K_{ij} \), is that

\[
K_{ij}(x) = \text{velocity of fluid at } x \text{ in direction } i \text{ induced by a pressure gradient of unit size in direction } j.
\]

Physical arguments lead to the conclusion that \( K(x) = [K_{ij}(x)] \) should be a symmetric, non-negative definite matrix for each \( x \).

For a stochastic anisotropic medium it is natural to represent the stochastic permeability matrix as follows (Gjerde (1995a), Øksendal (1994b)):

Let \( W(x) \in (\mathcal{S})_{0}^{N} \) be \( N \)-dimensional, \( d \)-parameter white noise with the value \( N = 1/2d(d+1) \). Define

\[
K(x) := \exp^\phi [W(x)];
\tag{2.6.62}
\]

where

\begin{center}
\text{Fig. 2.2 Two sample paths of the Wick exponential of the 1-parameter white noise process.}
\end{center}
2.7 The \((\mathcal{S})_{\rho,r}^N\) Spaces and the \(\mathcal{S}\)-Transform

Sometimes the following spaces, which are intermediate spaces to the spaces \((\mathcal{S})_{\rho}^N\), \((\mathcal{S})_{-\rho}^N\), are convenient to work in (see Våge (1996a)).

**Definition 2.7.1.** For \(\rho \in [-1,1]\) and \(r \in \mathbb{R}\), let \((\mathcal{S})_{\rho,r}^N\) consist of those \(F = \sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})_\rho^N\) (with \(a_\alpha \in \mathbb{R}^N\) for all \(\alpha\)) such that

\[
\|F\|_{\rho,r}^2 := \sum_\alpha a_\alpha^2 (\alpha!)^{1+\rho} (2N)^r < \infty.
\]  

\(\mathcal{W}(x) = \begin{bmatrix} W_{1,1}(x) & W_{1,2}(x) & \cdots & W_{1,d}(x) \\ W_{1,2}(x) & W_{2,2}(x) & \cdots & W_{2,d}(x) \\ \vdots & \vdots & \ddots & \vdots \\ W_{1,d}(x) & \cdots & W_{d,d}(x) \end{bmatrix}\)  

(2.6.63)

and \(W_{ij}(x); 1 \leq i \leq j \leq d\) are the \(1/2d(d+1)\) independent components of \(\mathcal{W}(x)\), in some (arbitrary) order.

Here the Wick exponential is to be interpreted in the Wick matrix sense, i.e.,

\[
\exp[\mathcal{M}] = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{M}^n
\]  

(2.6.64)

when \(\mathcal{M} \in (\mathcal{S})_{-1}^{m;k \times k}\) is a stochastic distribution matrix. It follows from Theorem 2.6.12 that \(\exp[\mathcal{M}]\) exists as an element of \((\mathcal{S})_{-1}^{m;k \times k}\).

We call \(K(x)\) the (singular) positive noise matrix. It will be used in Section 4.7.

Similarly, one can define the smoothed positive noise matrix

\[
K_\phi(x) = \exp[\mathcal{W}_\phi(x)],
\]  

(2.6.65)

where the entries of the matrix \(\mathcal{W}_\phi(x)\) are the components of the \(1/2d(d+1)\)-dimensional smoothed white noise process \(\mathcal{W}_\phi(x)\).
If \( F = \sum_{\alpha} a_{\alpha} H_{\alpha}, G = \sum_{\alpha} b_{\alpha} H_{\alpha} \) belong to \( (S)^{N}_{\rho, r} \), then we define the inner product \( (F, G)_{\rho, r} \) of \( F \) and \( G \) by

\[
(F, G)_{\rho, r} = \sum_{\alpha} (a_{\alpha}, b_{\alpha})(\alpha!)^{1+\rho}(2N)^{r\alpha},
\]

where \((a_{\alpha}, b_{\alpha})\) is the inner product on \( \mathbb{R}^N \).

Note that if \( \rho \in [0, 1] \), then \( (S)^{N}_{\rho} \) is the projective limit (intersection) of the spaces \( \{(S)^{N}_{\rho, r}\}_{r \geq 0} \), while \((S)^{N}_{-\rho}\) is the inductive limit (union) of \( \{(S)^{N}_{-\rho, -r}\}_{r \geq 0} \).

**Lemma 2.7.2 Våge (1996a).** For every pair \((\rho, r) \in [-1, 1] \times \mathbb{R}\) the space \( (S)^{N}_{\rho, r} \) equipped with the inner product \((2.7.2)\) is a separable Hilbert space.

**Proof** We first prove completeness: Fix \( \rho, r \) and suppose \( F_k = \sum_{\alpha} a_{\alpha}^{(k)} H_{\alpha} \) is a Cauchy sequence in \( (S)^{N}_{\rho, r}, k = 1, 2, \ldots \). Then \( \{a_{\alpha}^{(k)}\}_{k=1}^{\infty} \) is a Cauchy sequence in \( \mathbb{R}^N \) (with the usual norm), so \( a_{\alpha}^{(k)} \to a_{\alpha} \), say, as \( k \to \infty \). Define

\[
F = \sum_{\alpha} a_{\alpha} H_{\alpha}.
\]

We must prove that \( f \in (S)^{N}_{\rho, r} \) and that \( F_k \to F \) in \( (S)^{N}_{\rho, r} \). To this end let \( \epsilon > 0 \) and \( n \in \mathbb{N} \). Then there exists \( M \in \mathbb{N} \) such that

\[
\sum_{\alpha \in \Gamma_n} \sum_{i,j} (a_{\alpha}^{(i)} - a_{\alpha}^{(j)})^2 (\alpha!)^{1+\rho}(2N)^{r\alpha} < \epsilon^2
\]

for \( i, j \geq M \), where

\[
\Gamma_n = \{\alpha = (\alpha_1, \ldots, \alpha_n); \alpha_j \in \{0, 1, \ldots, n\}, j = 1, \ldots, n\}. \quad (2.7.3)
\]

If we let \( i \to \infty \), we see that

\[
\sum_{\alpha \in \Gamma_n} (a_{\alpha} - a_{\alpha}^{(j)})^2 (\alpha!)^{1+\rho}(2N)^{r\alpha} < \epsilon^2 \quad \text{for} \quad j \geq M.
\]

Letting \( n \to \infty \), we obtain that

\[
F - F_j \in (S)^{N}_{\rho, r}
\]

and that

\[
F_j \to F \quad \text{in} \quad (S)^{N}_{\rho, r}.
\]

Finally, the separability follows from the fact that \( \{H_{\alpha}\} \) is a countable dense subset of \( (S)^{N}_{\rho, r} \). \( \square \)
Example 2.7.3. Singular white noise $W(x, \omega)$ belongs to $(S)_{-0,-q}$ for all $q > 1$. This follows from the proof of Proposition 2.3.10.

The $\mathcal{S}$-Transform

The Hermite transform is closely related to the $\mathcal{S}$-transform. See Hida et al. (1993), and the references therein. For completeness, we give a short introduction to the $\mathcal{S}$-transform here.

Earlier we saw that if $\phi \in \mathcal{S}(\mathbb{R}^d)$, then $\langle \omega, \phi \rangle \circ_n = w(\phi, \omega) \circ_n \in (\mathcal{S})_1$, for all natural numbers $n$ (Example 2.3.4). It is natural to ask if we also have \( \exp \circ \left[ w(\phi, \cdot) \right] \in (\mathcal{S})_1 \), at least if $\| \phi \|_{L^2(\mathbb{R}^d)}$ is small enough. This is not the case. However, we have the following:

Lemma 2.7.4. a) Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $q < \infty$. Then there exists $\epsilon > 0$ such that for $\lambda \in \mathbb{R}$ with $|\lambda| < \epsilon$ we have

$$
\exp \circ \left[ \lambda w(\phi, \cdot) \right] \in (\mathcal{S})_{1,q}.
$$

b) For all $\rho < 1$ we have

$$
\exp \circ \left[ \lambda w(\phi, \cdot) \right] \in (\mathcal{S})_\rho
$$

for all $\lambda \in \mathbb{R}$.

Proof Choose $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and consider

$$
\exp \circ \left[ \langle \omega, \lambda_1 \eta_1 + \cdots + \lambda_k \eta_k \rangle \right]
$$

$$
= \exp \circ \left( \sum_{j=1}^{k} \lambda_j \langle \omega, \eta_j \rangle \right)
$$

$$
= \exp \circ \left( \sum_{j=1}^{k} \lambda_j H_{\epsilon(j)}(\omega) \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{j=1}^{k} \lambda_j H_{\epsilon(j)} \right)^{\circ_n}
$$

$$
= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{\alpha_1=1}^{n} \cdots \sum_{\alpha_k=1}^{n} \frac{n!}{\alpha_1! \cdots \alpha_k!} \lambda_1^{\alpha_1} \cdots \lambda_k^{\alpha_k} H_{\alpha_1 \epsilon_1 + \cdots + \alpha_k \epsilon(k)} \right)
$$

$$
= \sum_{n=0}^{\infty} \sum_{\alpha_1=1}^{n} \cdots \sum_{\alpha_k=1}^{n} \frac{1}{\alpha_1! \cdots \alpha_k!} \lambda_1^{\alpha_1} \cdots \lambda_k^{\alpha_k} H_{\alpha_1 \epsilon_1 + \cdots + \alpha_k \epsilon(k)}
$$

$$
= \sum_{n=0}^{\infty} \sum_{|\alpha|=n}^{\text{Index } \alpha \leq k} \frac{1}{\alpha!} \lambda^\alpha H_\alpha = \sum_{\text{Index } \alpha \leq k} \frac{1}{\alpha!} \lambda^\alpha H_\alpha =: \sum_{\alpha} a_\alpha^{(k)} H_\alpha,
$$

(2.7.6)
where $\lambda^\alpha = \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_k^{\alpha_k}$. Hence by Lemma 2.3.4

$$
\sum_{\alpha} (a_\alpha^{(k)})^2 (\alpha!)^2 (2N)^{q\alpha}
= \sum_{n=0}^\infty \sum_{|\alpha| = n} \binom{1}{\alpha!}^2 \lambda^{2\alpha} (\alpha!)^2 (2N)^{q\alpha}
= \sum_{\text{Index } \alpha \leq k} \lambda^{2\alpha} (2N)^{q\alpha}
= \sum_{\text{Index } \alpha \leq k} \lambda_1^{2\alpha_1} \cdots \lambda_k^{2\alpha_k} 2^{q\alpha_1} 4^{q\alpha_2} \cdots (2k)^{q\alpha_k}
\leq \left( \sum_{\alpha_1 = 0}^\infty (\lambda_1^2 (2^d \delta_1^{(1)} \cdots \delta_d^{(1)})^{q_1})^{\alpha_1} \right)
\cdots \left( \sum_{\alpha_k = 1}^\infty (\lambda_k^2 (2^d \delta_1^{(k)} \cdots \delta_d^{(k)})^{q_1})^{\alpha_k} \right)
= \prod_{j=1}^k \frac{1}{1 - \Lambda_j} < \infty, \quad (2.7.7)
$$

if

$$
\Lambda_j := \lambda_j^2 (2^d \delta_1^{(j)} \cdots \delta_d^{(j)})^{q'} < 1, \quad (2.7.8)
$$

where $q' = (d/d - 1)q$ if $d \geq 2$, $q' = q$ if $d = 1$. Now choose $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then by Theorem 2.3.1 there exists $M < \infty$ such that

$$(\phi, \eta_j)^2 \leq M^2 (2^d \delta_1^{(j)} \cdots \delta_d^{(j)})^{-q'} \quad \text{for all } j.$$

Hence, if $\lambda \in \mathbb{R}$ with $|\lambda|$ small enough, we have

$$
\lambda^2 (\phi, \eta_j)^2 \leq \frac{1}{2} \left( 2^d \delta_1^{(j)} \cdots \delta_d^{(j)} \right)^{-q'} \quad \text{for all } j. \quad (2.7.9)
$$

Therefore, if we define

$$
\lambda^2_j := \lambda^2 (\phi, \eta_j)^2,
$$

we see that (2.7.8) holds, and we can apply the above argument.

Then, if we write $(\phi, \eta)^\alpha = (\phi, \eta_1)^{\alpha_1} \cdots (\delta, \eta_k)^{\alpha_k}$ when $\alpha = (\alpha_1, \ldots, \alpha_k)$, we get

$$
\exp[\lambda(\omega, \phi)] = \sum_{\alpha} \frac{1}{\alpha!} \lambda^{\alpha} (\phi, \eta)^\alpha H_\alpha
=: \sum_{\alpha} c_\alpha^{(\lambda)} H_\alpha, \quad (2.7.10)
$$
2.7 The \((S)_{p,r}^N\) Spaces and the \(S\)-Transform

and hence, by (2.7.7) and (2.7.9),

\[
\sum_{\alpha} (c_\alpha^\lambda)^2 (\alpha!) (2N)^{\lambda\alpha} = \lim_{k \to \infty} \prod_{j=1}^{k} \frac{1}{1 - \Lambda_j} \leq \prod_{j=1}^{\infty} (1 + 2\Lambda_j)
\]

\[
= \exp \left[ \sum_{j=1}^{\infty} \log(1 + 2\Lambda_j) \right] \leq \exp \left[ \sum_{j=1}^{\infty} 2\Lambda_j \right] < \infty,
\]

by (2.3.3). \(\Box\)

If \(F \in (S)_{-1}\), then there exists \(q < \infty\) such that \(F \in (S)_{-1,-q}\). Hence we can make the following definition:

**Definition 2.7.5 (The \(S\)-transform).** (i) Let \(F \in (S)_{-1}\) and let \(\phi \in S(\mathbb{R}^d)\). Then the \(S\)-transform of \(F\) at \(\lambda\phi\), \((SF)(\lambda\phi)\), is defined, for all real numbers \(\lambda\) with \(|\lambda|\) small enough, by

\[
(SF)(\lambda\phi) = \langle F, \exp^\phi[w(\lambda\phi, \cdot)] \rangle, \quad (2.7.11)
\]

where \(\langle \cdot, \cdot \rangle\) denotes the action of \(F \in (S)_{-1,-q}\) on \(\exp^\phi[w(\lambda\phi, \cdot)]\), which belongs to 

\(((S)_{-1,-q})^* = (S)_{1,q}\) for \(|\lambda|\) small enough, by Lemma 2.7.4.

(ii) Let \(F \in (S)_{-\rho}\) for some \(\rho < 1\) and let \(\phi \in S(\mathbb{R}^d)\). Then the \(S\)-transform of \(F\) at \(\lambda\phi\) is defined by

\[
(SF)(\lambda\phi) = \langle F, \exp^\phi[w(\lambda\phi, \cdot)] \rangle \quad (2.7.12)
\]

for all \(\lambda \in \mathbb{R}\).

In terms of the chaos expansion we can express the \(S\)-transform as follows:

**Proposition 2.7.6.** Suppose \(F = \sum_{\alpha} a_{\alpha} H_{\alpha} \in (S)_{-1}\), and let \(\phi \in S(\mathbb{R}^d)\). Then, if \(\lambda \in \mathbb{R}\) with \(|\lambda|\) small enough, we have

\[
(SF)(\lambda\phi) = \sum_{\alpha} \lambda^{\alpha} a_{\alpha} (\phi, \eta)^{\alpha}; \quad \phi \in S(\mathbb{R}^d), \quad (2.7.13)
\]

where \((\phi, \eta)^{\alpha} = (\phi, \eta_1)^{\alpha_1} (\phi, \eta_2)^{\alpha_2} \cdots \).

**Proof** By (2.7.10) and (2.3.11) we have

\[
(SF)(\lambda\phi) = \langle F, \exp^\phi[w(\lambda\phi, \cdot)] \rangle
\]

\[
= \sum_{\alpha} a_{\alpha} \left( \frac{1}{\alpha!} \lambda^{\alpha} (\phi, \eta)^{\alpha} \right) \alpha! = \sum_{\alpha} \lambda^{\alpha} a_{\alpha} (\phi, \eta)^{\alpha}.
\]

\(\Box\)
Corollary 2.7.7. As a function of \( \lambda \) the expression \((SF)(\lambda \phi)\) is real analytic and hence has an analytic extension to all \( \lambda \in \mathbb{C} \) with \( |\lambda| \) small enough. If \( F \in (S)_{-\rho} \) for some \( \rho < 1 \), then \((SF)(\lambda \phi)\) extends to an entire function of \( \lambda \in \mathbb{C} \).

From now on we will consider the \( S \)-transforms \((SF)(\lambda \phi)\) to be these analytic extensions.

Example 2.7.8. i) The \( S \)-transform of smoothed white noise \( F = w(\psi, \cdot) \), where \( \psi \in L^2(\mathbb{R}^d) \), is, by (2.2.23) and (2.7.13),

\[
(Sw(\psi, \cdot))(\lambda \phi) = \sum_{j=1}^{\infty} \lambda \psi(\eta_j) \phi, a_j = (\lambda \phi, \psi); \quad \phi \in \mathcal{S}(\mathbb{R}^d), \lambda \in \mathbb{C}. \tag{2.7.14}
\]

ii) The \( S \)-transform of singular white noise \( F = W(x, \cdot) \) is, by (2.3.33) and (2.7.13),

\[
(SW(x))(\lambda \phi) = \sum_{j=1}^{\infty} \lambda \eta_j(x) \phi, \eta_j = \lambda \phi(x); \quad \phi \in \mathcal{S}(\mathbb{R}^d), \lambda \in \mathbb{C}. \tag{2.7.15}
\]

An important property of the \( S \)-transform follows: (Compare it with Proposition 2.6.6.)

Proposition 2.7.9. Suppose \( F, G \in (S)_{-1} \) and \( \phi \in \mathcal{S}(\mathbb{R}^d) \). Then, if \( |\lambda| \) is small enough,

\[
S(F \diamond G)(\lambda \phi) = (SF)(\lambda \phi) \cdot (SG)(\lambda \phi). \tag{2.7.16}
\]

Proof Suppose \( F = \sum_{\alpha} a_{\alpha} H_{\alpha}, \ G = \sum_{\beta} b_{\beta} H_{\beta} \). Then by (2.7.13)

\[
(SF)(\lambda \phi) \cdot (SG)(\lambda \phi) = \left( \sum_{\alpha} \lambda^{\alpha} a_{\alpha} (\phi, \eta)^{\alpha} \right) \left( \sum_{\beta} \lambda^{\beta} b_{\beta} (\phi, \eta)^{\beta} \right) = \sum_{\alpha, \beta} \lambda^{\alpha+\beta} a_{\alpha} b_{\beta} (\phi, \eta)^{\alpha+\beta} = \sum_{\gamma} \lambda^{\gamma} \left( \sum_{\alpha+\beta=\gamma} a_{\alpha} b_{\beta} \right) (\phi, \eta)^{\gamma} = S(F \diamond G)(\lambda \phi).
\]

The relation between the \( S \)-transform and the \( H \)-transform is the following:

Theorem 2.7.10. Let \( F \in (S)_{-1} \). Then

\[
(HF)(z_1, z_2, \ldots, z_k) = (SF)(z_1 \eta_1 + z_2 \eta_2 + \cdots + z_k \eta_k) \tag{2.7.17}
\]
for all \( (z_1, \ldots, z_k) \in \mathbb{C}^k \) with \( |z_j| < (2^d \delta_1^{(j)} \cdots \delta_d^{(j)})^{-q'} \); \( 1 \leq j \leq k \), where \( q < \infty \) is so large that

\[ F \in (S)_{-1,-q} \]

with \( q' = d/d - 1 \) if \( d \leq 2 \), \( q' = q \) if \( d = 1 \).

**Proof** By (2.6.18) and (2.7.7), both sides of (2.7.11) are defined for all such \( z = (z_1, \ldots, z_k) \in \mathbb{C}^k \). Suppose \( F \) has the chaos expansion

\[ F = \sum_{\alpha} b_\alpha H_\alpha. \]

Then by (2.7.13) we have

\[
(SF)(z_1 \eta_1 + \cdots + z_k \eta_k) = \sum_{\alpha} b_\alpha (z_1 \eta_1 + \cdots + z_k \eta_k, \eta)^\alpha
\]

\[
= \sum_{\alpha} b_\alpha z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k} = \sum_{\alpha} b_\alpha z_1^{\alpha_1} \cdots z_k^{\alpha_k} = (HF)(z),
\]

as claimed. \( \Box \)

### 2.8 The Topology of \((S)^{N}_{-1}\)

The topologies of \((S)^{N}_{\rho}\) and \((S)^{N}_{-\rho}; 0 \leq \rho \leq 1\) are defined by the corresponding families of seminorms given in (2.3.9) and (2.3.10), respectively. Since we will often be working with the \( H\)-transforms of elements of \((S)^{N}_{-1}\), it is useful to have a description of the topology in terms of the transforms. Such a description is

**Theorem 2.8.1.** The following are equivalent:

- **a)** \( X_n \to X \) in \((S)^{N}_{-1}\);
- **b)** there exist \( \delta > 0, q < \infty \) such that \( \tilde{X}_n(z) \to \tilde{X}(z) \) uniformly in \( K_q(\delta) \);
- **c)** there exist \( \delta > 0, q < \infty \) such that \( \tilde{X}_n(z) \to \tilde{X}(z) \) pointwise boundedly in \( K_q(\delta) \).

It suffices to prove this when \( N = 1 \). We need the following result (Recall that a bounded linear operator \( A : H_1 \to H_2 \) where \( H_i \) are Hilbert spaces \( i = 1, 2 \), is called a \*Hilbert-Schmidt operator\* if the series \( \sum_{i,j} |(Ae_i, f_j)|^2 \) converges whenever \( \{e_i\} \) and \( \{f_j\} \) are orthonormal bases for \( H_1 \) and \( H_2 \), respectively.):

**Lemma 2.8.2.** \((S)_1\) is a nuclear space.

**Proof** Define

\[
(S)_{1,r} = \left\{ f = \sum_{\alpha} c_\alpha H_\alpha; \|f\|^2_{1,r} < \infty \right\}
\]

(2.8.1)
where \(\|f\|_{\rho,r}\) is defined by Definition 2.7.1, so that
\[
\|f\|_{1,r}^2 = \sum_{\alpha} c_{\alpha}^2 (\alpha!)^2 (2N)^{r\alpha}.
\] (2.8.2)

Then \((S)_{1,r}\) is a Hilbert space with inner product
\[
\langle f, g \rangle_{1,r} = \sum_{\alpha} a_{\alpha} b_{\alpha} (\alpha!)^2 (2N)^{r\alpha}
\] (2.8.3)
when \(f = \sum_{\alpha} a_{\alpha} H_{\alpha} \in (S)_{1,r}, g = \sum_{\beta} b_{\beta} H_{\beta} \in (S)_{1,r}\).

Therefore the family of functions
\[
H_{\alpha,r} = \frac{1}{\alpha!} (2N)^{-\frac{r}{2}} H_{\alpha}; \alpha \in J
\]
constitutes an orthonormal basis for \((S)_{1,r}\). By definition, \((S)_{1}\) is the projective limit of \((S)_{1,r}\), i.e.,
\[
(S)_{1} = \bigcap_{r=1}^{\infty} (S)_{1,r}.
\]

If \(r_2 > r_1 + 1\), then
\[
\sum_{\alpha} \|H_{\alpha,r_2}\|_{1,r_1}^2 = \sum_{\alpha} \frac{1}{(\alpha!)^2} (2N)^{-r_2\alpha} (\alpha!)^2 (2N)^{r_1\alpha}
\]
\[
= \sum_{\alpha} (2N)^{-\alpha} < \infty,
\]
by Proposition 2.3.3. This means that the imbedding \((S)_{1,r_2} \subset (S)_{1,r_1}\) is Hilbert–Schmidt if \(r_2 > r_1 + 1\) and hence \((S)_{1}\) is a nuclear space. \(\square\)

**Proof of Theorem 2.8.1.** a) \(\Rightarrow\) b). First note that the dual \((S)_{-1,-r}\) of the space \((S)_{1,r}\) is defined by
\[
(S)_{-1,-r} = \left\{ F = \sum_{\alpha} c_{\alpha} H_{\alpha}; \|F\|_{-1,-r}^2 := \sum_{\alpha} c_{\alpha}^2 (2N)^{-r\alpha} < \infty \right\}.
\]

Assume that \(X_n = \sum_{\alpha} b_{\alpha}^{(n)} H_{\alpha} \to X = \sum_{\alpha} b_{\alpha} H_{\alpha}\) in \((S)_{-1}\). Since \((S)_{1}\) is nuclear (Hida (1980)), this implies that there exists \(r_0\) such that \(X_n \to X\) in \((S)_{-1,-r_0}\) as \(n \to \infty\). From this we deduce that
\[
M^2 := \sup_n \{\|X_n\|_{-1,-r_0}^2\}
\]
\[
= \sup_n \left\{ \sum_{\alpha} |b_{\alpha}^{(n)}|^2 (2N)^{-r_0\alpha} \right\} < \infty.
\]
Hence
\[ |\tilde{X}_n(z)| = \left| \sum_\alpha b_\alpha^{(n)} z^\alpha \right| = \left| \sum_\alpha b_\alpha^{(n)} (2N)^{-r_\alpha} (2N)^{\frac{r_\alpha}{2}} z^\alpha \right| \]
\[ \leq \left( \sum_\alpha |b_\alpha^{(n)}|^2 (2N)^{-r_\alpha} \right)^{\frac{1}{2}} \cdot \left( \sum_\alpha |z^\alpha|^2 (2N)^{r_\alpha} \right)^{\frac{1}{2}} \]
\[ \leq M(1 + R) \]
if \( z \in \mathbb{K}_{r_0}(R) \), so \( \{ \tilde{X}_n(z) \} \) is a bounded sequence on \( \mathbb{K}_{r_0}(R) \) for all \( R \).

Moreover, since \( X_n \to X \) in \( (S)_{-1,r_0} \), we have, by the same procedure as above, that
\[ |\tilde{X}_n(z) - \tilde{X}(z)| = \left| \sum_\alpha (b_\alpha^{(n)} - b_\alpha) z^\alpha \right| \]
\[ \leq \left( \sum_\alpha |b_\alpha^{(n)} - b_\alpha|^2 (2N)^{-r_\alpha} \right)^{\frac{1}{2}} \cdot \left( \sum_\alpha |z^\alpha|^2 (2N)^{r_\alpha} \right)^{\frac{1}{2}} \]
\[ \leq (1 + R)\|X_n - X\|_{-1,r_0} \to 0 \quad \text{as} \quad n \to \infty, \]
uniformly for \( z \in \mathbb{K}_{r_0}(R) \), for each \( R < \infty \).

b) ⇒ a). Suppose there exist \( \delta > 0, q < \infty, M < \infty \) such that \( \tilde{X}_n(z) \to \tilde{X}(z) \) for \( z \in \mathbb{K}_q(\delta) \) and \( |\tilde{X}_n(z)| \leq M \) for all \( n = 1, 2, \ldots, z \in \mathbb{K}_q(\delta) \).

For \( r < \infty \) and \( k \) a natural number, choose \( \zeta = \zeta^{(r,k)} = (\zeta_1, \ldots, \zeta_k) \) with
\[ \zeta_j = (2j)^{-r} \quad \text{for} \quad j = 1, \ldots, k. \]

Then
\[ \sum_\alpha (2N)^{r_\alpha} |\zeta^\alpha|^2 \leq \sum_\alpha (2N)^{-r_\alpha} < \delta^2 \]
for \( r \) large enough, say \( r \geq q_1 \).

Hence \( \zeta \in \mathbb{K}_r(\delta) \) for \( r \geq q_1 \). Write \( X_n = \sum_\alpha b_\alpha^{(n)} H_\alpha \). Since \( |\tilde{X}_n(z)| \leq M \) for \( z \in \mathbb{K}_q(\delta) \), we have by Proposition 2.6.8
\[ \sum_\alpha |b_\alpha^{(n)}||z^\alpha| \leq MA(q) \quad \text{for all} \quad z \in \mathbb{K}_{3q}(\delta). \]

Thus if \( r \geq \max(3q, q_1) \), we get
\[ \sum_{\text{Index } \alpha \leq k} |b_\alpha^{(n)}|(2N)^{-r_\alpha} = \sum_{\text{Index } \alpha \leq k} |b_\alpha^{(n)}|\zeta^\alpha \]
\[ \sum_{\text{Index } \alpha \leq k} |b_\alpha^{(n)}||\zeta^\alpha| \leq \sum_\alpha |b_\alpha^{(n)}||\zeta^\alpha| \leq MA(q). \]
Letting $k \to \infty$ we deduce that

$$K := \sup_{\alpha} |b_{\alpha}^{(n)}|(2N)^{-r\alpha} < \infty,$$

which implies

$$\sum_{\alpha} |b_{\alpha}^{(n)}|^2 (2N)^{-2r\alpha} \leq K \sum_{\alpha} |b_{\alpha}^{(n)}|(2N)^{-r\alpha} < KMA(q).$$

So

$$\|X_n\|_{-1,-2r} \leq KMA(q) \quad \text{for all} \quad n.$$

A similar argument applied to $X_n - X$ instead of $X_n$ gives the estimate

$$\|X_n - X\|_{-1,-2r} \leq KA(q) \sup_{z \in \mathbb{R}_+^d} |\widetilde{X}_n(z) - \widetilde{X}(z)|.$$

The proof of the equivalence of b) and c) follows the familiar argument from the finite-dimensional case and is left as an exercise.

\[\square\]

**Stochastic Distribution Processes**

As mentioned in the introduction, one advantage of working in the general space $(S)^N_{-1}$ of stochastic distributions is that it contains the solutions of many stochastic differential equations, both ordinary and partial and in arbitrary dimension. Moreover, if the objects of such equations are regarded as $(S)^N_{-1}$-valued, then differentiation can be interpreted in the usual strong sense in $(S)^N_{-1}$. This makes the following definition natural.

**Definition 2.8.3.** A measurable function

$$u : \mathbb{R}^d \to (S)^N_{-1}$$

is called a *stochastic distribution process* or an $(S)^N_{-1}$-process.

The process $u$ is called continuous, differentiable, $C^1$, $C^k$, etc., if the $(S)^N_{-1}$-valued function $u$ has these properties, respectively. For example, the partial derivative $\partial u / \partial x_k (x)$ of an $(S)_{-1}$-process $u$ is defined by

$$\frac{\partial u}{\partial x_k} (x_1, \ldots, x_d) = \lim_{\Delta x_k \to 0} \frac{u(x_1, \ldots, x_k + \Delta x_k, \ldots, x_d) - u(x_1, \ldots, x_k, \ldots, x_d)}{\Delta x_k}, \quad (2.8.4)$$

provided the limit exists in $(S)_{-1}$.

In terms of the Hermite transform $\tilde{u}(x)(z) = \tilde{u}(x; z)$, the limit on the right hand side of (2.8.4) exists if and only if there exists an element $Y \in (S)_{-1}$
such that

$$\frac{1}{\Delta x_k} [\tilde{u}(x_1, \ldots, x_k + \Delta x_k, \ldots, x_d; z) - \tilde{u}(x_1, \ldots, x_k, \ldots, x_d; z)] \rightarrow \tilde{Y}(z)$$

(2.8.5)

pointwise boundedly (or uniformly) in $\mathbb{K}_q(\delta)$ for some $q < \infty, \delta > 0$, according to Theorem 2.8.1. If this is the case, then $Y$ is denoted by $\partial u/\partial x_k$.

When we apply the Hermite transform to solve stochastic differential equations the following observation is important.

For simplicity of notation, choose $N = d = 1$ and consider a differentiable $(S)_{-1}$-process $X(t, \omega)$. The statement that

$$\frac{dX(t, \omega)}{dt} = F(t, \omega) \quad \text{in} \quad (S)^{-1}$$

is then equivalent to saying that

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left( \tilde{X}(t + \Delta t; z) - \tilde{X}(t; z) \right) = \tilde{F}(t; z)$$

pointwise boundedly for $z \in \mathbb{K}_q(\delta)$ for some $q < \infty, \delta > 0$. For this it is clearly necessary that

$$\frac{d\tilde{X}(t; z)}{dt} = \tilde{F}(t; z) \quad \text{for each} \quad z \in \mathbb{K}_q(\delta),$$

but apparently not sufficient, because we also need that the pointwise convergence is bounded for $z \in \mathbb{K}_q(\delta)$. The following result is sufficient for our purposes.

**Lemma 2.8.4 (Differentiation of $(S)_{-1}$-processes).** Suppose $X(t, \omega)$ and $F(t, \omega)$ are $(S)_{-1}$-processes such that

$$\frac{d\tilde{X}(t; z)}{dt} = \tilde{F}(t; z) \quad \text{for each} \quad t \in (a, b), z \in \mathbb{K}_q(\delta) \quad (2.8.6)$$

and that

$$\tilde{F}(t; z) \quad \text{is a bounded function of} \quad (t, z) \in (a, b) \times \mathbb{K}_q(\delta),$$

continuous in $t \in (a, b)$ for each $z \in \mathbb{K}_q(\delta).$ \hspace{1cm} (2.8.7)

Then $X(t, \omega)$ is a differentiable $(S)_{-1}$-process and

$$\frac{dX(t, \omega)}{dt} = F(t, \omega) \quad \text{for all} \quad t \in (a, b). \quad (2.8.8)$$
Proof. By the mean value theorem we have

\[ \frac{1}{\Delta t} \left( \tilde{X}(t + \Delta t; z) - \tilde{X}(t; z) \right) = \tilde{F}(t + \theta \Delta t; z) \]

for some \( \theta \in [0, 1] \), for each \( z \in \mathbb{K}_q(\delta) \). So if (2.8.6) and (2.8.7) hold, then

\[ \frac{1}{\Delta t} \left( \tilde{X}(t + \Delta t; z) - \tilde{X}(t; z) \right) \to \tilde{F}(t; z) \quad \text{as} \quad \Delta t \to 0, \]

pointwise boundedly for \( z \in \mathbb{K}_q(\delta) \).

Similarly we can relate the integrability of an \((S)_{-1}\)-process to the integrability of its \( H \)-transform as follows:

We say that an \((S)_{-1}\)-process \( X(t) \) is (strongly) integrable in \((S)_{-1}\) over the interval \([a, b]\) if

\[ \int_a^b X(t, \omega) dt := \lim_{\Delta t_k \to 0} \sum_{k=0}^{n-1} X(t_k^*, \omega) \Delta t_k \quad (2.8.9) \]

exists in \((S)_{-1}\), for all partitions \( a = t_0 < t_1 < \cdots < t_n = b \) of \([a, b]\), \( \Delta t_k = t_{k+1} - t_k \) and \( t_k^* \in [t_k, t_{k+1}] \) for \( k = 1, \ldots, n - 1 \).

Taking \( H \)-transforms and using Theorem 2.8.1, we get the following result:

**Lemma 2.8.5.** Let \( X(t) \) be an \((S)_{-1}\)-process. Suppose there exist \( q < \infty \), \( \delta > 0 \) such that

\[ \sup \{ \tilde{X}(t; z); t \in [a, b], z \in \mathbb{K}_q(\delta) \} < \infty \quad (2.8.10) \]

and

\[ \tilde{X}(t; z) \quad \text{is a continuous function} \quad \text{of} \quad t \in [a, b] \quad \text{for each} \quad z \in \mathbb{K}_q(\delta). \quad (2.8.11) \]

Then \( X(t) \) is strongly integrable and

\[ H \left[ \int_a^b X(t) dt \right] = \int_a^b \tilde{X}(t) dt. \quad (2.8.12) \]

**Example 2.8.6.** Choose \( N = m = d = 1 \) and let \( B(t, \omega) \) be Brownian motion. Then (see Example 2.2.5)

\[ B(t, \omega) = \sum_{j=1}^{\infty} \int_0^t \xi_j(s) ds H_{\epsilon(j)}(\omega), \]
and so
\[ \tilde{B}(t; z) = \sum_{j=1}^{\infty} \int_0^t \xi_j(s) ds z_j; \quad z \in (\mathbb{C}^N)_c. \]

Hence
\[ \frac{d\tilde{B}(t; z)}{dt} = \sum_{j=1}^{\infty} \xi_j(t) z_j, \quad \text{for each } z \in (\mathbb{C}^N)_c. \]

Moreover,
\[
\left| \sum_{j=1}^{\infty} \xi_j(t) z_j \right|^2 \leq \left( \sum_{j=1}^{\infty} \xi_j^2(t) (2N)^{-2\epsilon(j)} \right) \left( \sum_{j=1}^{\infty} |z^{(j)}|^2 (2N)^{2\epsilon(j)} \right) \\
\leq \sup_{j,t} |\xi_j^2(t)| \sum_{j=1}^{\infty} (2j)^{-2} \sum_{\alpha} |z^{\alpha}|^2 (2N)^{2\alpha} \leq CR^2
\]

for some constant $C$ if $z \in \mathbb{K}_2(R)$. We also have that
\[ t \to \sum_{j=1}^{\infty} \xi_j(t) z_j \quad \text{is continuous.} \]

Since $\sum_{j=1}^{\infty} \xi_j(t) z_j$ is the $\mathcal{H}$-transform of white noise $W(t, \omega)$ (see (2.6.8)), we conclude by Lemma 2.8.4 that
\[ \frac{dB(t, \omega)}{dt} = W(t, \omega) \quad \text{in } (\mathcal{S})_{-1}. \quad (2.8.13) \]

(Compare with (2.5.27).)

**Example 2.8.7.** Let us proceed one step further from the previous example and try to differentiate white noise $W(t, \omega)$. (Again we assume $m = d = 1$.) Since
\[ \tilde{W}(t; z) = \sum_{j=1}^{\infty} \xi_j(t) z_j; \quad z \in (\mathbb{C}^N)_c, \]
we get
\[ \frac{d\tilde{W}(t; z)}{dt} = \sum_{j=1}^{\infty} \xi'_j(t) z_j; \quad z \in (\mathbb{C}^N)_c. \]

Here the right hand side is clearly a continuous function of $t$ for each $z$. It remains to prove boundedness for $z \in \mathbb{K}_q(\delta)$ for some $q < \infty, \delta > 0$. From the definition (2.2.1) of the Hermite functions $\xi_j$ together with the estimate (2.2.5) we conclude that
\[ \sup_{t \in [a,b]} |\xi'_j(t)| \leq Cj, \quad (2.8.14) \]
where \( C = C_{a,b} \) is a constant depending only on \( a, b \). Hence
\[
\left| \sum_{j=1}^{\infty} \xi_j'(t)z_j \right|^2 \leq \left( \sum_{j=1}^{\infty} |\xi_j'(t)|^2(2N)^{-4\epsilon(j)} \right) \cdot \left( \sum_{j=1}^{\infty} |z_j|^2(2N)^{4\epsilon(j)} \right)
\]
\[
\leq C^2 \sum_{j=1}^{\infty} j^2(2j)^{-4} \cdot \sum_{\alpha} |z^{\alpha}|^2(2N)^{4\alpha} \leq C_1 R^2
\]
if \( z \in K_4(R); \ t \in [a, b] \). From Lemma 2.8.4 we conclude that
\[
\frac{dW(t, \omega)}{dt} = \sum_{j=1}^{\infty} \xi_j'(t)H_{\epsilon(j)}(\omega) \in (S)_{-1}.
\] (2.8.15)

2.9 The \( F \)-Transform and the Wick Product on \( L^1(\mu) \)

The \( S \)-transform is closely related to the Fourier transform or \( F \)-transform, which is defined on \( L^1(\mu) \) as follows:

**Definition 2.9.1.** Let \( g \in L^1(\mu_m), \phi \in S(\mathbb{R}^d) \). Then the \( F \)-transform, \( F[g](\phi) \), of \( g \) at \( \phi \), is defined by
\[
F[g](\phi) = \int_{S'(\mathbb{R}^d)} e^{i\langle \omega, \phi \rangle} g(\omega)d\mu(\omega).
\] (2.9.1)

Note that if \( g \in L^p(\mu_m) \) for some \( p > 1 \), then \( g \in (S)^* \) (Corollary 2.3.8) and hence, with \( i \) denoting the imaginary unit,
\[
(Sg)(i\phi) = \langle g, \exp^i[w(i\phi, \cdot)] \rangle = \int_{S'(\mathbb{R}^d)} \exp^i[w(i\phi, \omega)]g(\omega)d\mu(\omega)
\]
\[
= \int_{S'(\mathbb{R}^d)} \exp^i[i\langle \omega, \phi \rangle]g(\omega)d\mu(\omega)
\]
\[
= e^{\frac{1}{2}||\phi||^2} \int_{S'(\mathbb{R}^d)} \exp[i\langle \omega, \phi \rangle]g(\omega)d\mu(\omega)
\]
\[
= e^{\frac{1}{2}||\phi||^2} F[g](\phi).
\]

This gives

**Lemma 2.9.2.** a) Suppose \( g \in L^1(\mu_m) \cap (S)_{-\rho} \) for some \( \rho < 1 \). Then
\[
F[g](\phi) = e^{-\frac{1}{2}||\phi||^2}(Sg)(i\phi)
\] (2.9.2)
for all \( \phi \in S(\mathbb{R}^d) \).
b) Suppose \( h \in L^1(\mu_m) \cap (S)_{-1} \). Then for all \( \phi \in S(\mathbb{R}^d) \) we have
\[
\mathcal{F}[h](\lambda \phi) = e^{-\frac{1}{2} \lambda^2 \| \phi \|^2} (Sh)(i \lambda \phi) \text{ for } |\lambda| \text{ small enough.} \tag{2.9.3}
\]

**Proof**  
\( a) \) We have proved that (2.9.2) holds if \( g \in L^2(\mu_m) \). Since \( L^2(\mu) \) is dense in both \((S)_{-\rho}\) and \( L^1(\mu_m) \), the result follows.

\( b) \) Choose \( h_n \in L^1(\mu_m) \cap (S)_{-\rho} \) (for some fixed \( \rho < 1 \)) such that \( h_n \to h \) in \( L^1(\mu_m) \) and in \((S)_{-1}\). Then (2.9.2) holds for \( h_n \) for all \( n \). Taking the limit as \( n \to \infty \) we get (2.9.3). \( \square \)

This result gives the following connection between \( \mathcal{F} \)-transforms and Wick products.

**Lemma 2.9.3.**  
\( a) \) Suppose \( X,Y \) and \( X \circ Y \in L^1(\mu_m) \cap (S)_{-\rho} \) for some \( \rho < 1 \).
Then
\[
\mathcal{F}[X \circ Y](\lambda \phi) = e^{-\frac{1}{2} \lambda^2 \| \phi \|^2} \mathcal{F}[X](\phi) \cdot \mathcal{F}[Y](\phi); \quad \phi \in S(\mathbb{R}^d). \tag{2.9.4}
\]

\( b) \) Suppose \( X,Y \) and \( X \circ Y \) all belong to \( L^1(\mu_m) \cap (S)_{-1} \). Then for all \( \phi \in S(\mathbb{R}^d) \) we have
\[
\mathcal{F}[X \circ Y](\lambda \phi) = e^{-\frac{1}{2} \lambda^2 \| \phi \|^2} \mathcal{F}[X](\lambda \phi) \mathcal{F}[Y](\lambda \phi) \tag{2.9.5}
\]
for \( |\lambda| \) small enough.

**Proof**  
\( a) \) By Lemma 2.9.2 \( a) \) and Proposition 2.7.9 we have, for \( \phi \in S(\mathbb{R}^d) \),
\[
\mathcal{F}[X \circ Y](\phi) = e^{-\frac{1}{2} \| \phi \|^2} (SX \circ SY)(i \phi) = e^{-\frac{1}{2} \| \phi \|^2} (SX)(i \phi) \cdot (SY)(i \phi) = e^{-\frac{1}{2} \| \phi \|^2} e^{\frac{1}{2} \| \phi \|^2} \mathcal{F}[X](\phi) e^{\frac{1}{2} \| \phi \|^2} \mathcal{F}[Y](\phi) = e^{\frac{1}{2} \| \phi \|^2} \mathcal{F}[X](\phi) \mathcal{F}[Y](\phi).
\]

\( b) \) This follows from Lemma 2.9.2 \( b) \) in the same way. \( \square \)

Using the \( \mathcal{F} \)-transform we can now (partially) extend the Wick product to \( L^1(\mu_m) \) as follows:

**Definition 2.9.4 (The Wick product on \( L^1(\mu_m) \)).**  
Let \( X,Y \in L^1(\mu_m) \). Suppose there exist \( X_n,Y_n \in L^2(\mu_m) \) such that
\[
X_n \to X \text{ in } L^1(\mu_m) \text{ and } Y_n \to Y \text{ in } L^1(\mu_m) \text{ as } n \to \infty \tag{2.9.6}
\]
and such that
\[
\lim_{n \to \infty} X_n \circ Y_n \text{ exists in } L^1(\mu_m). \tag{2.9.7}
\]
Then we define the Wick product of $X$ and $Y$ in $L^1(\mu_m)$, denoted $X \hat{\diamond} Y$, by

$$X \hat{\diamond} Y = \lim_{n \to \infty} X_n \diamond Y_n.$$  
(2.9.8)

We must show that $X \hat{\diamond} Y$ is well defined, i.e., we must show that $\lim_{n \to \infty} X_n \diamond Y_n$ does not depend on the actual sequences $\{X_n\}, \{Y_n\}$. This is done in the following lemma.

**Lemma 2.9.5.** Let $X_n, Y_n$ be as in Definition 2.9.4 and assume that $X'_n, Y'_n$ also satisfy

$$X'_n \to X \text{ in } L^1(\mu_m) \text{ and } Y'_n \to Y \text{ in } L^1(\mu_m) \text{ as } n \to \infty$$  
(2.9.9)

and

$$\lim_{n \to \infty} X'_n \diamond Y'_n \text{ exists in } L^1(\mu_m).$$  
(2.9.10)

Then

$$\lim_{n \to \infty} X'_n \diamond Y'_n = \lim_{n \to \infty} X_n \diamond Y_n = X \hat{\diamond} Y.$$  
(2.9.11)

Moreover, we have

$$\mathcal{F}[X \hat{\diamond} Y](\phi) = e^{\frac{1}{2} \|\phi\|^2} \mathcal{F}[X](\phi) \mathcal{F}[Y](\phi)$$  
(2.9.12)

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$.

**Proof** Set $Z = \lim_{n \to \infty} X_n \diamond Y_n$. Then by Lemma 2.9.3 we have

$$\mathcal{F}[Z](\phi) = \lim_{n \to \infty} \mathcal{F}[X_n \diamond Y_n](\phi)$$

$$= \lim_{n \to \infty} e^{\frac{1}{2} \|\phi\|^2} \mathcal{F}[X_n](\phi) \mathcal{F}[Y_n](\phi)$$

$$= e^{\frac{1}{2} \|\phi\|^2} \mathcal{F}[X](\phi) \mathcal{F}[Y](\phi), \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

Similarly, we get, with $Z' = \lim_{n \to \infty} X'_n \diamond Y'_n$,

$$\mathcal{F}[Z'](\phi) = e^{\frac{1}{2} \|\phi\|^2} \mathcal{F}[X](\phi) \cdot \mathcal{F}[Y](\phi),$$

hence $\mathcal{F}[Z](\phi) = \mathcal{F}[Z'](\phi)$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$. Since the algebra $\mathcal{E}$ generated by the stochastic exponentials $\exp[i\langle \omega, \phi \rangle]$; $\phi \in \mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mu_m)$ (see Theorem 2.1.3), a function in $L^1(\mu_m)$ is uniquely determined by its Fourier transform. Therefore $Z = Z'$. This proves (2.9.11) and also (2.9.12). \[\Box\]

We can now verify that the two Wick products $\hat{\diamond}$ and $\diamond$ coincide on the intersection $L^1(\mu_m) \cap (\mathcal{S})_-$. 
Theorem 2.9.6. Let $X, Y \in L^1(\mu_m) \cap (S)_{-1}$. Assume that $X \hat{\diamond} Y$ exists in $L^1(\mu_m)$ and that $X \diamond Y$ (the Wick product in $(S)_{-1}$) belongs to $L^1(\mu_m)$. Then $X \hat{\diamond} Y = X \diamond Y$.

Proof. By Lemma 2.9.3 (ii) we have that
\[
\mathcal{F}[X \diamond Y](\lambda \phi) = e^{\frac{i}{2} \lambda^2 \|\phi\|^2} \mathcal{F}[X](\lambda \phi) \cdot \mathcal{F}[Y](\lambda \phi)
\]
for all $\phi \in S(\mathbb{R}^d)$ if $|\lambda|$ is small enough. On the other hand, from Lemma 2.9.5 we have that
\[
\mathcal{F}[X \hat{\diamond} Y](\psi) = e^{\frac{i}{2} \|\psi\|^2} \mathcal{F}[X](\psi) \cdot \mathcal{F}[Y](\psi)
\]
for all $\psi \in S(\mathbb{R}^d)$.

This is sufficient to conclude that $X \hat{\diamond} Y = X \diamond Y$. 

Remark. In view of Theorem 2.9.6 we can – and will – from now on write $X \diamond Y$ for the Wick product in $L^1(\mu_m)$.

Corollary 2.9.7. Let $X, Y \in L^1(\mu_m)$, and assume that $X \diamond Y \in L^1(\mu_m)$ exists (in the sense of Definition 2.9.4). Then
\[
E[X \diamond Y] = E[X] \cdot E[Y].
\]

Proof. Choose $\phi = 0$ in (2.9.12).

Functional Processes
As pointed out in the introduction, it is sometimes useful to smooth the singular white noise $W(x, \omega)$ by a test function $\phi \in S(\mathbb{R}^d)$, thereby obtaining the smoothed white noise process
\[
W_\phi(x, \omega) = w(\phi_x, \omega),
\]
where $\phi_x(y) = \phi(y - x)$; $x, y \in \mathbb{R}^d$ (see (2.1.19)).

The reason for doing this could be simply technical: By smoothing the white noise we get less singular equations to work with and therefore (we hope) less singular solutions.

But the reason could also come from the model: In some cases the smoothed process (2.9.14) simply gives a more realistic model for the noise we consider. In these cases the choice of $\phi$ may have a physical significance. For example, in the modeling of fluid flow in a porous, random medium the smoothed positive noise
\[
K_\phi(x, \omega) = \exp^\phi[W_\phi(x, \omega)]
\]
will be a natural model for the (stochastic) permeability of the medium, and then the size of the support of \( \phi \) will give the distance beyond which the permeability values at different points are independent. (See Chapter 4.)

In view of this, the following concept is useful:

**Definition 2.9.8 (Functional processes).** A functional process is a map

\[
X : \mathcal{S}(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow L^1(\mu_m).
\]

If there exists \( p \geq 1 \) such that

\[
X(\phi, x) \in L^p(\mu_m) \quad \text{for all} \quad \phi \in \mathcal{S}(\mathbb{R}^d), \quad x \in \mathbb{R}^d,
\]

then \( X \) is called an \( L^p \)-functional process.

**Example 2.9.9.** The processes \( W_\phi(x), K_\phi(x) \) given in (2.9.14) and (2.9.15) are both \( L^p \)-functional processes for all \( p < \infty \).

In Chapters 3 and 4 we will give examples of smoothed stochastic differential equations with solutions \( X(\phi, x) \) that are \( L^p \)-functional processes for \( p = 1 \) but not for any \( p > 1 \).

### 2.10 The Wick Product and Translation

There is a striking relation between Wick products, Wick exponentials of white noise and translation. This relation was first formulated on the Wiener space in Gjessing (1994), Theorem 2.10, and applied there to solve quasilinear anticipating stochastic differential equations. Subsequently the relation was generalized by Benth and Gjessing (2000), and applied to a class of nonlinear parabolic stochastic partial differential equations. The relation has also been applied to prove positivity of solutions of stochastic heat transport equations in Benth (1995).

In this section we will prove an \((\mathcal{S})^{-1}\)-version of this relation (Theorem 2.10.2). Then in Chapter 3 we present a variation of the SDE application in Gjessing (1994), and in Chapter 4 we will look at some of the above-mentioned applications to SPDEs.

We first consider the translation on functions in \((\mathcal{S})_1\).

**Theorem 2.10.1.** For \( f \in (\mathcal{S})_1 \) and \( \omega_0 \in \mathcal{S}'(\mathbb{R}^d) \), define the function \( T_{\omega_0}f : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathbb{R} \) by

\[
T_{\omega_0}f(\omega) = f(\omega + \omega_0); \omega \in \mathcal{S}'(\mathbb{R}^d).
\]

Then the map \( f \rightarrow T_{\omega_0}f \) is a continuous linear map from \((\mathcal{S})_1\) into \((\mathcal{S})_1\).
Proof Suppose \( f \in (S)_1 \) has the expansion
\[
f(\omega) = \sum_\beta c_\beta H_\beta(\omega) = \sum_\beta c_\beta \langle \omega, \eta \rangle^{\diamond \beta},
\]
where
\[
\langle \omega, \eta \rangle^{\diamond \beta} = \langle \omega, \eta_1 \rangle^{\diamond \beta_1} \circ \langle \omega, \eta_2 \rangle^{\diamond \beta_2} \circ \ldots
\]
(see (2.4.17)). Then
\[
f(\omega + \omega_0) = \sum_\beta c_\beta \langle \omega + \omega_0, \eta \rangle^{\diamond \beta} = \sum_\beta c_\beta (\langle \omega, \eta \rangle + \langle \omega_0, \eta \rangle)^{\diamond \beta}
\]
\[
= \sum_\beta c_\beta \prod_{j=1}^{\infty} (\langle \omega, \eta_j \rangle + \langle \omega_0, \eta_j \rangle)^{\diamond \beta_j}
\]
\[
= \sum_\beta c_\beta \prod_{j=1}^{\infty} \left( \sum_{\gamma_j=0}^{\beta_j} \binom{\beta_j}{\gamma_j} \langle \omega, \eta_j \rangle^{\diamond \gamma_j} \langle \omega_0, \eta_j \rangle^{(\beta_j - \gamma_j)} \right)
\]
\[
= \sum_\beta c_\beta \sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} \langle \omega, \eta \rangle^{\diamond \gamma} \langle \omega_0, \eta \rangle^{\beta - \gamma},
\]
where we have used the multi-index notation
\[
\binom{\beta}{\gamma} = \binom{\beta_1}{\gamma_1} \binom{\beta_2}{\gamma_2} \cdots = \frac{\beta_1!}{\gamma_1!(\beta_1 - \gamma_1)!} \cdot \frac{\beta_2!}{\gamma_2!(\beta_2 - \gamma_2)!} \cdots = \frac{\beta!}{\gamma!(\beta - \gamma)!}.
\]
Hence the expansion of \( f(\omega + \omega_0) \) is
\[
f(\omega + \omega_0) = \sum_\gamma \sum_{\beta \geq \gamma} c_\beta \binom{\beta}{\gamma} \langle \omega_0, \eta \rangle^{\beta - \gamma} \langle \omega, \eta \rangle^{\diamond \gamma}.
\]
Introduce
\[
b_\gamma = \sum_{\beta \geq \gamma} c_\beta \binom{\beta}{\gamma} \langle \omega_0, \gamma \rangle^{\beta - \gamma}.
\]
To show that \( f(\omega + \omega_0) \in (S)_1 \), we must verify that
\[
J(q) := \sum_{\gamma} b_\gamma^2 (\gamma!)^2 (2N)^{q\gamma} < \infty \quad \text{for all} \quad q \in \mathbb{N}. \quad (2.10.2)
\]
Choose \( q > 2 \). Since we have \( f \in (S)_1 \), we know that for all \( r \in \mathbb{N} \) there exists \( M(r) \in (0, \infty) \) such that

\[
c^2_{\beta} (\beta!)^2 (2N)^{2r\beta} \leq M^2(r) \quad \text{for all } \beta,
\]
i.e.,

\[
|c_{\beta}| \leq M(r)(\beta!)^{-1}(2N)^{-r\beta} \quad \text{for all } \beta.
\]

Therefore, with \( r \) to be determined later,

\[
J(q) \leq \sum_{\gamma} \left( \sum_{\beta \geq \gamma} M(r)(\beta!)^{-1}(2N)^{-r\beta} \langle \omega_0, \eta \rangle_{\beta-\gamma} \right)^2 (\gamma!)^2 (2N)^{q\gamma}
\leq M(r)^2 \sum_{\gamma} \left( \sum_{\beta \geq \gamma} \langle \omega_0, \eta \rangle_{\beta-\gamma}(2N)^{-r\beta} \right)^2 (2N)^{q\gamma}. \tag{2.10.3}
\]

By Theorem 2.3.1 we can write \( \omega_0 = \sum_{j=1}^{\infty} b_j \eta_j \), where

\[
\sum_{j=1}^{\infty} b_j^2 (\delta_j^{(1)})^{-\theta_1} (\delta_j^{(2)})^{-\theta_2} \cdots (\delta_j^{(d)})^{-\theta_d} < \infty
\]

for some \( \theta = (\theta_1, \ldots, \theta_d) \). Setting \( \theta_0 = \max\{\theta_j; 1 \leq j \leq d\} \), we get

\[
\sum_{j=1}^{\infty} b_j^2 (\delta_j^{(1)} \cdots \delta_j^{(d)})^{-\theta_0} < \infty.
\]

By Lemma 2.3.4 this implies that

\[
\sum_{j=1}^{\infty} b_j^2 j^{-\theta_0 d} < \infty.
\]

In particular, there exists \( K \in (1, \infty) \) such that

\[
|\langle \omega_0, \eta_j \rangle| = |b_j| \leq K \cdot j^{\theta_0 d}. \tag{2.10.4}
\]

Using this in (2.10.3), we get

\[
J(q) \leq M(r)^2 \sum_{\gamma} \left( \sum_{\beta \geq \gamma} (K\!\!N)^{\theta_0 d(\beta-\gamma)} (2N)^{-r\beta} \right)^2 (2N)^{q\gamma}.
\]

Now choose

\[
r = \theta_0 d(1 + \log_2 K) + q + 2. \tag{2.10.5}
\]
Then we get
\[ J(q) \leq M(r)^2 \sum_{\gamma} \left( \sum_{\beta \geq \gamma} K^{\theta_0 d|\beta| N\theta_0 d|\beta| 2^{-r|\beta| N-\gamma}} \right)^2 (2N)^q\gamma \]
\[ \leq M(r)^2 \sum_{\gamma} \left( \sum_{\beta \geq \gamma} N^{\theta_0 d|\beta| 2^{-(q+2)|\beta| N-\theta_0 d|\beta|-(q+2)\beta}} \right)^2 (2N)^q\gamma \]
\[ = M(r)^2 \sum_{\gamma} \left( \sum_{\beta \geq \gamma} (2N)^{-q\beta-2\beta} \right)^2 (2N)^q\gamma \]
\[ \leq M(r)^2 \sum_{\gamma} \left( \sum_{\beta \geq 0} (2N)^{-2\beta} \right) (2N)^{-q\gamma} \]
\[ = M(r)^2 \sum_{\beta \geq 0} (2N)^{-2\beta} \sum_{\gamma \geq 0} (2N)^{-q\gamma} < \infty. \]

This proves (2.10.2), and we conclude that \( T_{\omega_0} f \in (S)_1 \).

It is clear that the map \( f \to T_{\omega_0} f \) is linear. Finally, to prove that \( f \to T_{\omega_0} f \) is continuous from \((S)_1 \) into \((S)_1 \), note that the argument above actually shows that \( T_{\omega_0} \) maps \((S)_{1,r} \) into \((S)_{1,q} \) when \( r \) is given by (2.10.5). This proves the continuity, for if \( f_n \) is a sequence in \((S)_1 \) converging to 0 and
\[ N_{1,q,R} := \{ f \in (S)_{1,q}; \| f \|_{1,q} < R \} \]
is a neighborhood of 0 in \((S)_1 \), then \( T_{\omega_0} f_n \in N_{1,q,R} \) if \( n \) is so large that we have \( f_n \in (S)_{1,r} \).

\( \square \)

**Remark** It is proved in Hida et al. (1993), Theorem 4.15, that \( T_{\omega_0} \) is a continuous linear map from \((S) (= (S)_0) \) into \((S) \). In Potthoff and Timpel (1995), the same is proved for the translation operator on the space \((G) \).

**Definition 2.10.2.** Fix \( \omega_0 \in S'(R^d) \).

a) The map \( T_{\omega_0} : (S)_1 \to (S)_1 \) is called the **translation operator**.

b) The **adjoint translation operator** is the map
\[ T_{\omega_0}^*: (S)_{-1} \to (S)_{-1} \]
defined by
\[ \langle T_{\omega_0}^* X, f \rangle = \langle X, T_{\omega_0} f \rangle; \, f \in (S)_1, X \in (S)_{-1}. \] (2.10.6)
Remark  Note that $T_{\omega_0}^*$ maps $(S)^{-1}$ into $(S)^{-1}$ because of Theorem 2.10.1.

The following result is the $(S)^{-1}$-version of Lemma 5.3 in Benth and Gjessing (2000) (see also Prop. 9.4 in Benth (1995)).

**Theorem 2.10.3 Benth and Gjessing (2000).** Let $\omega_0 \in S'(R^d)$ and $X \in (S)^{-1}$. Then

$$T_{\omega_0}^* X = X \circ \exp^\circ [w(\omega_0)],$$

(2.10.7)

where

$$w(\omega_0, \omega) := \sum_{j=1}^{\infty} \langle \omega_0, \eta_j \rangle H_{\epsilon(j)}(\omega) \in (S)^*. \tag{2.10.8}$$

is the generalized smoothed white noise.

**Proof** First note that from (2.10.4) it follows that $w(\omega_0, \cdot) \in (S)^*$. We verify (2.10.7) by proving that $S$-transforms of the two sides are equal: For $\phi \in S(R^d)$ and $|\lambda|$ small enough we have (see (2.7.11))

$$(ST_{\omega_0}^* X)(\lambda \phi) = \langle T_{\omega_0}^* X, \exp^\circ [w(\lambda \phi, \cdot)] \rangle$$

$$= \langle X, T_{\omega_0} (\exp^\circ [w(\lambda \phi, \cdot)]) \rangle$$

$$= \langle X, \exp^\circ [\langle \omega + \omega_0, \lambda \phi \rangle] \rangle$$

$$= \langle X, \exp^\circ [\langle \omega, \lambda \phi \rangle] \rangle \cdot \exp [\langle \omega_0, \lambda \phi \rangle]$$

$$= (SX)(\lambda \phi) \cdot (Sw_{\omega_0})(\lambda \phi)$$

$$= S(X \circ w_{\omega_0})(\lambda \phi).$$

By Theorem 2.7.10 and the uniqueness of the Hermite transform on $(S)^{-1}$, the theorem is proved. □

**Corollary 2.10.4 Benth and Gjessing (2000).** a) If $X \in (S)^{-1}$ and $\omega_0 \in S'(R^d)$, then

$$\langle \exp^\circ [w(\omega_0)] \circ X, f \rangle = \langle X, T_{\omega_0} f \rangle; f \in (S)_1. \tag{2.10.9}$$

b) If $X \in (S)_1$, $f \in (S)_1$ and $\omega_0 = \phi \in L^2(R^d)$, then

$$\int_{S'} f(\omega) \cdot (\exp^\circ [w(\phi)] \circ X)(\omega) d\mu(\omega) = \int_{S'} X(\omega) f(\omega + \phi) d\mu(\omega). \tag{2.10.10}$$

**Proof** a) follows directly from (2.10.7). Version b) is an $(S)_1$-version of a). □

In particular, as observed in Benth and Gjessing (2000), if we choose $X \equiv 1$, we recover a version of the Girsanov formula.
Corollary 2.10.5. Let \( f \in L^p(\mu_1) \) for some \( p > 2 \) and let \( \phi \in L^2(\mathbb{R}^d) \). Then \( f(\omega + \phi) \in L^2(\mu_1) \) and

\[
\int_{\mathcal{S}'} f(\omega) \cdot \exp^\phi[w(\phi)](\omega) d\mu_1(\omega) = \int_{\mathcal{S}'} f(\omega + \phi) d\mu_1(\omega).
\]

(2.10.11)

Proof Fix \( p > 2 \) and \( \phi \in L^2(\mathbb{R}^d) \). Choose \( f_n \in (S)_1 \) such that \( f_n \to f \) in \( L^p(\mu_1) \). Then by (2.10.10) we get that (2.10.11) holds for each \( f_n \) and with \( \phi = 1/2 \phi \), i.e.

\[
\int_{\mathcal{S}'} f_n(\omega) \cdot \exp^\phi\left[w\left(\frac{1}{2} \phi\right)\right](\omega) d\mu_1(\omega) = \int_{\mathcal{S}'} f_n(\omega + \frac{1}{2} \phi) d\mu_1(\omega); \quad n = 1, 2, \ldots
\]

Since \( \exp^\phi[w(1/2\phi)] = \exp[w(1/2\phi)] - 1/8||\phi||^2_{L^2(\mathbb{R}^d)} \) is in \( L^q(\mu_1) \) for all \( q < \infty \), we have by the Hölder inequality that \( f_n \cdot \exp^\phi[w(1/2\phi)] \to f \cdot \exp^\phi[w(1/2\phi)] \) in \( L^2(\mu_1) \). Therefore

\[
\int_{\mathcal{S}'} (f_n - f_m)^2 \left(\exp^\phi\left[w\left(\frac{1}{2} \phi\right)\right]\right)^2 d\mu_1(\omega) \to 0 \quad \text{as } m, n \to \infty
\]

This is equivalent to

\[
\int_{\mathcal{S}'} (f_n - f_m)^2 \exp^\phi[w(\phi)] d\mu_1(\omega) \to 0 \quad \text{as } m, n \to \infty
\]

By (2.10.10) this implies that

\[
\int_{\mathcal{S}'} (f_n(\omega + \varphi) - f_m(\omega + \varphi))^2 d\mu_1(\omega) \to 0 \quad \text{as } m, n \to \infty
\]

and hence \( \{f_n(\cdot + \varphi)\}_{n=1}^\infty \) is convergent in \( L^2(\mu_1) \). Since a subsequence of \( \{f_n\} \) converges to \( f \) a.e., we conclude that the \( L^2(\mu_1) \) limit of \( f_n(\cdot + \varphi) \) must be \( f(\cdot + \varphi) \).

The following useful result first appeared in Gjessing (1994), Theorem 2.10, in the Wiener space setting, and subsequently in Benth and Gjessing (2000), Lemma 5.6, in a white noise setting (for the spaces \( \mathcal{G}, \mathcal{G}^* \)). We will here present the \( L^2(\mu_1) \)-version of their result.

Theorem 2.10.6 (Gjessing’s Lemma). Let \( \phi \in L^2(\mathbb{R}^d) \) and \( X \in L^p(\mu_1) \) for some \( p > 1 \). Then \( X \diamond \exp^\phi[w(\phi)] \in L^p(\mu_1) \) for all \( p < p \), and almost surely we have

\[
(X \diamond \exp^\phi[w(\phi)])(\omega) = T_{-\phi}X(\omega) \cdot \exp^\phi[w(\phi)](\omega).
\]

(2.10.13)
Proof First assume $X \in (\mathcal{S})_1$ and choose $f \in (\mathcal{S})_1$. Then, by (2.10.10) and (2.10.11),
\[
\int_{\mathcal{S}'} f(\omega) \cdot (X \circ \exp^\phi[w(\phi)])(\omega) d\mu_1(\omega)
= \int_{\mathcal{S}'} X(\omega) f(\omega + \phi) d\mu_1(\omega)
= \int_{\mathcal{S}'} X(\omega - \phi) f(\omega) \exp^\phi[w(\phi)](\omega) d\mu_1(\omega).
\] (2.10.14)

By Corollary 2.10.5 we know that $X(\cdot - \phi) \in L^\rho(\mu_1)$ for all $\rho < p$ and hence the same is true for $X(\cdot - \phi) \cdot \exp^\phi[w(\phi)]$. Since (2.10.14) holds for all $f \in (\mathcal{S})_1$ and $(\mathcal{S})_1$ is dense in $L^q(\mu_1)$ for all $q < \infty$, we conclude that
\[
X \circ \exp^\phi[w(\phi)] = X(\omega - \phi) \cdot \exp^\phi[w(\phi)],
\]
as claimed. \(\square\)

2.11 Positivity

In many applications the noise that occurs is not white. The following example illustrates this.

If we consider fluid flow in a porous rock, we often lack exact information about the permeability of the rock at each point. The lack of information makes it natural to model the permeability as a (multiparameter) noise (see Chapter 1). This noise will, of course, not be white, but positive, since permeability is always a non-negative quantity. In this section we will discuss the positivity in the case of distributions and also in the case of functional processes. Let $(\mathcal{S})_1$ and $(\mathcal{S})_{-1}$ be the spaces defined in Section 2.3.

Definition 2.11.1. An element $\Phi \in (\mathcal{S})_{-1}$ is called positive if for any positive $\phi \in (\mathcal{S})_1$ we have $\langle \Phi, \phi \rangle \geq 0$. The collection of positive elements in $(\mathcal{S})_{-1}$ is denoted by $(\mathcal{S})^+_1$.

Before we state an important characterization of positive distributions, we must provide some preparatory results. For simplicity we assume $d = 1$. Let $A$ be an operator on $L^2(\mathbb{R})$ given by
\[
A = -\left(\frac{d}{dx}\right)^2 + (x^2 + 1).
\] (2.11.1)

Then the Hermite function $\xi_n, n \geq 1$ is an eigenfunction of $A$ with eigenvalue $2n$. Let $\mathcal{S}_p(\mathbb{R})$ be the completion of $\mathcal{S}(\mathbb{R})$ under the norm \[ \cdot \|_2, p := \|A^p \cdot \|_{L^2(\mathbb{R})}. \]
Denote by $S_{-p}(\mathbb{R})$ the dual space of $S_p(\mathbb{R})$, with the norm $|\cdot|_{2,-p}$. It is well known that $S(\mathbb{R})$ is the projective limit of $S_p(\mathbb{R}), p > 0$ and $S'(\mathbb{R})$ is the union of $S_{-p}(\mathbb{R}), p > 0$, with inductive topology.

**Lemma 2.11.2** Kondratiev et al. (1995a), Corollary 1. Let $p > 0$ be a constant such that the embedding $S_p(\mathbb{R}) \to L^2(\mathbb{R})$ is Hilbert–Schmidt. Assume that $\phi \in (S)_1$. Then for any $\varepsilon > 0$, there exists a constant $C_{\varepsilon,p}$ such that
\[
|\phi(x)| \leq C_{\varepsilon,p}\|\phi\|_{1,p}\varepsilon^{\varepsilon|x|_{2,-p}}; \ x \in S_{-p}(\mathbb{R}).
\] (2.11.2)

**Proof** See Corollary 1 in Kondratiev et al. (1995a).

**Theorem 2.11.3** Kondratiev et al. (1995a), Theorem 2. Let $\Phi \in (S)_{-1}$. Then there exists a unique positive measure $\nu$ on $(S'(\mathbb{R}), B(S'(\mathbb{R})))$ such that for all $\phi \in (S)_1$,
\[
\langle \Phi, \phi \rangle = \int_{S'(\mathbb{R})} \phi(x)\nu(dx).
\] (2.11.3)

**Proof** We will construct the measure $\nu$ by estimating the moments of the distributions. Since the polynomials $P \subset (S)_1$, we can define the moments of the distribution $\Phi$ as
\[
M_n(\zeta_1, \zeta_2, \ldots, \zeta_n) = \langle \Phi, \prod_{j=1}^{n} \langle \cdot, \zeta_j \rangle \rangle; \ n \in \mathbb{N}, 1 \leq j \leq n, \ \zeta_j \in S(\mathbb{R})
\] (2.11.4)
\[
M_0 = \langle \Phi, 1 \rangle.
\]

First assume $\zeta_1 = \zeta_2 = \cdots = \zeta_n = \zeta \in S(\mathbb{R})$. Since $\Phi \in (S)_{-p}$ for some $p > 0$, we have
\[
|\langle \Phi, \langle \cdot, \zeta \rangle^n \rangle| \leq \|\Phi\|_{-1,-p}\|\langle \cdot, \zeta \rangle^n \|_{1,p}.
\] (2.11.5)

To obtain a bound of $\|\langle \cdot, \zeta \rangle^n \|_{1,p}$, we use the well-known Hermite decomposition (see Appendix C)
\[
\langle \cdot, \zeta \rangle^n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n!}{k!(n-2k)!} \left( -\frac{1}{2}\|\zeta\|^2_{L^2(\mathbb{R})} \right)^k \int_{\mathbb{R}^{n-2k}} \zeta^{\otimes n-2k} dB^{\otimes n-2k}. \] (2.11.6)

But for any integer $n \geq 1$,
\[
\int_{\mathbb{R}^n} \zeta^{\otimes n} dB^{\otimes n} = \sum_{|\alpha|=n} \langle \zeta^{\otimes n}, \zeta^{\otimes \alpha} \rangle \int_{\mathbb{R}^n} \xi^{\otimes |\alpha|} dB^{\otimes |\alpha|} = \sum_{|\alpha|=n} \langle \zeta^{\otimes n}, \zeta^{\otimes \alpha} \rangle H_\alpha(\omega).
\] (2.11.7)
Hence,

\[
\left\| \int_{\mathbb{R}^n} \zeta \otimes^n dB \otimes^n \right\|_{1,p}^2 = \sum_{|\alpha|=n} \left\langle \zeta \otimes^n, \xi \otimes^\alpha \right\rangle^2 (2N)^p \alpha \\
\leq \left( \sum_{|\alpha|=n} \left\langle \zeta \otimes^n, \xi \otimes^\alpha \right\rangle^2 (2N)^p \right) (n!)^2 \\
= (n!)^2 \left( \sum_{|\alpha|=n} \left\langle A \xi \otimes^n, \xi \otimes^\alpha \right\rangle^2 \right) = (n!)^2 |\zeta|_{2,n}^{2n}. \tag{2.11.8}
\]

Observe that \( \int_{\mathbb{R}^n} \zeta \otimes^n dB \otimes^n, n \geq 1, \) are orthogonal in \( (S)^{-p} \). Thus we obtain from (2.11.6) and (2.11.8) that

\[
\left\| \left\langle \cdot, \zeta \right\rangle \right\|_{1,p}^2 = \sum_{k=0}^{\left[ \frac{n}{2} \right]} \left( \frac{n!}{k!(n-2k)!(2k)!} \right)^2 \|\zeta\|_{L^2(\mathbb{R})}^4 \left\| \int_{\mathbb{R}^n} \zeta \otimes n-2k dB \otimes n-2k \right\|_{1,p}^2 \\
\leq (n!)^2 |\zeta|_{2,n}^{2n} \sum_{k=0}^{\left[ \frac{n}{2} \right]} \frac{2^{2n}}{(k!)^2} \cdot 2^{-2k} \leq (n!)^2 4^n |\zeta|_{2,n}^{2n} C, \tag{2.11.9}
\]

where \( C = \sum_{k=1}^{\infty} 2^{-2k}/(k!)^2 \). By the polarization formula, this implies

\[
\left\| \prod_{j=1}^{n} \left\langle \cdot, \zeta_j \right\rangle \right\|_{1,p} \leq \sqrt{C} 2^n (n!) \prod_{j=1}^{n} |\zeta_j|_{2,n}, \tag{2.11.10}
\]

Hence we obtain from (2.11.4) that

\[
|M_n(\zeta_1, \ldots, \zeta_n)| \leq \|\Phi\|_{-1,-p} \sqrt{C} 2^n (n!) \prod_{j=1}^{n} |\zeta_j|_{2,n}. \tag{2.11.11}
\]

Due to the kernel theorem – see Gelfand and Vilenkin (1964) – the following decomposition holds:

\[
M_n(\zeta_1, \ldots, \zeta_n) = \langle M^{(n)}(\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n) \rangle, \tag{2.11.12}
\]

where \( M^{(n)} \in S'(\mathbb{R}) \hat{\otimes}^n \). The sequence \( \{M^{(n)}(\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n) \} \) has the following property of positivity: For any finite sequence of smooth kernels \( \{f^{(n)}(\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n) \}, \) i.e., that \( f^{(n)}(\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n) = 0, n \geq n_0 \)

\[
\sum_{k,j}^{n_0} \langle M^{(k+j)}(\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n), f^{(k)}(\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n) \rangle = \langle \Phi, |\phi| \rangle \geq 0, \tag{2.11.13}
\]

where \( \phi(x) = \sum_{n=0}^{n_0} \langle x \otimes f^{(n)}(\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n), x \in S'(\mathbb{R}) \rangle. \)
By the result in Berezansky and Kondratiev (1988), (2.11.11) and (2.11.13) are sufficient to ensure the existence of a uniquely defined measure $\nu$ on the probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})))$, such that for any $\phi \in \mathcal{P}$

$$
\langle \Phi, \phi \rangle = \int_{\mathcal{S}'(\mathbb{R})} \phi(\omega) \nu(d\omega).
$$

(2.11.14)

By Corollary 2.4 in Zhang (1992), it is known that any element $\phi \in \mathcal{S}'(\mathbb{R})$ is defined pointwise and continuous. Thus to show (2.11.14) also holds for any $\phi \in \mathcal{S}'(\mathbb{R})$, by Lemma 2.11.2 it suffices to prove that there exists $p' > 0$ and $\varepsilon > 0$ such that $\exp[\varepsilon |x|_{2,-p'}]$ is integrable with respect to $\nu$. Choose $p' > p/2$ such that the embedding $\iota_{p'} : \mathcal{S}_{p'}(\mathbb{R}) \to \mathcal{S}_{p/2}(\mathbb{R})$ is of the Hilbert–Schmidt type. Then we let $\{e_k, k \in \mathbb{N}\} \subset \mathcal{S}(\mathbb{R})$ be an orthonormal basis in $\mathcal{S}_{p'}(\mathbb{R})$. This gives

$$
|x|_{2,-p'}^2 = \sum_{k=1}^{\infty} \langle x, e_k \rangle^2, \quad x \in \mathcal{S}_{-p'}(\mathbb{R})
$$

(2.11.15)

and

$$
\int_{\mathcal{S}'(\mathbb{R})} \omega|_{2,-p'}^{2n} \nu(d\omega) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \sum_{\omega \in \mathcal{S}'(\mathbb{R})} \langle \omega, e_{k_1} \rangle^2 \cdots \langle \omega, e_{k_n} \rangle^2 \nu(d\omega).
$$

Using the bound (2.11.11), we have

$$
\int_{\mathcal{S}'(\mathbb{R})} \omega|_{2,-p'}^{2n} \nu(d\omega) \leq \frac{\|\Phi\|_{-1,-p} \sqrt{C} 2^{2n} (2n)!}{\|i\|^2_{-1,-p} \nu(\mathcal{S}'(\mathbb{R}))^{1/2}} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} |e_{k_1}|_{2,-p'}^2 \cdots |e_{k_n}|_{2,-p'}^2
$$

$$
= \frac{\|\Phi\|_{-1,-p} \sqrt{C} 2^{2n} (2n)!}{\|i\|^2_{-1,-p} \nu(\mathcal{S}'(\mathbb{R}))^{1/2}} \sum_{k=1}^{\infty} |e_k|_{2,-p'}^2
$$

$$
\leq \frac{\|\Phi\|_{-1,-p} \sqrt{C} 2^{2n} (2n)!}{\|i\|^2_{-1,-p} \nu(\mathcal{S}'(\mathbb{R}))^{1/2}} \sum_{k=1}^{\infty} |e_k|_{2,-p'}^2
$$

(2.11.16)

$$
\sum_{k=1}^{\infty} |e_k|_{2,-p'}^2 = \|i\|^2_{-1,-p} \nu(\mathcal{S}'(\mathbb{R}))^{1/2}.
$$

For an arbitrary integer $n \geq 1$,

$$
\int_{\mathcal{S}'(\mathbb{R})} \omega|_{2,-p'}^{2n} \nu(d\omega) \leq \frac{\|\Phi\|_{-1,-p} \sqrt{C} 2^{2n} (2n)!}{\|i\|^2_{-1,-p} \nu(\mathcal{S}'(\mathbb{R}))^{1/2}} \sum_{k=1}^{\infty} |e_k|_{2,-p'}^2
$$

$$
\leq \sqrt{\frac{\|\Phi\|_{-1,-p} C^{1/2} 2^{2n} \cdot 2^n n! (\|i\|^2_{-1,-p})^n M_0^{1/2}}{\nu(\mathcal{S}'(\mathbb{R}))^{1/2}}},
$$

(2.11.16)

where we have used that $(2n)! \leq 4^n n!^2$ and $\nu(\mathcal{S}'(\mathbb{R})) = M_0 < +\infty$. Choose $\varepsilon < 1/4\|i\|^2_{-1,-p}$.
\[
\int_{S'(\mathbb{R})} \exp[\varepsilon |\omega|_{2,-p'}] \nu(d\omega) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \int_{S'(\mathbb{R})} |\omega|_{2,-p'}^n \nu(d\omega) \\
\leq M_0^{\frac{1}{2}} C_{\frac{1}{2}} \sqrt{\|\Phi\|_{-1,-p}} \sum_{n=0}^{\infty} (\varepsilon 4\|j_{p'}\|_{HS})^n < +\infty.
\]

(2.11.17)

Let \( X = \sum_{\alpha} c_{\alpha} H_{\alpha} \in (S)^* \) (the Hida distribution space defined in Section 2.3). As we know, the Hermite transform of \( X \) is given by

\[
\mathcal{H}X(z) = \tilde{X}(z) = \sum_{\alpha} c_{\alpha} z^{\alpha}, \; z = (z_1, z_2, \ldots, z_n, \ldots) \in \mathbb{C}^n.
\]

(2.11.18)

By Lindstrøm et al. (1991), Lemma 5.3, and Zhang (1992), \( \tilde{X}(z) \) converges absolutely for \( z = (z_1, \ldots, z_n, 0, 0, \ldots) \) for each integer \( n \). Therefore, the function \( \tilde{X}^{(n)}(z_1, \ldots, z_n) := \tilde{X}(z_1, z_2, \ldots, z_n, 0 \ldots 0) \) is analytic on \( \mathbb{C}^n \) for each \( n \). Following Definition 2.11.1, we can define the positivity in \( (S)^* \). The following characterization is sometimes useful.

**Theorem 2.11.4 Lindstrøm et al. (1991a).** Let \( X \in (S)^* \). Then \( X \) is positive if and only if

\[
g_n(y) := \tilde{X}^{(n)}(iy)e^{-\frac{1}{2}|y|^2}; \; y \in \mathbb{R}^n
\]

is positive definite for all \( n \).

Before giving the proof, let us recall the definition of positive definiteness. A function \( g(y), y \in \mathbb{R}^n \), is called positive definite if for all positive integers \( m \) and all \( y^{(1)}, \ldots, y^{(m)} \in \mathbb{R}^n, a = (a_1, \ldots, a_m) \in \mathbb{C}^m, \)

\[
\sum_{j,k}^{m} a_j \bar{a}_k g(y^{(j)} - y^{(k)}) \geq 0.
\]

(2.11.20)

**Proof** Let \( d\lambda(x) \) be the standard Gaussian measure on \( \mathbb{R}^\infty \), i.e., the direct product of infinitely many copies of the normalized Gaussian measure on \( \mathbb{R} \). Set \( F(z) = \tilde{X}^{(n)}(z) \), for \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \). Define

\[
J_n(x) = J_n(x_1, \ldots, x_n) = \int \tilde{X}^{(n)}(x + iy)d\lambda(y) = \int F(x + iy)e^{-\frac{1}{2}|y|^2}(2\pi)^{-\frac{n}{2}}dy,
\]

where \( y = (y_1, \ldots, y_n), dy = dy_1 \ldots dy_n \).

(2.11.21)
We write this as
\[ J_n(x) = e^{-\frac{1}{2}|x|^2} \int F(z) e^{\frac{i}{2}z^2} \cdot e^{-i(x,y)(2\pi)^{-\frac{n}{2}}} dy \]
\[ = e^{-\frac{1}{2}|x|^2}(2\pi)^{-\frac{n}{2}} \int G(z) e^{-i(x,y)} dy, \quad (2.11.22) \]
where \( z = (z_1, \ldots, z_n), z_k = x_k + iy_k, z^2 = z_1^2 + \cdots + z_n^2, (x,y) = \sum_{k=1}^n x_k y_k, \) and \( G(z) := F(z)e^{\frac{i}{2}z^2} \) is analytic. Consider the function
\[ f(x, \eta) = \int G(x + iy)e^{-i(\eta,y)} dy, x, \eta \in \mathbb{R}^n. \quad (2.11.23) \]
Using the Cauchy–Riemann equations, we have
\[ \frac{\partial f}{\partial x_1} = \int \frac{\partial G}{\partial x_1} \cdot e^{-i(\eta,y)} dy = \int (-i) \frac{\partial G}{\partial y_1} e^{-i(\eta,y)} dy. \]
But
\[ \int_{-\infty}^{+\infty} (-i) \frac{\partial G}{\partial y_1} e^{-i\eta_1 y_1} dy_1 = i \int_{-\infty}^{+\infty} G(z) e^{-i\eta_1 y_1} (-i\eta_1) dy_1. \]
This gives
\[ \frac{\partial f(x, \eta)}{\partial x_1} = \eta_1 f(x, \eta). \quad (2.11.24) \]
Hence we have \( f(x_1, x_2, \ldots, x_n; \eta) = f(0, x_2, \ldots, x_n; \eta)e^{\eta_1 x_1}, \) and so on for \( x_2, \ldots, x_n. \) Therefore,
\[ f(x, \eta) = f(0, \eta)e^{\eta x} = e^{(\eta,x)} \int G(iy)e^{-i(\eta,y)} dy. \quad (2.11.25) \]
We conclude from (2.11.21)–(2.11.25) that
\[ J_n(x) = e^{\frac{1}{2}|x|^2}(2\pi)^{-\frac{n}{2}} \int \tilde{X}^{(n)}(iy)e^{-\frac{1}{2}|y|^2} e^{-i(x,y)} dy \]
\[ = e^{\frac{1}{2}|x|^2} \hat{g}_n(x), \quad (2.11.26) \]
where \( \hat{g}_n(x) = (2\pi)^{-n/2} \int g_n(y)e^{-i(x,y)} dy \) is the Fourier transform of \( g_n. \)
Note that \( g_n \in S(\mathbb{R}^n), \) and hence \( \hat{g}_n \in S(\mathbb{R}^n). \) Therefore, we can apply the Fourier inversion to obtain
\[ g_n(y) = (2\pi)^{-\frac{n}{2}} \int \hat{g}_n(-x)e^{i(x,y)} dx = (2\pi)^{-\frac{n}{2}} \int J_n(-x)e^{\frac{1}{2}|x|^2} e^{i(x,y)} dx \]
\[ = \int J_n(-x)e^{i(x,y)} d\lambda(x) = \int J_n(x)e^{-i(x,y)} d\lambda(x), \quad (2.11.27) \]
so if \( y^{(1)}, \ldots, y^{(m)} \in \mathbb{R}^n \) and \( a = (a_1, \ldots, a_m) \in \mathbb{C}^m \), then
\[
\sum_{j,k}^m a_j \bar{a}_k g_n(y^{(j)} - y^{(k)}) = \int |\gamma(x)|^2 J_n(x) d\lambda(x), \quad (2.11.28)
\]
where \( \gamma(x) = \sum_{j=1}^m a_j e^{-ixy^{(j)}} \). Since
\[
\int \eta dB = \left( \int \eta_1(t) dB(t), \int \eta_2(t) dB(t), \ldots, \int \eta_n(t) dB(t), \ldots \right)
\]
has distribution \( d\lambda(x) \), we can rewrite (2.11.28) as
\[
\sum_{j,k}^m a_j \bar{a}_k g_n(y^{(j)} - y^{(k)}) = E \left[ |\gamma(\int \eta dB)|^2 J_n\left( \int \eta dB \right) \right], \quad (2.11.29)
\]
since \((S)\) is an algebra (see, e.g., Zhang (1992)), we have that \( |\gamma(\int \eta dB)|^2 \in (S) \). Since \( J_n(\int \eta dB) \to X \) in \((S)^*\),
\[
\sum_{j,k}^m a_j \bar{a}_k g_n(y^{(j)} - y^{(k)}) \to <X, |\gamma(\int \eta dB)|^2 >, \quad \text{as} \quad n \to +\infty. \quad (2.11.30)
\]
So if \( X \) is positive for almost all \( \omega \), we deduce that
\[
\lim_{n \to +\infty} \sum_{j,k}^m a_j \bar{a}_k g_n(y^{(j)} - y^{(k)}) \geq 0. \quad (2.11.31)
\]
But with \( y^{(1)}, \ldots, y^{(m)} \in \mathbb{R}^n \) fixed, \( g_n(y^{(j)} - y^{(k)}) \) eventually becomes constant as \( n \to +\infty \), so (2.11.31) implies that \( g_n(y) \) is positive definite.
Conversely, if \( g_n \) is positive definite, then, by (2.11.27), \( J_n(x) \geq 0 \) for almost all \( x \) with respect to \( d\lambda \), and if this is true for all \( n \geq 1 \), we have that
\[
X(\omega) = \lim_n J_n\left( \int \eta dB \right) \text{ is positive.} \quad \square
\]

**Remark** Let \( X \in L^2(\mu_1) \subset (S)^* \). Then \( X \) is positive in \((S)^*\) if and only if \( X \) is a non-negative random variable.

**Definition 2.11.5.** A functional process \( X(\phi, x, \omega) \) is called **positive** or a **positive noise** if
\[
X(\phi, x, \omega) \geq 0 \quad \text{for almost all} \ \omega \quad (2.11.32)
\]
for all \( \phi \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d \).

**Example 2.11.6.** Let \( w(\phi) \) be the smoothed white noise defined as in Section 2.6. Then the Wick exponential \( \exp^\diamond[w(\phi, \omega)] \) is a positive noise.
This follows from the identity (see 2.6.55)

\[
\exp^{\diamond} [w(\phi, \omega)] = \exp \left[ w(\phi, \omega) - \frac{1}{2} \|\phi\|^2 \right].
\]

**Corollary 2.11.7.** Let \( X = X(\phi, \omega) \) and \( Y = Y(\phi, \omega) \) be positive \( L^2 \)-functional processes of the following form:

\[
X(\phi, \omega) = \sum_{\alpha} a_{\alpha} (\phi \otimes |\alpha|) H_{\alpha}(\omega), \quad Y(\phi, \omega) = \sum_{\alpha} b_{\alpha} (\phi \otimes |\alpha|) H_{\alpha}(\omega),
\]

where \( a_{\alpha}, b_{\alpha} \in H^{-s}(\mathbb{R}^n) \) for some \( s \). If \( X \diamond Y \) is well defined, then \( X \diamond Y \) is also positive.

**Proof** For \( \phi \in S(\mathbb{R}^d) \), consider \( \tilde{X}^{(n)}(\phi, iy)e^{-1/2|y|^2} \) as before and, similarly, \( \tilde{Y}^{(n)}(\phi, iy)e^{-1/2|y|^2} \). Replacing \( \phi \) by \( \rho \phi \) where \( \rho > 0 \), Theorem 2.11.3 yields that

\[
g_n^{(\rho)}(y) = \tilde{X}^{(n)}(\phi, iy)e^{-\frac{1}{2}|y|^2}
\]

is positive definite, hence

\[
\sigma_n(y) := \tilde{X}^{(n)}(\phi, iy)e^{-\frac{1}{2}y_1^2} \quad \text{is positive definite},
\]

and, similarly,

\[
\gamma_n(y) = \tilde{Y}^{(n)}(\phi, iy)e^{-\frac{1}{2}y_1^2} \quad \text{is positive definite}.
\]

Therefore the product \( \sigma_n \gamma_n(y) = (\tilde{X}^{(n)}(\phi) \tilde{Y}^{(n)}(\phi))(iy)e^{-1/2|y|^2} \) is positive definite. If we choose \( \rho = \sqrt{2} \), this gives that

\[
\mathcal{H}(X \diamond Y)^{(n)}(\phi, iy)e^{-\frac{1}{2}|y|^2} \quad \text{is positive definite}.
\]

So from Theorem 2.11.4, we have \( X \diamond Y \geq 0 \). \( \square \)

**Exercises**

2.1 To obtain a formula for \( E[\langle \cdot, \phi \rangle^n] \), replace \( \phi \) by \( \alpha \phi \) with \( \alpha \in \mathbb{R} \) in equation (2.1.3), and compute the \( n \)th derivative with respect to \( \alpha \) at \( \alpha = 0 \). Then use polarization to show that we have \( E[\langle \cdot, \phi \rangle \langle \cdot, \psi \rangle] = (\phi, \psi) \) for functions \( \phi, \psi \in S(\mathbb{R}^d) \).

2.2 Extend Lemma 2.1.2 to functions that are not necessarily orthogonal in \( L^2(\mathbb{R}^d) \).
2.3 Show that $E[|\tilde{B}(x_1) - \tilde{B}(x_2)|^4] = 3|x_1 - x_2|^2$.

2.4 Prove formula (2.1.7). (Hint: Set $F(\alpha, \beta) = \int_{\mathbb{R}} e^{i\alpha t - \beta t^2} dt$ for $\beta > 0$. Verify that $\partial F/\partial \beta = \partial^2 F/\partial \alpha^2$ and $F(0, \beta) = (\pi/\beta)^{1/2}$, and use this to conclude that $F$ must coincide with the right-hand side of (2.1.7).)

2.5 Give an alternative proof of Lemma 2.1.2. (Hint: Use (2.1.3) to prove that the characteristic function of the random variable $(\langle \omega, \xi_1 \rangle, \langle \omega, \xi_2 \rangle, \ldots, \langle \omega, \xi_n \rangle)$ coincides with that of the Gaussian measure $\lambda_n$ on $\mathbb{R}^n$.)

2.6 Prove statement (2.1.9): If $\phi \in L^2(\mathbb{R}^d)$ and we choose $\phi_n \in S(\mathbb{R}^d)$ such that $\phi_n \to \phi$ in $L^2(\mu_1)$, then

$$\langle \omega, \phi \rangle := \lim_{n \to \infty} \langle \omega, \phi_n \rangle$$

exists in $L^2(\mu_1)$ and is independent of the choice of $\{\phi_n\}$ (Hint: From Lemma 2.1.2 (or from Exercise 2.1), we get $E[\langle \omega, \phi \rangle^2] = ||\phi||^2$ for all $\phi \in S(\mathbb{R}^d)$. Hence $\{\langle \cdot, \phi_n \rangle\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(\mu_1)$ and therefore convergent.)

2.7 Use the Kolmogorov's continuity theorem (see, e.g., Stroock and Varadhan (1979), Theorem 2.1.6) to prove that the process $\tilde{B}(x) := \langle \omega, \chi_{[0,x_1] \times \cdots \times [0,x_d]} \rangle$ defined in (2.1.10) has a continuous version.

2.8 Find the Wiener–Itô chaos expansion (2.2.21),

$$f(\omega) = \sum_{\alpha \in J} c_\alpha H_{\alpha}(\omega); \ c_\alpha \in \mathbb{R}^N,$$

for the following $f \in L^2(\mu_m)$ (when nothing else is said, assume $N = m = 1$):

a) $f(\omega) = \omega^{\otimes 2}(\phi, \omega)$, $\phi \in S(\mathbb{R}^d)$. (Hint: Use (2.2.23) and (2.4.2).)

b) $f(\omega) = B^{\otimes 2}(x, \omega); \ x \in \mathbb{R}^d$. (Hint: Use (2.2.24) and (2.4.2).)

c) $f(\omega) = B^2(x, \omega); \ x \in \mathbb{R}^d$. (Hint: Use b) and (2.4.14).)

d) $f(\omega) = B^3(x, \omega); \ x \in \mathbb{R}^d$. (Hint: Use (2.4.17).)

e) $f(\omega) = \exp^2[w(\eta_1, \omega)]$. (Hint: Use (2.6.48).)

f) $f(\omega) = \exp[w(\eta_1, \omega)]$. (Hint: Use (2.6.55).)

g) $m \geq 1$, $f(\omega) = B_1(x, \omega) + \cdots + B_m(x, \omega); \ x \in \mathbb{R}^d$. (Hint: Use (2.2.26).)

h) $m \geq 1$, $f(\omega) = B^2_1(x, \omega) + \cdots + B^2_m(x, \omega); \ x \in \mathbb{R}^d$. (Hint: Use (2.2.26) and c).

i) $N \geq 1$, $f(\omega) = (2B(x, \omega) + 1, B^2(x, \omega))$.

2.9 Prove (2.4.15):

$$w(\phi) \circ w(\psi) = w(\phi) \cdot w(\psi) - (\phi, \psi)$$

for all $\phi, \psi \in L^2(\mathbb{R}^d)$. (Hint: Use (2.4.10) and (2.2.23).)
2.10 Let $F \in L^1(\mu) \cap (S)_{-1}$, with chaos expansion

$$F(\omega) = \sum_{\alpha} c_{\alpha} H_{\alpha}(\omega).$$

Prove that $E[F] = c_0$. (Hint: Combine (2.9.1) and (2.9.3) when $\phi = 0$).

2.11 Let $K_\phi(x, \omega) = \exp^\diamond [W_\phi(x, \omega)]$ be as in (2.6.56). Prove that

$$E[K_\phi(x, \cdot)] = 1 \text{ and } \text{Var}[K_\phi(x, \cdot)] = \exp[\|\phi\|^2] - 1.$$

(Hint: Use (2.6.54) and (2.6.55).)

2.12 Let $\phi$ be normally distributed with mean 0 and variance $\sigma^2$. Prove that

$$E[\phi^{2k}] = (2k - 1)(2k - 3) \cdots 3 \cdot 1 \cdot \sigma^{2k} \text{ for } k \in \mathbb{N}.$$

(Hint: We may assume that $\sigma = 1$. Use integration by parts and induction:

$$E[\phi^{2k}] = \int_{\mathbb{R}} x^{2k} d\lambda(x) = \int_{-\mathbb{R}} x \cdot x^{2k-1} d\lambda(x)$$

$$= \left[ -x^{2k-1} \cdot e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} + \int_{\mathbb{R}} (2k - 1)x^{2k-2} d\lambda(x),ight]$$

with $d\lambda(x) = d\lambda_1(x)$ as in (2.1.4).)

2.13 For $X \in (S)_{-1}$ we define the Wick-cosine of $X$, $\cos^\diamond[X]$, and the Wick-sine of $X$, $\sin^\diamond[X]$, by

$$\cos^\diamond[X] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} X^{\diamond(2n)}$$

and

$$\sin^\diamond[X] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} X^{\diamond(2n-1)},$$

respectively. Prove that

a) $\cos^\diamond[w(\phi)] = \exp^\diamond \left[ \frac{1}{2}\|\phi\|^2 \right] \cdot \cos[w(\phi)]$

b) $\sin^\diamond[w(\phi)] = \exp^\diamond \left[ \frac{1}{2}\|\phi\|^2 \right] \cdot \sin[w(\phi)].$

(Hint: Use Lemma 2.6.16 and the formulas

$$\cos^\diamond[w(\phi)] = \frac{1}{2} (\exp^\diamond[w(i\phi)] + \exp^\diamond[w(-i\phi)])$$

$$\sin^\diamond[w(\phi)] = \frac{1}{2i} (\exp^\diamond[w(i\phi)] - \exp^\diamond[w(-i\phi)]),$$

where $i = \sqrt{-1}$ is the imaginary unit.)
2.14

a) Prove that
\[
\exp^\diamond \left[ -\exp^\diamond \left[ w(\phi) \right] \right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \exp \left[ n w(\phi) - \frac{n^2}{2} \|\phi\|^2 \right],
\]
where the right-hand side converges in \(L^1(\mu)\).

b) Give an example to show that the Wick exponential \(\exp^\diamond X\) need not in general be positive. In fact, it may not even be bounded below.
(Hint: Consider \(f(x, y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \exp[nx - n^2y^2]\). If we have that \(x = 2y^2 > 2 \ln(3 + M)\) for \(M > 0\), then \(f(x, y) < -M\).)

2.15

a) Show the following generating formula for the Hermite polynomials:
\[
\exp \left[ tx - \frac{1}{2} t^2 \right] = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x).
\]
(Hint: Write \(\exp[tx - 1/2t^2] = \exp[1/2tx^2] \cdot \exp[-1/2(x - t)^2]\) and use Taylor’s Theorem at \(t = 0\) on the last factor. Then combine with Definition (C.1).)

b) Show that
\[
\exp \left[ w(\phi) - \frac{1}{2} \|\phi\|^2 \right] = \sum_{n=0}^{\infty} \frac{\|\phi\|^n}{n!} h_n \left( \frac{w(\phi)}{\|\phi\|} \right)
\]
for all \(\phi \in L^2(\mathbb{R}^d)\), where \(\|\phi\| = \|\phi\|_{L^2(\mathbb{R}^d)}\).

c) Deduce that
\[
\exp \left[ B(t) - \frac{1}{2} t \right] = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n \left( \frac{B(t)}{\sqrt{t}} \right)
\]
for all \(t \geq 0\).

d) Combine b) with Lemma 2.6.16 and (2.6.48) to give an alternative proof of (2.4.17):
\[
w(\phi)^{\circ n} = \|\phi\|^n h_n \left( \frac{w(\phi)}{\|\phi\|} \right),
\]
for all \(n \in \mathbb{N}\) and all \(\phi \in L^2(\mathbb{R}^d)\).

2.16 Show that \(\exp^\diamond [w(\phi)^{\circ 2}]\) is not positive.

2.17 Show that
\[
\| \exp^\diamond [nw(\phi)] \|_{L^p(\mu)} = \exp \left[ (p-1)n^2\|\phi\|^2_{L^2(\mathbb{R}^d)} \right]
\]
for all \(n \in \mathbb{N}, \phi \in L^2(\mathbb{R}^d), p \geq 1\).
2.18
a) Show that $\exp^\circ[\exp^\circ[w(\phi)]] \in (S)_{-1} \cap L^1(\mu)$.
b) Show that if $p > 1$, then $\exp^\circ[\exp^\circ[w(\phi)]] \notin L^p(\mu)$.

2.19 Let $\psi = \chi_{[0,t]}$. Use Itô’s formula to show that
a) $\int_{\mathbb{R}} \int_{\mathbb{R}} \psi \hat{\otimes}^2 dB \otimes^2 = B^2(t) - t.$
b) $\int_{\mathbb{R}^3} \psi \hat{\otimes}^3 dB \otimes^3 = B^3(t) - 3tB(t).$
(Compare with (2.2.29).)

2.20 Let $\phi \in (S_1)$. Prove that $\phi$ is pointwise defined and continuous on $S'(\mathbb{R})$.
(Hint: Use Definition 2.3.2 and formula (C.2).)

2.21 Consider the space $L^2(\mathbb{R}, \lambda)$ where
$$d\lambda(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$ Let $h_n(x); n = 0,1,2, \ldots$ be the Hermite polynomials defined in (2.2.1). In this 1-dimensional situation we can construct $(S)$ and $(S)^*$ as follows: For $p \geq 1$, define
$$(S)_p = \left\{ u(x) = \sum_{n=0}^{\infty} c_n h_n(x) \in L^2(\mathbb{R}, \lambda); \sum_{n=0}^{\infty} c_n^2 n! 2^{np} < \infty \right\};$$
set $(S) = \bigcap_{p \geq 1} (S)_p$ and $(S)^* = (S)'$, the dual of $(S)$. Prove that if $f \in (S)_1$, then
$$f(x) \exp \left[ -\frac{1}{2} x^2 \right] \in S(\mathbb{R}).$$

2.22 Let $G$ be a Borel subset of $\mathbb{R}$. Let $\mathcal{F}_G$ be the $\sigma$-algebra generated by all random variables of the form
$$\int_{\mathbb{R}} \chi_A(t)dB(t) = \int_A dB(t); \ A \subset G \text{ Borel set.}$$
Thus if $G = [0,t]$, we have, with $\mathcal{F}_t$ as in Appendix B, $\mathcal{F}_{[0,t]} = \mathcal{F}_t$ for $t \geq 0$.

a) Let $g \in L^2(\mathbb{R})$ be deterministic. Show that
$$E \left[ \int_{\mathbb{R}} g(t)dB(t) | \mathcal{F}_G \right] = \int_{\mathbb{R}} \chi_G(t)g(t)dB(t).$$
b) Let $v(t, \omega) \in \mathbb{R}$ be a stochastic process such that $v(t, \cdot)$ is $\mathcal{F}_t \cap \mathcal{F}_{G}$-measurable for all $t$ and
$$E \left[ \int_{\mathbb{R}} v^2(t, \omega)dt \right] < \infty.$$
Show that $\int_G v(t, \omega) dB(t)$ is $\mathcal{F}_G$-measurable. (Hint: We can assume that $v(t, \omega)$ is a step function $v(t, \omega) = \sum_i v_i(\omega) \chi_{[t_i, t_{i+1})}(t)$ where $v_i(\omega)$ is $\mathcal{F}_{t_i} \cap \mathcal{F}_G$-measurable. Then

$$
\int_G v(t, \omega) dB(t) = \sum_i \int_{G \cap [t_i, t_{i+1})} v_i(\omega) dB(t).
$$

\[c\) Let $u(t, \omega)$ be an $\mathcal{F}_t$-adapted process such that

$$
E\left[ \int_{\mathbb{R}} u^2(t, \omega) dt \right] < \infty.
$$

Show that

$$
E\left[ \int_{\mathbb{R}} u(t, \omega) dB(t) | \mathcal{F}_G \right] = \int_{\mathbb{R}} E[u(t, \omega) | \mathcal{F}_G] dB(t).
$$

(Hint: By b) it suffices to verify that

$$
E\left[ f(\omega) \cdot \int_{\mathbb{R}} u(t, \omega) dB(t) \right] = E\left[ f(\omega) \cdot \int_{\mathbb{R}} E[u(t, \omega) | \mathcal{F}_G] dB(t) \right]
$$

for all $f(\omega) = \int_A dB(t), A \subset G$.)

\[d\) Let $f_n \in \hat{L}^2(\mathbb{R}^n)$. Show that

$$
E\left[ \int_{\mathbb{R}^n} f_n dB^{\otimes n} | \mathcal{F}_G \right] = \int_{\mathbb{R}^n} f_n(t_1, \ldots, t_n) \chi_G(t_1) \ldots \chi_G(t_n) dB^{\otimes n}(t_1, \ldots, t_n).
$$

(Hint: Apply induction to c).)

\[e\) We say that two random variables $\phi_1, \phi_2 \in L^2(\mu)$ are strongy independent if there exist two Borel sets $G_1, G_2 \subset \mathbb{R}$ such that $\phi_i$ is $\mathcal{F}_{G_i}$-measurable for $i = 1, 2$ and $G_1 \cap G_2$ has Lebesgue measure 0. Suppose $\phi_1, \phi_2$ are strongly independent. Show that $\phi_1 \diamond \phi_2 = \phi_1 \cdot \phi_2$. (See Example 2.4.9.) (Hint: Use Proposition 2.4.2 and d).)

2.23 Find the Wiener–Itô expansion (2.2.35)

$$
f(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n dB^{\otimes n}; f_n \in \hat{L}^2(\mathbb{R}^n)
$$

of the following random variables $f(\omega) \in L^2(\mu)$ ($N = m = d = 1$):

\[a\) $f(\omega) = B(t_0, \omega); t_0 > 0$
b) \( f(\omega) = B^2(t_0, \omega); \ t_0 > 0 \)

c) \( f(\omega) = \exp[\int_{\mathbb{R}} g(s) dB(s, \omega)]; \ g \in L^2(\mathbb{R}) \) deterministic

d) \( f(\omega) = B(t, \omega)(B(T, \omega) - B(t, \omega)); \ 0 \leq t \leq T. \)

(Answers:

a) \( f_0 = 0, \ f_1 = \chi_{[0,t_0]}, \ f_n = 0 \) for \( n \geq 2. \)

b) \( f_0 = t_0, \ f_1 = 0, \ f_2(t_1, t_2) = \chi_{[0,t_0]}(t_1) \cdot \chi_{[0,t_0]}(t_2), \ f_n = 0 \) for \( n \geq 3. \)

c) \( f_n = \frac{1}{n!} \exp \left[ \frac{1}{2} \|g\|_{L^2(\mathbb{R})}^2 \right] g^\otimes n \) for all \( n \geq 0. \)

d) \( f_0 = 0, \ f_1 = 0, \ f_2(t_1, t_2) = \frac{1}{2} (\chi_{\{t_1 < t_2 < T\}} + \chi_{\{t_2 < t_1 < T\}}), \ f_n = 0 \) for \( n \geq 3. \)

2.24 Find the following Skorohod integrals using Definition 2.5.1:

a) \( \int_0^T B(t_0, \omega) \delta B(t); \ 0 \leq t_0 \leq T \)

b) \( \int_0^T \int_0^T g(s) dB(s) \delta B(t), \) where \( g \in L^2(\mathbb{R}) \) is deterministic.

c) \( \int_0^T B^2(t_0, \omega) \delta B(t); \ 0 \leq t_0 \leq T. \)

d) \( \int_0^T \exp[B(T, \omega)] \delta B(t). \)

e) \( \int_0^T B(t, \omega)(B(T, \omega) - B(t, \omega)) \delta B(t). \)

(Hint: Use the expansions you found in Exercise 2.23.)

(Answers:

a) \( B(t_0)B(T) - t_0. \)

b) \( B(T) \cdot \int_0^T g(s) dB(s) - \int_0^T g(s) ds. \)

c) \( B^2(t_0)B(T) - 2t_0B(t_0). \)

d) \( \exp \left[ \frac{1}{2} T \right] \sum_{n=0}^{\infty} \frac{1}{n!} T^{n+1} h_{n+1} \left( \frac{B(T)}{\sqrt{T}} \right) \)

e) \( \frac{1}{6} (B(T)^3 - 3T B(T)). \)

2.25 Compute the Skorohod integrals in Exercise 2.24 by using the Wick product representation in Theorem 2.5.9. (Hint: In e) apply Exercise 2.12 e.)

Remark: Note how much easier the calculation is with Wick products!

2.26 Let

\[ \mathbf{w}(\phi, \omega) = (\langle \omega_1, \phi_1 \rangle, \langle \omega_2, \phi_2 \rangle) \]

be the 2-dimensional smoothed white noise vector defined by (2.1.27). Define

\[ \mathbf{w}_c(\phi, \omega) = \langle \omega_1, \phi_2 \rangle + i \langle \omega_2, \phi_2 \rangle, \]
where \( i = \sqrt{-1} \) is the imaginary unit. We call \( w_c \) the \textit{complex smoothed white noise}.

Prove that

\[
\delta^2_c(\phi, \omega) = \delta^2_c(\phi, \omega).
\]

For generalizations of this curious result, see Benth et al. (1996).

\section*{2.27} Let

\[
w(\omega_1) = w(\omega_1, \omega) = \sum_{j=1}^{\infty} \langle \omega_1, \eta_j \rangle H_{\varepsilon_j}(\omega) \in (S)^*; \ \omega_1 \in S'(\mathbb{R}^d)
\]

be the generalized smoothed white noise defined in (2.10.8). Prove that

\[
\exp^\Diamond[w(\omega_1 + \omega_2)] = \exp^\Diamond[w(\omega_1)] \circ \exp^\Diamond[w(\omega_2)].
\]

(Note that both sides are functions of \( \omega \in S'(\mathbb{R}^d) \).)

\section*{2.28} Let \( \omega_1, \omega_2 \in S'(\mathbb{R}^d) \). Prove that

\[
T^*_{\omega_1 + \omega_2} = T^*_{\omega_1} T^*_{\omega_2} = T^*_{\omega_2} T^*_{\omega_1}.
\]

(Hint: See Theorem 2.10.3 and use Exercise 2.24.)

\section*{2.29} In this exercise, we let \( \phi \) denote the Hermite function of order \( k \in \mathbb{N} \), i.e., \( \phi(x) = \xi_k(x) \) where \( \xi_k(x) \) is given by (2.2.2).

a) Define \( X(\omega) = \sum_{n=0}^{\infty} a_n w(\phi)^{\circ n} \), where

\[
\sum_{n=0}^{\infty} (n!)^2 a_n^2 < \infty.
\]

Show that

\[
\psi_X(x) := \sum_{n=0}^{\infty} a_n h_n(x) \in L^2(\mathbb{R}, d\lambda),
\]

where

\[
d\lambda(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx
\]

and that

\[
X(\omega) = \psi_X(w(\phi)).
\]

(Hint: See Exercise 2.15.)

b) Define

\[
\delta(w(\phi)) := \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n! \sqrt{2\pi}} w(\phi)^{\circ 2n} = \frac{1}{\sqrt{2\pi}} \exp^\Diamond \left( -\frac{1}{2} w(\phi)^{\circ 2} \right).
\]
This is called the Donsker delta function. Note that $\delta(w(\phi)) \in (S)^*$. Show that $X \in (S)_1$ and that

$$\langle \delta(w(\phi)), X \rangle = \psi_X(0).$$

c) Show that no element $Z \in (S)_{-1}$ can satisfy the relation

$$Z \diamond w(\phi) = 1.$$  

d) In spite of the result in c), we can come close to a Wick inverse of $w(\phi)$ proceeding as follows:

With a slight abuse of notation, define

$$w(\phi)^{-\diamond 1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 2)k!} w(\phi)^{\diamond (2k+1)}.$$  

Show that $w(\phi)^{-\diamond 1} \in (S)_{-1}$ and that

$$w(\phi)^{-\diamond 1} \diamond w(\phi) = 1 - \sqrt{2\pi} \delta(w(\phi)).$$

$w(\phi)^{-\diamond 1}$ is called the Wick inverse of white noise. See Hu et al. (1995) for more details.

2.30 Prove (2.3.38), i.e., that $W(t) = \frac{d}{dt} B(t)$ in $(S)^*$. 

Stochastic Partial Differential Equations
A Modeling, White Noise Functional Approach
Holden, H.; Øksendal, B.; Ubøe, J.; Zhang, T.
2010, XV, 305 p. 17 illus., Softcover
ISBN: 978-0-387-89487-4