Chapter 2

Operators and Systems in the Plane

2.1 Operators in the Plane: First Definitions

We will work in the plane $\mathbf{R}^2$ with coordinates (x,t).

Definition 2.1. *A differential operator P of order $m \in \mathbf{N}$ is defined by*

$$(Pu)(x,t) = \Sigma_{k+l \leq m} a_{kl}(x,t)\partial_x^k \partial_t^l u(x,t).$$

Here, the coefficients a_{kl} are C^∞, given functions (to simplify). The operator

$$P_m = \Sigma_{k+l=m} a_{kl}(x,t)\partial_x^k \partial_t^l$$

is the "principal part" of P, the rest $P - P_m$ being the "lower order terms."

Definition 2.2. *Assume $a_{0m} \neq 0$. The Cauchy problem for the differential operator P with initial surface $\Sigma = \{t = 0\}$ and data $(u_0, \dots, u_{m-1}, f)$ is the problem of finding $u \in C^m$ such that*

$$Pu = f,\ u(x,0) = u_0(x), \dots,\ (\partial_t^{m-1} u)(x,0) = u_{m-1}(x).$$

Here $f \in C^0$ and the m functions $u_0, \dots, u_{m-1}$ are given with $u_k \in C^{m-k}$.

We remark that if $u \in C^\infty(\mathbf{R} \times [0,T[)$ is a solution of the Cauchy problem, *all traces* $(\partial_t^k u)(x,0)$ are known from the data; in fact, using the equation

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for $t = 0$, we obtain $(\partial_t^m u)(x, 0)$ from

$$a_{0m}(x,0)(\partial_t^m u)(x,0) + \Sigma_{k+l\le m, l<m} a_{kl}(x,0)\partial_x^k u_l(x) = f(x,0).$$

Differentiating any number of times with respect to x yields $\partial_t^m \partial_x^p u(x, 0)$. Differentiating the equation once with respect to t gives us $\partial_t^{m+1}(x, 0)$, and so on.

Definition 2.3. *For fixed (x, t), the roots of the polynomial equation in τ*

$$\Sigma_{k+l=m} a_{kl}(x,t)\tau^l = 0$$

are denoted by $-\lambda_1(x, t), \ldots, -\lambda_m(x, t)$, and the λ_i are called the characteristic speeds of P.

This terminology can be understood from Example 1.16. Note that this equation involves only the coefficients of the principal part of P.

Definition 2.4 (Hyperbolicity). *We say that P is hyperbolic in $\Omega \subset \mathbf{R}^2$ if all the characteristic speeds λ_i are real in Ω. We call it strictly hyperbolic if they are also distinct.*

In dealing with the Cauchy problem, we will *always* make the assumption that P is hyperbolic. If P is strictly hyperbolic, the functions λ_i are C^∞ by the implicit function theorem, and we will order them

$$\lambda_1(x,t) < \cdots < \lambda_m(x,t).$$

We can also write $P = a_{0m}\Pi(\partial_t + \lambda_i(x,t)\partial_x) + Q$, where Q is an operator of order $m-1$. Hence the principal part of P is just a product of *real* vector fields (modulo lower order terms).

Example 2.5. The one-dimensional wave equation (also called “vibrating string” equation) is (c being a positive constant)

$$P = \partial_t^2 - c^2\partial_x^2 = (\partial_t - c\partial_x)(\partial_t + c\partial_x) = (\partial_t + c\partial_x)(\partial_t - c\partial_x).$$

The operator P is associated to the quadratic form $\tau^2 - c^2\xi^2$, the level sets of which are hyperbola in the plane (ξ, τ). This explains the denomination “hyperbolic.” More generally, for a and b real constants satisfying $a^2 - 4b > 0$, the operator

$$P = \partial_t^2 + a\partial_{xt}^2 + b\partial_x^2$$

is strictly hyperbolic.

Example 2.6. The operator $P = \partial_t^2 - x^2\partial_x^2 = (\partial_t + x\partial_x)(\partial_t - x\partial_x) + x\partial_x$ is hyperbolic, but not strictly hyperbolic for $x = 0$.

Example 2.7. The operator $P = \partial_t^2 - t^2\partial_x^2 = (\partial_t + t\partial_x)(\partial_t - t\partial_x) + \partial_x$ is hyperbolic, but not strictly hyperbolic for $t = 0$.

Example 2.8. The Tricomi operator $P = \partial_t^2 + t\partial_x^2$ is strictly hyperbolic for $t < 0$, hyperbolic but nonstrictly for $t = 0$, and not hyperbolic for $t > 0$.

2.2 Systems in the Plane: First Definitions

Definition 2.9. *A first order system is an operator of the form*

$$L = S(x,t)\partial_t + A(x,t)\partial_x + B(x,t),$$

where S, A, and B are C^∞ $N \times N$ matrices, and L acts on C^1 vectors in $\mathbf{C}^N$ by

$$LU = S\partial_t U + A\partial_x U + BU.$$

Definition 2.10. *Assume S invertible. The Cauchy problem for the system L with initial surface $\Sigma = \{t = 0\}$ and data (U_0, F) is the problem of finding $U \in C^1$ such that*

$$LU = F,\ U(x,0) = U_0(x),$$

where $F \in C^0$ and $U_0 \in C^1$ are given.

Definition 2.11 (Hyperbolicity). *The system L is hyperbolic if all the eigenvalues λ_i of $S^{-1}A$ are real. These eigenvalues are called the characteristic speeds of L. We call L strictly hyperbolic if the characteristic speeds are distinct. The system is symmetric hyperbolic if S and A are hermitian and S is positive definite.*

The importance of symmetry is not obvious and is explained in Exercise 13. See also Chapter 7, Section 7.3, where the notion is discussed in detail.

If L is strictly hyperbolic, the eigenvalues $\lambda_i(x,t)$ are C^∞, since they are simple roots of a polynomial (the characteristic polynomial) with C^∞ coefficients. Assume that, in the domain D where we work, we can choose a basis of smooth eigenvectors $(r_1(x,t), \ldots, r_N(x,t))$ of $S^{-1}A$. Then

$$P(x,t)^{-1}S^{-1}(x,t)A(x,t)P(x,t) = \Lambda(x,t),$$

where P has the eigenvectors r_i as its columns, and Λ is diagonal. Setting $U = PV$, we find that the Cauchy problem

$$LU = F,\ U(x,0) = U_0(x)$$

is equivalent to the Cauchy problem

$$\partial_t V + \Lambda \partial_x V + CV = G,\ V(x,0) = V_0(x) = P^{-1}(x,0)U_0(x),$$

with

$$C = P^{-1}\partial_t P + \Lambda P^{-1}\partial_x P + P^{-1}S^{-1}BP,\ G = P^{-1}S^{-1}F.$$

The principal part of the system is now diagonal, the functions V_i satisfying the N equations

$$(\partial_t + \lambda_i(x,t)\partial_x)V_i(x,t) + \Sigma C_{ij}(x,t)V_j(x,t) = G_i(x,t),\ i = 1,\ldots,N.$$

We think of this new system as *scalar equations coupled* through the coefficients C_{ij}. If L has constant coefficients S and A and is homogeneous (that is, $B \equiv 0$), then $C \equiv 0$ and we just have a collection of N scalar equations, which can be solved as explained in Chapter 1.

Important Remark: We explained in Chapter 1 why the Cauchy problem for a nonreal field could not be well-posed in the sense of Hadamard. Since a system with different speeds λ_k can be diagonalized, it follows that hyperbolicity is a necessary condition for the Cauchy problem for the system L to be well-posed.

2.3 Reducing an Operator to a System

Just like one does for ordinary differential equations, one can reduce scalar operators of order m to $m \times m$ first order systems. Assume that the operator P contains no terms of order less than $m-1$.

- If u is a C^m solution of the Cauchy problem

$$Pu = f,\ u(x,0) = u_0(x),\ldots,\ (\partial_t^{m-1}u)(x,0) = u_{m-1}(x),$$

we introduce as new unknowns the m functions

$$U_0 = \partial_x^{m-1}u,\ldots,\ U_k = \partial_t^k \partial_x^{m-1-k}u,\ldots,\ U_{m-1} = \partial_t^{m-1}u.$$

Then U is a C^1 solution of the Cauchy problem

$$\begin{aligned}
\partial_t U_0 = \partial_x U_1,\ldots,\ \partial_t U_{m-2} = \partial_x U_{m-1},\\
\partial_t U_{m-1} = -(a_{0m})^{-1}\Sigma_{k\geq 1} a_{kl}\partial_x U_l + (a_{0m})^{-1} f,\\
U_0(x,0) = \partial_x^{m-1}u_0(x),\ldots,\ U_{m-1}(x,0) = u_{m-1}(x).
\end{aligned}$$

• Conversely, if U is a C^1 solution of the above Cauchy problem, we define u by

$$\partial_t^m u = \partial_t U_{m-1},\ u(x,0) = u_0(x), \ldots,\ (\partial_t^{m-1} u)(x,0) = u_{m-1}(x).$$

Then we obtain successively

$$\partial_t^{m-1} u = U_{m-1},\ \partial_x \partial_t^{m-2} u = U_{m-2}, \ldots,\ \partial_x^{m-1} u = U_0,$$

and u turns out to be a C^m solution of the Cauchy problem for P. Just as we did in Section 2.3, we emphasize the fact that, since an operator can be reduced to a system with the same characteristic speeds (see Exercise 9), these speeds must be real in order for the Cauchy problem to be well-posed.

Example 2.12. In the case $m = 2$, $U_0 = \partial_x u$, $U_1 = \partial_t u$, we obtain from the wave equation $P = \partial_t^2 - c^2 \partial_x^2$ the system

$$\partial_t U_0 = \partial_x U_1,\ \partial_t U_1 = c^2 \partial_x U_0 + f.$$

If we modify the procedure slightly by setting

$$U_0 = c\partial_x u,\ U_1 = \partial_t u,$$

we obtain a *symmetric* system. We can even try right away

$$U_0 = \partial_t u + c\partial_x u,\ U_1 = \partial_t u - c\partial_x u,$$

and obtain a *diagonal* system. We note that U_0 and U_1 are just then the factors of P.

Example 2.13. For $P = \partial_t^2 - x^2 \partial_x^2$ of Example 2.6 above, we can try the same approach, setting

$$U_0 = x\partial_x u,\ U_1 = \partial_t u.$$

Then

$$\partial_t U_0 = x\partial_x U_1, \partial_t U_1 = x\partial_x U_0 - U_0 + f,$$

and again we obtain a *symmetric* system.

Example 2.14. If we try the same procedure for $P = \partial_t^2 - t^2 \partial_x^2$, setting

$$U_0 = t\partial_x u, U_1 = \partial_t u,$$

we obtain now the system

$$\partial_t U_0 = t\partial_x U_1 + \frac{U_0}{t},\ \partial_t U_1 = t\partial_x U_0 + f,$$

which is *singular* on $\{t = 0\}$. The difference with Example 2.13 is not just a consequence of our awkwardness: It reflects a true difference in the behavior of the solutions of the Cauchy problems.

Example 2.15. For the Tricomi operator we use $U_0 = \partial_x u$, $U_1 = \partial_t u$; To get a nice system, we multiply the first line by $-t$ and obtain the symmetric system

$$-t\partial_t U_0 + t\partial_x U_1 = 0,\ \partial_t U_1 + t\partial_x U_0 = f.$$

Note that the system is symmetric hyperbolic exactly when $t < 0$.

If the operator P has terms of order less than $m - 1$, one can try to express them in terms of the new unknowns. For instance, if $m = 2$, $u(x,t) = u_0(x) + \int_0^t U_1(x,s)ds$, etc. The obtained system will not be strictly speaking a first order system, but the additional (integral) terms can be handled as zero order terms and cause no trouble.

For the operator in Example 2.13, if one chooses U_0 and U_1 as indicated in order to obtain a symmetric system, it will not be possible to express smoothly a lower order term such as $a(x,t)\partial_x u$ with the help of U, unless $a(0,t) = 0$. In fact, it can be shown that the well-posedness of the Cauchy problem for $P = \partial_t^2 - x^2\partial_x^2 + a(x,t)\partial_x$ requires precisely this condition. Thus, turning a nonstricly hyperbolic operator into a hyperbolic symmetric system is a subtle issue, one that requires sometimes additional conditions on the lower order terms, called "Levy conditions."

2.4 Gronwall Lemma

The following elementary lemma will be useful here and later on.

Lemma 2.16 (Gronwall Lemma). *Let $A, \phi \in C^0([0,T[)$ such that, for $0 \le t < T$,*

$$\phi(t) \le C + \int_0^t A(s)\phi(s)ds.$$

Assume that $A \ge 0$. Then $\phi(t) \le C\exp(\int_0^t A(s)ds)$.

The proof is left as Exercise 3.

2.5 Domains of Determination I (A priori Estimate)

Definition 2.17. *For a hyperbolic operator P, the field $\partial_t + \lambda_i \partial_x$ is called the i-characteristic field, and its integral curves are called i-characteristics of P. The same definition holds for first order systems.*

Note that we have shown that P is equal to the product of its characteristic fields (up to lower order terms) and that a system can be reduced to the diagonal system of its characteristic fields (modulo zero order coupling terms).

Definition 2.18. *A closed domain $D \subset \mathbf{R}_x \times [0, \infty[$ with base*

$$\omega = D \cap \{t = 0\}$$

is a domain of determination of ω for an operator P (or a system L) if for any $m = (x_0, t_0) \in D$, and all i, the backward i-characteristic (that is, for $t \leq t_0$) drawn from m reaches ω while remaining in D.

Example 2.19. Consider the wave equation, and take $\omega = [a, b]$ on the x-axis. A triangle D bounded by a line through $(a, 0)$ (with positive slope) and a line through $(b, 0)$ (with negative slope) is a domain of determination if the lines have slopes respectively less than c and greater than $-c$. The biggest possible D is bounded by lines with slopes c and $-c$, respectively. More generally, as a consequence of the usual comparison theorem for solutions of ordinary differential equations (see Appendix, Theorem A.7), we have the following theorem.

Theorem 2.20. *For a strictly hyperbolic operator or system, the biggest domain of determination D with base $\omega = [a, b]$ on the x-axis is the curved triangle bounded by the x-axis, the fastest characteristic (corresponding to λ_m) from $(a, 0)$, and the slowest characteristic (corresponding to λ_1) from $(b, 0)$.*

For a domain of determination D, we will denote by $p_i(m)$ the point where the backward i-characteristic $\gamma_i(m) = \{(x_i(t, m), t)\}$ drawn from m meets ω. We can now prove the following *a priori estimate*.

Theorem 2.21. *Let D be a compact domain of determination with base ω on the x-axis for a first order strictly hyperbolic system L. Set $D_t = \{x, (x, t) \in D\}$. Then there exists a constant C such that, for any $U \in C^1(\bar{D})$,*

$$\max_{0 \leq s \leq t} ||U(\cdot, s)||_{L^\infty(D_s)} \leq C\{||U_0||_{L^\infty(\omega)} + \int_0^t ||(LU)(\cdot, s)||_{L^\infty(D_s)} ds\}.$$

Proof: As explained in Section 1.6, we reduce the Cauchy problem $LU = F, U(x,0) = U_0(x)$ to the problem

$$\partial_t V + \Lambda \partial_x V + CV = G, V(x,0) = V_0(x).$$

Integrating the equation for V_i along the i-characteristic between 0 and t, we obtain

$$V_i(m) = (V_0)_i(p_i(m)) + \int_0^t [G_i - (CV)_i](x_i(s,m),s)ds.$$

We fix t and take the sup norm in x to get, for some numerical constant C_1,

$$||V_i(\cdot,t)||_{L^\infty(D_t)} \leq ||V_0||_{L^\infty(\omega)} + C_1 \int_0^t \{||F(\cdot,s)||_{L^\infty(D_s)} + ||V(\cdot,s)||_{L^\infty(D_s)}\}ds.$$

We set now $\phi(t) = \max_{0\leq s\leq t} ||V(\cdot,s)||_{L^\infty(D_s)}$. Summing the above inequalities over i, we obtain for $0 \leq t' \leq t \leq T$ (with another constant C_2)

$$\begin{aligned} ||V(\cdot,t')||_{L^\infty(D_{t'})} \leq C_2||V_0||_{L^\infty(\omega)} + C_2 \int_0^T ||F(\cdot,s)||_{L^\infty(D_s)}ds \\ + C_2 \int_0^t ||V(\cdot,s)||_{L^\infty(D_s)}ds. \end{aligned}$$

Taking the supremum in t' we get for $t \leq T$

$$\phi(t) \leq A + C_2 \int_0^t \phi(s)ds, A = C_2||V_0||_{L^\infty(\omega)} + C_2 \int_0^T ||F(\cdot,s)||_{L^\infty(D_s)}ds.$$

Using the Gronwall lemma, we finally get $\phi(t) \leq C_3 A$, which is the desired result. □

In particular, the theorem implies the uniqueness of a possible solution to the Cauchy problem in D. From the proof of the theorem, we see that it can be extended to a noncompact domain (for instance, a strip $\{0 \leq t \leq T\}$), provided the appropriate obvious assumptions on the coefficients of L have been made. Such a theorem is called an **a priori estimate**, since it applies to any U.

2.6 Domains of Determination II (Existence)

We prove now an existence theorem in a domain of determination D, chosen as in Section 2.5.

Theorem 2.22. *Let D be a compact domain of determination with base ω on the x-axis for a first order strictly hyperbolic system L. Let $F \in C^1(D)$ and $U_0 \in C^1(\omega)$. Then there exists a unique solution $U \in C^1(D)$ of the Cauchy problem*

$$LU = F, \, U(x,0) = U_0(x).$$

Proof: *Step 1.* We resume the notation of the proof of Theorem 2.21. We first prove that the system on V, written in integral form

$$V_i(m) = (V_0)_i(p_i(m)) + \int_0^t [G_i - (CV)_i](x_i(s,m),s)ds,$$

has a C^0 solution in D. To this aim, we define a sequence $V^n \in C^0(D)$ by

$$V_i^{n+1}(m) = (V_0)_i(p_i(m)) + \int_0^t [G_i - (CV^n)_i](x_i(s,m),s)ds, \; V^0 = 0.$$

Introducing $\delta^n(t) = ||V^{n+1}(\cdot,t) - V^n(\cdot,t)||_{L^\infty(D_t)}$, we obtain by subtracting the equations for $n+1$ and n and taking the supremum for fixed t as before,

$$\delta^n(t) \leq C_1 \int_0^t \delta^{n-1}(s)ds.$$

We claim now that for some constants c_0 and c_1, we have for all n, $\delta^n(t) \leq c_0 c_1^n t^n/n!$. For $n = 0$, this is certainly true for c_0 big enough, which we now fix accordingly. Assume that this is true for n: then we get from the above inequality and the induction hypothesis

$$\delta^{n+1}(t) \leq C_1 \int_0^t c_0 c_1^n \frac{s^n}{n!} ds = C_1 c_0 c_1^n \frac{t^{n+1}}{(n+1)!}.$$

This shows that the claim is true if $c_1 \geq C_1$. If $t \leq T$ in D, we obtain then

$$||V^{n+1} - V^n||_{L^\infty(D)} \leq c_0 \frac{(c_1 T)^n}{n!},$$

which is the general term of a convergent series. Hence, V^n converges uniformly in D to some $V \in C^0(D)$, which is a solution of the system on V written in integral form.

Step 2. However, this does not imply that V is C^1 and satisfies the differential system! To handle this difficulty, we set $W^n = \partial_x V^n$, which is allowed since in fact V^n belongs to $C^1(D)$ if F and U_0 do. Differentiating with respect to x the integral expression of V^{n+1}, we obtain

$$W_i^{n+1}(m) = \partial_x[(V_0)_i(p_i(m)) + \int_0^t G_i(x_i(s,m),s)ds] - \int_0^t [(\partial_x C)V^n + CW^n]_i(x_i(s,m),s)(\partial_x x_i(s,m))ds.$$

Just as before, we prove that V^n and W^n converge uniformly in D to continuous functions V and W. This implies that V admits a continuous partial derivative $\partial_x V = W$. Since

$$\partial_t V^{n+1} + \Lambda \partial_x V^{n+1} + CV^n = G,$$

$\partial_t V^n$ also converges uniformly to a continuous function. Hence V admits continuous partial derivatives and is in C^1. We can then differentiate the system in integral form satisfied by V to recover the original system, and this finishes the proof. □

2.7 Exercises

1.(a) Consider in the plane $\mathbf{R}^2_{x,t}$ the wave operator $P = \partial_t^2 - \partial_x^2$. Prove that any C^2 function u of the form $u(x,t) = \phi(x+t)$ or $u(x,t) = \psi(x-t)$ satisfies $Pu = 0$. Deduce from this an explicit formula for the solution u of the homogeneous Cauchy problem in a domain

$$D = \{(x,t), t \geq 0, t + |x| \leq a\}.$$

(b) Find explicitly the solution of the Cauchy problem $Pu = f$ in D with zero Cauchy data on $\{t = 0\}$.

2. Let D be the unit closed disc in the plane with coordinates (x,y), and ∂D the unit circle. What are all the C^2 solutions of the equation $\partial^2_{xy} u = 0$ in $\mathbf{R}^2$? in D? Show that the boundary value problem in D

$$\partial^2_{xy} u = f, u|\partial D = u_0$$

does not have a unique solution. If we impose the stronger boundary conditions $u = \nabla u = 0$ on ∂D, show that the corresponding boundary value problem in D has at most one solution. Write down necessary conditions on f for such a solution to exist.

3. Prove the Gronwall lemma (Section 2.4)
(Hint: Set $\psi(t) = C + \int_0^t A(s)\phi(s)ds$, and solve the differential inequality on ψ).

4. We consider a C^2 real solution u of the wave equation

$$Pu = (\partial_t^2 - \partial_x^2)u = 0$$

in the cylinder $\mathcal{C} = \{(x,t), t \geq 0, a \leq x \leq b\} \subset \mathbf{R}^2_{x,t}$. Assume that u satisfies the boundary conditions

$$u(a,t) = 0,\ (\partial_t u + \partial_x u)(b,t) = 0.$$

(a) Define the energy of u at time t by

$$E(t) = \frac{1}{2}\int_a^b [(\partial_t u)^2 + (\partial_x u)^2](x,t)dx.$$

By computing $\int_{\mathcal{C}\cap\{0\leq t\leq T\}}(Pu)(\partial_t u)dxdt$, show

$$E(T) - E(0) = -\int_0^T (\partial_t u)^2(b,t)dt.$$

The energy is said to "dissipate" along the boundary $\{x = b\}$.

(b) Show that for $t \geq 2(b-a)$, $u \equiv 0$ (so much energy dissipated that there is nothing left!).

5. Prove an a priori estimate analogous to that of Theorem 2.21 for a second order strictly hyperbolic operator P.

6. Prove an existence theorem analogous to that of Theorem 2.22 for a second order strictly hyperbolic operator P.

7. Let P be a strictly hyperbolic operator of order two in $\mathbf{R}^2_{x,t}$, and $u \in C^2(\mathbf{R}_x \times \mathbf{R}_t^+)$ be a solution of $Pu = 0$. Assume that the Cauchy data of u vanish outside $[a,b]$. Let $x = x_1(t)$ be the 1-characteristic of P through $(a,0)$, and $x = x_2(t)$ the 2-characteristic through $(b,0)$. Prove that the support of u is contained in the set

$$\{(x,t), t \geq 0, x_1(t) \leq x \leq x_2(t)\}.$$

8. Consider a strictly hyperbolic homogeneous operator P with constant coefficients. Show that if D is not a domain of determination of its base $[a,b]$ for P, no uniqueness can hold for the Cauchy problem in D.

9. Prove that when an operator P is reduced to a first order system L as in Section 2.3 the characteristic speeds are the same for P and L.

10. Let A be a real square matrix. Show that if there exists a hermitian positive definite S such that SA is hermitian, then the eigenvalues of A are

real. Conversely, if all eigenvalues of A are real and distinct, there exists such an S. Explain why this is relevant for hyperbolic systems.

11.(a) Let P be the wave operator with real coefficient $c \in C^1(\mathbf{R}^2)$

$$P = \partial_t^2 - c^2(x,t)\partial_x^2,\ 1/2 \le c \le 2.$$

Prove for all $u \in C^2(\mathbf{R}^2)$ the identity

$$\begin{aligned}(Pu)(\partial_t u) &= \frac{1}{2}\partial_t[c^2(\partial_x u)^2 + (\partial_t u)^2] - \partial_x[c^2(\partial_x u)(\partial_t u)] \\ &\quad + 2c(\partial_x c)(\partial_x u)(\partial_t u) - c(\partial_t c)(\partial_x u)^2.\end{aligned}$$

(b) Assume that in the strip $S_T = \{0 \le t \le T\}$ for some constant C,

$$|\partial_x c| + |\partial_t c| \le C.$$

Assume for simplicity that u is real and that $u(\cdot,t)$ has compact support for all t. Using the formula of (a) to compute $\int_{S_t}(Pu)(\partial_t u)dxds$, prove for $t \le T$ the inequality

$$E(t) \le E(0) + C_1 \int_0^t E(s)ds + C_1 \int_0^t ||f(\cdot,s)||_{L^2} E^{1/2}(s)ds,$$

where $Pu = f$ and $E(t) = (1/2)\int[c^2(\partial_x u)^2 + (\partial_t u)^2]dx$. Proceed then as in Exercise 17 of Chapter 1, using the Gronwall lemma, to establish the a priori L^2 inequality

$$\max_{0\le s\le t} E^{1/2}(s) \le C_2 E^{1/2}(0) + C_2 \int_0^t ||f(\cdot,s)||_{L^2} ds,\ t \le T.$$

Such an a priori inequality in L^2 norm is called an “energy inequality.”

12. We keep the notation of Exercise 11. Let D be a compact domain of determination for P, and set $D_T = \{(x,t) \in D, 0 \le t \le T\}$. On the nonhorizontal part Λ of the boundary of D_T, we denote the components of the unit outgoing normal by $(n_x, n_t > 0)$. Proceeding as in Exercise 11, prove the a priori inequality

$$\begin{aligned}E(T) + \int_\Lambda (n_t^2 - c^2 n_x^2)\frac{(\partial_t u)^2}{2n_t}d\sigma &\le E(0) + C_1 \int_0^T E(t)dt \\ &\quad + C_1 \int_0^T ||f(\cdot,t)||_{L^2} E^{1/2}(t)dt,\end{aligned}$$

where $d\sigma$ is the length element on Λ and E is now defined by an integration on $D \cap \{t = T\}$. If $|n_x| \leq n_t/c$ on Λ, this yields exactly the same energy inequality as in Exercise 11. Show that this condition on ∂D is always satisfied for a domain of determination (this is a remarkable fact, since it shows that the method of proof does not require more assumptions than what is known to be necessary anyway).

13.(a) Let $L = S\partial_t + A\partial_x + B$ be a symmetric hyperbolic system, where we take for simplicity S and A to be real. Prove, for all real $U \in C^1(\mathbf{R}^2)$, the identity

$$2{}^tULU = \partial_t({}^tUSU) + \partial_x({}^tUAU) - {}^tU(\partial_t S + \partial_x A - 2B)U.$$

Give appropriate conditions on the coefficients of L in a strip $S_T = \{0 \leq t \leq T\}$ to obtain, as in Exercise 11, the energy inequality

$$\max_{0 \leq s \leq t} ||U(\cdot, s)||_{L^2} \leq C_1||U_0||_{L^2} + C_1 \int_0^t ||f(\cdot, s)||_{L^2} ds.$$

(b) We keep the notation of Exercise 12 and set $E(t) = \int_{(x,t)\in D} |U(x,t)|^2 dx$. Prove the inequality

$$||U(\cdot, T)||_{L^2}^2 + \int_\Lambda {}^tU(n_t S + n_x A)U d\sigma \leq C_2||U_0||_{L^2}^2 + C_2 \int_0^T E(t)dt$$
$$+ C_2 \int_0^T ||f(\cdot, t)||_{L^2} E^{1/2}(t)dt.$$

Show that the conditions

$$n_x > 0 \Rightarrow n_t + \lambda_1 n_x \geq 0, \;\; n_x < 0 \Rightarrow n_t + \lambda_N n_x \geq 0$$

imply that the matrix $n_t S + n_x A$ is nonnegative. Prove then an energy inequality analogous to that of (a). Are these conditions always satisfied for a domain of determination?

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