Chapter 5
Asymptotics in Fluid Mechanics

5.1 Introduction

As we pointed out in some detail in Chapter 1, the concept of intermediate asymptotics first arose in two physical contexts: propagation in a certain spatial region of a gene, whose carriers have an advantage in the struggle for existence, and fluid mechanics, in particular, explosion and implosion phenomena involving shocks. In Chapters 3 and 4, we studied the asymptotic behaviour of solutions of nonlinear PDEs of parabolic type, which include those describing gene propagation. In the present chapter, we discuss some physical problems which arise from fluid mechanics and which are governed by hyperbolic or parabolic systems of equations. These systems admit similarity solutions of the first or second kind. The latter, in general, enjoy intermediate asymptotic character in some parametric regimes. We recall that the self-similar solutions of the first kind are fully determined by the dimensional considerations and require the solution of the resulting nonlinear ODEs with appropriate conditions at the shock and the centre of the explosion in this context, say, whereas those of the second kind involve solution of an eigenvalue problem in the reduced phase plane, the implosion problem, for example.

In Section 5.2, we study the propagation of a strong shock produced by a large explosion into a medium for which the density varies according to the power law \( \rho_0(r) = kr^{-\omega} \), where \( r \) is the distance measured from the centre of the explosion and \( k > 0 \) and \( \omega > 0 \) are constants. It is first shown that the self-similar solutions describing this phenomenon change their character from one of the first kind for \( \omega < 3 \) to that of the second kind for \( \omega > 3 \). The intermediate asymptotic character of the solutions for \( \omega_g(\gamma) < \omega < \omega_c(\gamma) \) is brought out by comparison with the numerical solution of the original system of nonlinear PDEs with appropriate initial conditions; here, \( \omega_g(\gamma) \) and \( \omega_c(\gamma) \) depend on the ratio of specific heats, \( \gamma = C_p/C_v \). Section 5.3 deals with self-similar solutions of the second kind which describe a collapsing spherical cavity. Here, we show by reference to the basic work of Hunter (1960) that the numerical solution of the governing system of nonlinear PDEs with appropriate initial/boundary conditions tends, for different sets of \( \gamma \), to the relevant
self-similar solution of the second kind as the radius of the cavity tends to zero. More recent work on this problem is also summarised. Section 5.4 concerns large time behaviour of compressible flow equations with damping. It is shown, by following the work of Liu (1996), that the solutions of this hyperbolic system of equations tend, for large time, to those of a nonlinear parabolic equation; this is brought out by referring to a special class of solutions of each of these systems. This study justifies, in a limited sense, the so-called Darcy’s law which is used to describe compressible flow in a porous medium. Sections 5.5 and 5.6 deal with the systems of nonlinear PDEs holding in the Prandtl boundary layer. In the former section, we take up the study of unsteady boundary layer equations governing the flow in an incompressible medium and rigourously show, following the work of Oleinik (1966a), that, under a certain set of conditions, the solutions of an unsteady system of equations tend to those of the corresponding steady equations as time becomes large. In the latter section, we deal with the basic work of Serrin (1967) which proves that the similarity solution of the steady boundary layer equations governed by the Falkner–Skan equation, a nonlinear ODE of third order, subject to relevant boundary conditions at 0 and ∞, describes the asymptotic behaviour of solutions of (steady) boundary layer equations with appropriate initial/boundary conditions on the spatial domain 0 < x < ∞. This is accomplished in a rigorous analytical manner. We may mention that the present problem is parabolic in character.

5.2 Strong explosion in a power law density medium – Self-similar solutions of first and second kind

One of the most important examples of self-similar solutions of the first kind in fluid mechanics relates to a point source explosion into a uniform medium which was analysed by Taylor (1950), Sedov (1946), and von Neumann (1947); see Sachdev (2004) for a detailed account. This explosion results from a sudden release of a large amount of energy in a small volume. We consider here the case of a spherical explosion; it is headed by a strong shock which propagates into a medium with uniform density and zero pressure. This problem was solved by the exact similarity reduction of the governing system of nonlinear PDEs to ODEs and the solution of the latter subject to Rankine–Hugoniot conditions at the shock and zero particle velocity at the centre of the explosion. It was dealt with in Eulerian coordinates by Taylor (1950) and Sedov (1946) and Lagrangian coordinates by von Neumann (1947).

Here we consider a more general problem: a strong shock produced by a large explosion propagates into a medium for which the density varies according to power law

\[ \rho_0(r) = Kr^{-\omega}, \]  

where \( r \) is the distance measured from the centre of the explosion and \( K > 0, \omega > 0 \) are constants. The medium ahead is assumed to be an ideal gas with zero pressure.
This is a curious example where the self-similar solution changes its character from one of the first kind for $\omega < 3$ to the (so-called) second kind for $\omega > 3$ (see Chapter 1). The case $\omega < 3$, which is a straightforward generalisation of the Taylor–Sedov solution, was first treated by Korobeinikov and Riazanov (1959). They reduced the system of nonlinear PDEs governing the inviscid gas dynamic equations in spherical, cylindrical, and plane symmetries to a system of nonlinear ODEs by assuming that pressure, density, and particle velocity behind the shock may be written in the form $v = v_2 f(\lambda), \rho = \rho_2 g(\lambda), p = p_2 h(\lambda), \lambda = r/r_2$, where the subscript '2' denotes conditions immediately behind the shock, $r_2 = r_2(t)$. The nonlinear ODEs for $f, g,$ and $h$ were solved subject to the conditions $f(1) = g(1) = h(1) = 1$ at the shock and the particle velocity $f(0) = 0$ at the centre; the latter arises from the spherical symmetry of the problem. Indeed the system of ODEs with the above conditions was solved in a closed form. Korobeinikov and Riazanov (1959) considered only those parameters for which the solution could be extended to the centre of symmetry. The solution, however, exhibited singularities when the density exponent $\omega$ in (5.2.1) assumes values $\omega = \omega_1, \omega_2, \omega_3$, where $\omega_1 = (7 - \gamma)/(\gamma + 1), \omega_2 = (2\gamma + 1)/\gamma$, and $\omega_3 = 3(2 - \gamma)$ for the spherically symmetric case that we consider here. For these singular cases, Korobeinikov and Riazanov (1959) found limiting behaviour of the solutions by solving the governing system of ODEs directly.

Unaware of the above work, Waxman and Shvarts (1993) considered this problem in a different fashion. They did not directly impose the symmetry condition $u(0, t) = 0$ at the centre of explosion. Instead they enquired for what values of the parameter $\omega$ the similarity solution was a straightforward generalisation of the Taylor–Sedov solution. They arrived at a very interesting conclusion, namely, that Taylor–Sedov type solutions exist only for the case $\omega < 3$. The value $\omega = 3$ is exactly the point where the singularities in the solution of Korobeinikov and Riazanov (1959) appear if we assume that $\gamma > 1$.

Waxman and Shvarts (1993) showed that self-similar solutions of the first kind fail to describe the asymptotic behaviour as $t \to \infty$ for $3 \leq \omega < 5$. They discovered new solutions belonging to the so-called second kind for $3 < \omega < 5$ and for $\omega \geq 5$. These solutions are distinct from the Taylor–Sedov type because they describe flows with accelerating shocks. This is in contrast to Taylor–Sedov solutions for $\omega < 3$, which are headed by decelerating shocks. The new class of solutions was found in the manner of the solutions for the converging shocks for which the shock exponent is found, not from dimensional considerations alone, but by requiring that the solution, for a given $\gamma$, starting from the shock, passes through an appropriate singular point of the reduced ODE in the sound speed square-particle velocity plane, the so-called Guderley map. In the present case, the solution must pass through a ‘new’ singular point in the Guderley map (see Sachdev 2004).

Now we follow the work of Waxman and Shvarts (1993). The main purpose here is to identify self-similar solutions of the second kind for $\omega > 3$, which describe the limiting behaviour which is approached asymptotically for $t \to \infty$ by flows that are initially non-self-similar. A large class of problems, which differ in boundary and initial conditions, tends in the above limit to the same asymptotic self-similar behaviour of the second kind.
The spherically symmetric flows with shocks are governed by

\begin{align}
\frac{\partial}{\partial t} \ln \rho + u \frac{\partial}{\partial r} \ln \rho + \frac{\partial u}{\partial r} + \frac{2u}{r} &= 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{c^2}{\gamma} \frac{\partial}{\partial r} \ln \rho + \frac{1}{\gamma} \frac{\partial c^2}{\partial r} &= 0, \\
\frac{\partial}{\partial t} \ln (c^2 \rho^{1-\gamma}) + u \frac{\partial}{\partial r} \ln (c^2 \rho^{1-\gamma}) &= 0,
\end{align}

where \(u, \rho,\) and \(c\) are particle velocity, density, and sound speed, respectively. We assume that, at later time, the solution of the system (5.2.2)–(5.2.4) does not depend upon constants with the dimension of length or time deriving from the boundary conditions; the flow here must depend on only two-dimensional parameters. The typical length scale is the radius of the shock given by

\[ R(t) = A t^\alpha, \]

where \(A\) and \(\alpha\) are constants. The flow variables may now be expressed in the self-similar form

\begin{align}
u(r, t) &= \hat{R} \xi U(\xi), \quad c(r, t) = \hat{R} \xi C(\xi), \\
\rho(r, t) &= B t^\beta G(\xi), \quad \xi = \frac{r}{R(t)}.
\end{align}

Here, \(\xi\) is the similarity variable. For sufficiently late times the length scale of the flow is described by \(R = R(t)\).

In the Taylor–Sedov form of the solution, the early flow is described by two-dimensional parameters, the parameter \(K\) in the density law (5.2.1) and the energy of the flow which is assumed to be equal to the explosion energy \(E\). By simple dimensional considerations we may find the dimensional constants \(A\) and \(B\) and the parameters \(\alpha\) and \(\beta\) in (5.2.5) and (5.2.6) as

\begin{align}
A &= \varphi(\gamma, \omega) \left( \frac{E}{K} \right)^{\alpha/2}, \quad \alpha = \frac{2}{5 - \omega} \\
B &= KA^{-\omega}, \quad \beta = -\alpha \omega,
\end{align}

where \(\varphi\) is a dimensionless function of the (dimensionless) parameters \(\gamma\) and \(\omega\), which is determined from the constancy of the explosion energy behind the shock. Korobeinikov and Riazanov (1959) considered the case \(\omega \leq 3\).

Waxman and Shvarts (1993) showed that the self-similar solutions of Taylor–Sedov type discussed above cease to hold for large \(R\) for two reasons: (i) the energy of explosion in this case tends to infinity and therefore cannot be the second-dimensional parameter, and (ii) for \(\omega \geq 3\) there must exist a region around the origin with a non-self-similar behaviour. For large time, when \(R(t)\) diverges, the initial length and time scales which arise from the initial conditions do not characterise the physical process; these scales characterise flows at early times.
The solution in the outer region is in fact described by a self-similar solution of the second kind, as we show presently. The substitution of (5.2.5) and (5.2.6) into (5.2.2)–(5.2.4) reduces the latter to the system of ODEs,

\[
\xi(U-1)G' + \xi GU' = - \left( \frac{\beta + 3\alpha U}{\alpha} \right) G, \tag{5.2.8}
\]

\[
\xi G(U-1)U' + \frac{C}{\gamma'} \xi G' + \frac{2}{\gamma} \xi CGC' = \left[ \frac{1}{\alpha - U} \right] UG - \frac{2}{\gamma} C^2 G, \tag{5.2.9}
\]

\[
2\alpha \xi(U-1)GC' + \alpha(1-\gamma)\xi(U-1)CG' = [\beta(\gamma-1) + 2 - 2\alpha U] CG, \tag{5.2.10}
\]

which, in the \((U, C)\) plane, becomes

\[
\frac{dU}{dC} = \frac{\triangle_1(U,C)}{\triangle_2(U,C)}, \tag{5.2.11}
\]

\(U\) and \(C\) are related to \(\xi\) by

\[
\frac{d \log \xi}{dU} = \frac{\triangle(U,C)}{\triangle_1(U,C)} \tag{5.2.12}
\]

and

\[
\frac{d \log \xi}{dC} = \frac{\triangle(U,C)}{\triangle_2(U,C)}. \tag{5.2.13}
\]

Here,

\[
\triangle = C^2 - (1-U)^2,
\]

\[
\triangle_1 = U(1-U) \left( 1 - U - \frac{\alpha-1}{\alpha} \right) - C^2 \left( 3U - \frac{\omega - 2 \left[ \frac{\alpha-1}{\alpha} \right]}{\gamma} \right),
\]

\[
\triangle_2 = C \left[ (1-U) \left( 1 - U - \frac{\alpha-1}{\alpha} \right) - \frac{\gamma-1}{2} U \left( 2(1-U) + \frac{\alpha-1}{\alpha} \right) \right. \\
\left. - C^2 + \frac{(\gamma-1) \omega + 2 \left[ \frac{\alpha-1}{\alpha} \right] }{2\gamma} \frac{C^2}{1-U} \right]. \tag{5.2.14}
\]

The function \(G\) in (5.2.6) may then be obtained from a quadrature of the system (5.2.8)–(5.2.10), namely,

\[
C^{-2}(1-U)^{\lambda} G^{\gamma-1+\lambda} \xi^{3\lambda-2} = \text{constant}, \tag{5.2.15}
\]
where
\[ \lambda = \frac{(\gamma - 1)\omega + 2 \left[ \frac{(\alpha - 1)}{\alpha} \right]}{3 - \omega}. \] (5.2.16)

It is customary to first examine the equation (5.2.11) in the \((U, C)\)-plane to identify the required integral and then use (5.2.12) and (5.2.13) to relate \(U\) and \(C\) to the variable \(\xi\). The strong shock conditions
\[ u = \frac{2}{\gamma + 1} \hat{R}, \quad \rho = \frac{\gamma + 1}{\gamma - 1} \rho_0, \quad p = \frac{2}{\gamma + 1} \rho_0 \hat{R}^2, \] (5.2.17)
in view of (5.2.6), become
\[ U(1) = \frac{2}{\gamma + 1}, \quad C(1) = \sqrt{\frac{2\gamma(\gamma - 1)}{\gamma + 1}}, \quad G(1) = \frac{\gamma + 1}{\gamma - 1}. \] (5.2.18)

For the Taylor–Sedov type of solution, the energy \(E_1\) contained in the region \(\xi_1 \leq \xi \leq 1\) corresponding to \([\xi_1 R(t) \leq r \leq R(t)]\) is given by
\[ E_1 = \int_{\xi_1 R}^{R} dr 4\pi r^2 \rho \left\{ \frac{1}{2} u^2 + \frac{1}{\gamma(\gamma - 1)} c^2 \right\} = 4\pi KR^{3-\omega} \hat{R}^2 \int_{\xi_1}^{1} d\xi \xi^4 G \left( \frac{1}{2} U^2 + \frac{1}{\gamma(\gamma - 1)} C^2 \right) \] (5.2.19)
which, in view of (5.2.5)–(5.2.7), becomes
\[ E_1 = \left[ 4\pi \left( \frac{2}{5 - \omega} \right)^2 \varphi^{\gamma - \omega - 1} \right] \int_{\xi_1}^{1} d\xi \xi^4 G \left( \frac{1}{2} U^2 + \frac{1}{\gamma(\gamma - 1)} C^2 \right) E. \] (5.2.20)

In this case, \(E_1\) is independent of time. In view of this constancy, the self-similar contour in the \((U, C)\)-plane may be obtained by the energy–work done principle, namely, the work done during the interval \(dt\) by a fluid element which is at \(\xi = \xi_1\) at time \(t\) on the fluid that lies in \(\xi > \xi_1\) at time \(t\) equals the energy that leaves the region \(\xi_1 \leq \xi \leq 1\) during the same time interval via the energy flux through the surface \(\xi_1 = \text{constant}\). Thus, the work done by a fluid element that lies at \(\xi_1\) at the time \(t\) on the fluid that occupies the region \(r > \xi_1 R(t)\) at that time during the interval \(dt\) is
\[ 4\pi r_1^2 u(r_1, t) dt \gamma^{-1} \rho(r_1, t) c^2(r_1, t) = 4\pi \gamma^{-1} K R^{2-\omega} \hat{R}^4 \xi_1^5 U(\xi_1) G(\xi_1) C^2(\xi_1) dt. \] (5.2.21)
The corresponding energy that leaves the region \(\xi > \xi_1\) during the time \(dt\) due to the energy flux across the surface \(\xi_1 = \text{constant}\) is given by
\[ 4\pi r_1^2 \left[ \xi_1 \hat{R} - u(r_1, t) \right] dt \left( \frac{1}{2} \rho(r_1, t) u^2(r_1, t) + \frac{1}{\gamma(\gamma - 1)} c^2(r_1, t) \right). \]
\[ 4\pi \gamma^{-1} KR^{2-\omega} \xi_1^3 [1 - U(\xi_1)] G(\xi_1) \left( \frac{\gamma U^2(\xi_1) + C^2(\xi_1)}{\gamma - 1} \right) dt. \] (5.2.22)

Equation (5.2.21) and (5.2.22), we arrive at the integral

\[ C^2 = \frac{\gamma(\gamma - 1) U^2(1 - U)}{2 \gamma U - 1}. \] (5.2.23)

In view of (5.2.23), equation (5.2.12) becomes

\[ \frac{d \log \xi}{dU} = (\gamma + 1) \frac{\gamma U^2 - 2U + \frac{2}{\gamma + 1}}{U(\gamma U - 1)[5 - \omega - (3\gamma - 1)U]} \] (5.2.24)

which involves only \( U \) on the RHS.

Equation (5.2.24) was analysed by Waxman and Shvarts (1993) in the appendix to their paper. It was found that, for \( \omega < (7 - \gamma)/(\gamma + 1) \), \( U \) tends to \( 1/\gamma \) and \( C \) tends to infinity as \( \xi \to 0 \). On the other hand, for \( \omega > (7 - \gamma)/(\gamma + 1) \), a vacuum is formed in \( 0 < \xi < \xi_{\text{in}} \) where \( \xi = \xi_{\text{in}} > 0 \) is the outer boundary of this region. The self-similar solution holds in \( \xi_{\text{in}} \leq \xi \leq 1 \) and is separated from the evacuated region \( \xi < \xi_{\text{in}} \) by a particle path. We observe that the value \( (7 - \gamma)/(\gamma + 1) \) of \( \omega \) coincides with the singularity \( \omega = \omega_1 \) of Korobeinikov and Riazanov (1959) for spherical symmetry.

The case \( \omega > (7 - \gamma)/(\gamma + 1) \) includes \( \omega \geq 3 \) for \( \gamma \geq 1 \). Here, the solution curve approaches the point \((C = 0, U = 1)\) as \( \xi \) tends to \( \xi_{\text{in}} \). A simple local analysis of (5.2.15), (5.2.23), and (5.2.24) shows that

\[ U(\xi) \approx 1 - \frac{3\gamma + \omega - 6}{\gamma} \log \left( \frac{\xi}{\xi_{\text{in}}} \right), \]
\[ C(\xi) \approx \left[ \frac{3\gamma + \omega - 6}{2} \log \left( \frac{\xi}{\xi_{\text{in}}} \right) \right]^{1/2}, \] (5.2.25)
\[ G(\xi) \approx \text{constant} \times \left[ \log \left( \frac{\xi}{\xi_{\text{in}}} \right) \right]^{-(\gamma(\omega + \omega - 6)/(3\gamma + \omega - 6))} \]

for \( \xi \gtrsim \xi_{\text{in}} \). Insertion of (5.2.25) in the expression (5.2.20) for the energy \( E_1 \) in the region \( \xi_{\text{in}} \leq \xi \leq 1 \) corresponding to \( \xi_{\text{in}} R(t) \leq r \leq R(t) \) shows that, for \( \omega \geq 3, E_1 \to \infty \) as \( \xi_1 \to \xi_{\text{in}} \). The (constant) energy of the blast is therefore contained in a region bounded by some point \( \xi_{\text{in}} = \xi_0 \) and the shock. It follows that the flow resulting from the finite energy of the blast is described by a Taylor–Sedov solution in a smaller region \( \xi_{\text{in}} \leq \xi_0 \leq \xi \leq 1 \) and is different from the self-similar solution in the intermediate layer \( \xi_{\text{in}} \leq \xi \leq \xi_0 \). This implies that some initial length and time scales influence the flow behaviour over the region \( O(R) \) as \( R \to \infty \), contradicting its self-similar nature. It is this fact and not the infinite energy of the blast for \( \omega > 3 \) alone which makes the self-similar solution of Taylor–Sedov type untenable. In any case, for \( \omega > 3 \), (infinite) energy of the explosion is not the relevant second parameter.
For $\omega > 3$, Waxman and Shvarts (1993) constructed the asymptotic similarity solution as follows. They considered two flow regions: the outer region lying in $r_1(t) < r(t) < R(t)$ and the inner one lying between $r = r_1(t)$ and $r = 0$. They assumed that the flow in the outer region is independent of that in the inner region and is determined for $t > t_0$ entirely by the initial flow conditions in the former at some time $t = t_0$. They also assumed that $r_1(t)/R(t)$ tends to zero as $R \to \infty$; furthermore, they surmised that the flow behaviour in the inner region does not affect the description of the flow over the scales of order $R$. Thus, the flow in the large outer region is assumed to be independent of that in the much smaller inner region. A self-similar solution is constructed for the outer region, which is not affected by the initial length and time scales; the latter influence only the smaller inner region.

It was shown that the inner region is bounded by a $C_+$ characteristic which starts from $r = r_1, t = t_0$ (Zel’dovich and Raizer 1967). The solution in the outer region is constructed in the manner self-similar solutions of the second kind are analysed (Zel’dovich and Raizer 1967). A physical solution starting from the shock must cross the sonic line $\Delta = 0$ at a singular point of (5.2.11) where $\Delta_1 = \Delta_2 = 0$, otherwise equations (5.2.12) and (5.2.13) would imply that one of the functions $U(\xi)$ and $C(\xi)$ is not single-valued. Thus the appropriate value of $\alpha$ for a given $\gamma$ is found by starting the integration from the shock point $(U(1), C(1))$ and continuing until the integral curve crosses the sonic line $\Delta = 0$ at a singular point of (5.2.11). It is now shown that such an integral curve exists for a certain range of $\omega$ values greater than 3.

Fortunately, it becomes possible to find an explicit self-similar solution of (5.2.12) and (5.2.13) for a particular value of $\omega = \omega_b \equiv 2(4\gamma - 1)/(\gamma + 1)$, satisfying the shock conditions $[U(1), C(1)]$ and passing through the singular point $U = 2/(\gamma + 1), C = (\gamma - 1)/(\gamma + 1)$. Because for $\gamma > 1, \omega_b > 3$, the exponent $\alpha = \alpha_b(\gamma)$ for this case is simply $(\gamma + 1)/2$. The explicit form of the solution for this special case is found to be

$$U(\xi) = U(1) = \frac{2}{\gamma + 1},$$

$$C(\xi) = C(1)\xi^3 = \frac{\sqrt{2\gamma(\gamma - 1)}}{\gamma + 1}\xi^3, \quad (5.2.26)$$

$$G(\xi) = \frac{\gamma + 1}{\gamma - 1}\xi^{-8}$$

and motivates the solution for other values of $\omega$. It also provides a valuable check on the veracity of the numerical solution for large time. We may observe that the exponent $\alpha_b = (\gamma + 1)/2$ differs considerably from the corresponding Taylor–Sedov value $2(\gamma + 1)/(7 - 3\gamma)$.

Before considering the solution for other $\omega > 3$, a numerical solution of the full flow equations (5.2.2)–(5.2.4) was found by Waxman and Shvarts (1993) for $\omega = \omega_b$, using the methods of artificial viscosity (see Sachdev 2004). The initial conditions for this particular example were zero particle velocity everywhere, constant density and pressure at the time of energy release in $r < d$, zero pressure,
and density proportional to \( r^{-\omega} \) for \( r > d \). The results for \( \omega_a = 4.25 \) corresponding to \( \gamma = 5/3 \) from the exact analytic solution, the numerical solution, and from the Taylor–Sedov form of the solution are shown in Figure 5.1. Here, \((t/\tau)^{\alpha}/(R/d)\) is plotted against \( R/d \) for large time with \( \tau = (Kd^{5-\omega}/E)^{1/2} \). It is easily seen that this quantity approaches a constant value for the explicit solution (5.2.26) with \( \alpha = 4/3 \) (solid line). This is in contrast to the Taylor–Sedov values for \( \alpha = 8/3 \) (dashed line) which continue to diverge.

The solution \( C = C(U) \) for \( \gamma = 5/3 \) and \( \omega_a = 4.25 \) according to (5.2.26) (solid line), numerical solution (dashed line), and the Taylor–Sedov type solution (dashed dot line) are shown in Figure 5.2. \( U(\xi) \) and \( C(\xi) \) were found from (5.2.5) and (5.2.6) by first computing \( R(t) \) and \( dR(t)/dt \) from the numerical results. It is clear from Figure 5.2 that the numerical solution for large times agrees very well with the exact asymptotic solution given by (5.2.26). The dotted line in the figure is \( U + C = 1 \). There is some departure of the analytic solution from the numerical solution when it is below this line (see Waxman and Shvarts 1993).

![Fig. 5.1](image-url)  

The exact solution (5.2.26) shows that it approaches the point \( U = 2/(\gamma + 1), C = 0 \) as \( \xi \to 0 \). This is a singular point of (5.2.11). It is a special case of the singular point \( P(U = 1/\alpha, C = 0) \) which exists for all \( \omega > 0 \) and \( \alpha \). This is the point where the self-similar solution starting from the shock (5.2.18) crosses the sonic line; it is approached in the limit \( \xi \to 0 \). Waxman and Shvarts (1993) also studied \((C,U)\) curves for \( \gamma = 5/3, \omega = 3.4 \) and \( \gamma = 5/3, \omega = 5.5 \). The agreement of the self-
similar solution with the numerical solution for large times was clearly observed. The Taylor–Sedov type of solution does not exist for \( \omega = 5.5 \). Because \( dU/dC = 0 \) when \( U = 1 \) (see (5.2.11)–(5.2.14)) and because \( U < 1 \) at the shock (see (5.2.18)), the solution being considered here must satisfy the inequality \( U < 1 \) behind the shock. Therefore, the singular point \( P(U = 1/\alpha, C = 0) \) where the integral curve crosses the sonic line is of interest only when \( \alpha > 1 \). Thus for this class of self-similar solutions the similarity exponent \( \alpha \) is greater than 1; this is in contrast to the self-similar solutions of the Taylor–Sedov type for \( \omega < 3 \) for which \( \alpha < 1 \).

It is of some interest to write the present self-similar solution in the neighbourhood of the singular point \( P \). We may approximate \( \Delta, \Delta_1, \text{ and } \Delta_2 \) near \( P \) as

\[
\Delta = - \left( 1 - \frac{1}{\alpha} \right)^2,
\]

\[
\Delta_1 = - \frac{1}{\alpha} \left( 1 - \frac{1}{\alpha} \right) \left( U - \frac{1}{\alpha} \right) + \left( \frac{\omega - 2 \left[ \frac{\alpha - 1}{\gamma} \right]}{\gamma} - \frac{3}{\alpha} \right) C^2,
\]

\[
\Delta_2 = - \frac{3}{\alpha} \left( 1 - \frac{1}{\alpha} \right) \frac{\gamma - 1}{2} C.
\]
We may, therefore, write an approximate solution of (5.2.11) near $P$ as

$$U = \frac{1}{\alpha} + \begin{cases} \text{constant } C^{2/(\gamma-1)}, & \gamma > 4/3, \\ f_1(\gamma, \omega, \alpha)C^2 \log C, & \gamma = 4/3, \\ f_2(\gamma, \omega, \alpha)C^2, & \gamma < 4/3, \end{cases} \tag{5.2.28}$$

where

$$f_1 = \left( \frac{3}{\alpha} - \frac{\omega - 2(\alpha - 1)/\alpha}{\gamma} \right) \frac{2\alpha}{\alpha - 1},$$
$$f_2 = \left( \frac{3}{\alpha} - \frac{\omega - 2(\alpha - 1)/\alpha}{\gamma} \right) \frac{\alpha^2}{(\alpha - 1)(3\gamma - 4)} \tag{5.2.29}.$$  

The approximate form of the functions $C(\xi)$ and $G(\xi)$ may hence be obtained from (5.2.13) and (5.2.15) as

$$C(\xi) = \text{constant } \times \xi^{3/(\gamma-1)/2(\alpha-1)}, \tag{5.2.30}$$
$$G(\xi) = \text{constant } \times \xi^{-(\alpha\omega - 3)/(\alpha-1)}. \tag{5.2.31}$$

A detailed numerical evaluation of the exponent $\alpha = \alpha(\omega)$ shows that $\alpha$ tends to unity as $\omega \downarrow \omega_g$ for some $\omega_g > 3(\omega_g = 3.256$ for $\gamma = 5/3)$; $\alpha$ tends to infinity as $\omega \uparrow \omega_c$ for some finite $\omega_c$ ($\omega_c = 7.686$ for $\gamma = 5/3$). For $3 \leq \omega \leq \omega_g(\gamma)$, there is no $\alpha$ for which the integral curve in the $(U, C)$-plane crosses the sonic line at the singular point $P$.

To summarise, the self-similar analysis of flows arising from a strong explosion into a nonuniform power law medium reveals some very fascinating features. These solutions change their character from the first kind to those of the second kind as the density exponent $\omega$ crosses the value 3. The asymptotic character of the solutions in the ranges $3 \leq \omega \leq \omega_g(\gamma)$ and $\omega \geq \omega_c(\gamma)$ needs further investigation.

### 5.3 Self-similar solutions for collapsing cavities

The generation and propagation of converging shock waves were first treated by Guderley (1942). Guderley’s self-similar solutions are self-similar solutions of the second kind and are descriptors of converging cylindrical shocks close to the axis of symmetry. This work was subsequently discussed by several authors (see Zel’dovich and Raizer 1967, Whitham 1974). In particular, their analytical asymptotic behaviour in the neighbourhood of the axis was elegantly treated by Van Dyke and Guttmann (1982). We have earlier discussed converging shocks in some detail (Sachdev 2004). Here we deal with a related problem, the collapse of an empty spherical or cylindrical cavity in water and air and the asymptotic behaviour of its solution near the centre (axis) of collapse. First we discuss briefly the collapse of an empty spherical cavity in water because the analysis here is quite close to that detailed in Section 5.2. Later, we study the same problem when the medium is air; the analytical results for the latter are quite distinct from those for water.
The major work concerning cavity collapse in water is due to Hunter (1960). Here, the effects of viscosity and surface tension were neglected but compressibility of the water was allowed for by using a suitable form of the equation of state. It is envisioned that a cavity of initially infinite size has been collapsing for an infinite time. Thus the pressure in the cavity is zero and that for the liquid far from it is $p_0$.

It is also assumed that there is a constant, finite energy $E$ associated with the flow. Following Rayleigh’s (1917) treatment of the same problem in an incompressible medium, the cavity wall was assumed to move according to the formula

$$E = 2\pi \rho' R'^3 \dot{R}^2,$$  
(5.3.1)

where $\dot{R} = \dot{R}(t)$ is the radius of the cavity. On integration of (5.3.1), we have

$$R'^{5/2} = \frac{5}{2} \sqrt{\frac{E}{2\pi \rho'}} (t'_0 - t'),$$  
(5.3.2)

where $t'_0$ is the instant of collapse. After appropriately scaling the variables by the conditions at infinity, the problem may be reduced to solving the system

$$u_t + uu_r + \frac{1}{\gamma - 1} (c^2)_r = 0,$$  
(5.3.3)

$$(c^2)_t + u(c^2)_r + (\gamma - 1)c^2 \left( u_r + \frac{2u}{r} \right) = 0.$$  
(5.3.4)

The boundary conditions are

$$c = 1 \quad \text{and} \quad u = \dot{R} \quad \text{at the cavity wall} \quad r = R,$$

$$c \to 1 \quad \text{as} \quad r \to \infty,$$

$$R \sim (t_0 - t)^{2/5}, \quad u \sim -\frac{2}{5} \frac{(t_0 - t)^{1/5}}{r^2},$$

$$c^2 = 1 + \frac{2(\gamma - 1)(t_0 - t)^{-6/5}}{25} \left[ \frac{(t_0 - t)^{2/5}}{r} - \frac{(t_0 - t)^{8/5}}{r^4} \right]$$  
(5.3.5)

as $t_0 - t \to \infty$. The expression for $c^2$ comes from the solution of this problem assuming the medium to be incompressible. The initial conditions for the numerical solution of this problem were chosen to be (5.3.5) when $\dot{R} = -0.1$. $\gamma = c_p/c_v$ in (5.3.3) and (5.3.4) was assumed to be 7. The method of characteristics was used to solve this problem. Two significant results emerged from this numerical study: (i) $R(-\dot{R})^t \to$ constant as $R \to 0$ where the constant $\tau$ was found to be 1.27. (ii) For each fixed value of $r/R$, $-u \to \infty$ and $c \to \infty$ as $R \to 0$ in such a way that $u/R$ and $c^2/R^2$ become functions of $r/R$ alone. In one of the first instances of similarity theory, Hunter (1960) was motivated by the numerical results to seek out the solution of this problem in the form
5.3 Self-similar solutions for collapsing cavities

\[ u = \dot{R} f(r/R), \quad c^2 = \dot{R}^2 g(r/R). \quad (5.3.6) \]

The problem of asymptotic collapse of the cavity was studied as its radius tends to zero. Thus, any length scale for the flow, which may be derived from the boundary conditions, will be too large to be relevant as \( R \to 0 \). Therefore, the only suitable length scale for the flow is the cavity radius \( R \) itself. The same is true for the velocity scale because any such scale derived from the boundary conditions will be too small compared to the velocity of the cavity wall. Thus, \( \dot{R} \) is the appropriate velocity scale.

When \((5.3.6)\) is substituted into \((5.3.3)\) and \((5.3.4)\), the latter reduce to ODEs only if \( R\ddot{R}/\dot{R}^2 = \text{constant} = 1 - n^{-1} \), say. Thus, \( R = A(-t)^n \) where \( A \) is a constant and \( t = 0 \) is the instant of the cavity collapse. The conditions \( u = \dot{R} \) and \( c^2 = 0 \) at \( r = R \) lead to the boundary conditions \( f(1) = 1 \) and \( g(1) = 0 \) at the cavity. The equation of state was assumed to be \( p \propto \rho^\gamma \) where \( \gamma = 7 \). The reduction of the system of ODEs to a single equation is similar to that described in Section 5.2. Hunter (1960) carried out a very detailed analysis of singularities in the \((Y, Z)\)-plane (see Section 5.2). The value of the similarity exponent \( n \) was determined by the regularity properties of the similarity solution. For \( \gamma = 7 \), it was found to be 0.5552. The corresponding value from the numerical solution was found to be 0.560. We may mention that \( \tau \) and \( n \) are related by \( \tau = n/(1 - n) \) (see discussion below (5.3.5)). The similarity analysis valid only for high pressures and velocities was continued beyond the instant of the cavity collapse to describe the formation and initial propagation of the shock wave after the collapse is completed.

In a later study, Thomas et al. (1986) considered the collapse of a spherical cavity surrounded by a perfect gas initially at rest. The cavity begins to move with uniform velocity \( -2c_0/(\gamma - 1) \), where \( c_0 \) is the speed of sound in the undisturbed gas. Thomas et al. (1986) showed that the cavity velocity remains practically uniform until the radius of the cavity, \( R \), becomes a small fraction \( \xi(\gamma) \) of the initial radius \( R_0 \). Then it begins to move according to the asymptotic self-similar behaviour \( \dot{R} \sim R^{-\tau(\gamma)} \). However, for \( 1 < \gamma < \gamma_{cr}, \gamma_{cr} \approx 1.5 \), the velocity \( \dot{R} \) of the cavity surface remains strictly uniform for the entire period of collapse. The effect of geometry seems to be overridden by the high compressibility of the gas.

We consider first the case when the radius \( R \) of the cavity remains sufficiently close to its initial value \( R_0 \). The flow is described by a plane rarefaction wave (Courant and Friedrichs 1948) with the modulus of the velocity profile given by

\[ u = \frac{2}{\gamma + 1} \left( \frac{R_0 - r}{r} + c_0 \right). \quad (5.3.7) \]

In this case, the velocity of the cavity is simply

\[ \dot{R} = -\frac{2c_0}{(\gamma - 1)} = -V_i, \quad \text{say}. \quad (5.3.8) \]

Writing \( u = dr_i/dt \) in \((5.3.7)\) and integrating we arrive at the result
where \( r_{io} = r_i(t = 0) \). The relation (5.3.9) holds provided that \( t \geq (r_{io} - R_0)/c_0 \).

The above analysis is valid until the geometrical compression begins to contribute significantly.

As we discussed earlier with reference to the work of Hunter (1960) for \( \gamma = 7 \) (see also Zel’dovich and Raizer 1967), the flow for \( R \ll R_0 \) is described by the self-similar solution. Here the radius of the cavity is given by

\[
R = A(-t)^n, \tag{5.3.10}
\]

where \( A \) is a constant which depends on the initial conditions and \( n = n(\gamma) \). We may rewrite (5.3.10) as

\[
\dot{R} = -C R^{-\dot{\tau}} \tag{5.3.11}
\]

where \( \dot{\tau} = \tau^{-1} = (1 - n)/n \) and \( C = nA^{1/n} \).

As we mentioned earlier, the law (5.3.10) was motivated by Rayleigh’s results for the cavity collapse in an incompressible fluid for which \( \dot{\tau} = 3/2 \) and \( C = (3p_0/2\rho_0)^{1/2}R_0^{3/2} \). For an ideal gas \( \dot{\tau} \) depends on the compressibility of the medium.

Thomas et al. (1986) carried out the numerical solution of this problem with a scheme different from Hunter’s and came up with some interesting conclusions. The results we describe here refer to the asymptotic stage when \( r \to R \) and \( R \to 0 \). Figure 5.3 shows \(-\dot{R}\) as a function of the cavity radius \( R/R_0 \). Here, the undisturbed pressure \( p_0 \) and density \( \rho_0 \) were taken to be \( 10^6 \) dyn cm\(^{-2}\) and \( 1.6 \times 10^{-4} \) g cm\(^{-3}\), respectively. The minimum value, \( R_{min}/R_0 \), for which ‘numerical saturation’ took place depended on the value of \( \gamma \). It was found to be \( R_{min}/R_0 = 10^{-2}, 2 \times 10^{-3}, 8 \times 10^{-4} \) for \( \gamma = 7, 4, 2.4 \), respectively. Figure 5.3 also shows an \((R/R_0, -\dot{R})\) relation for \( \gamma = \infty, 7, 4, 2.4, \) and 5/3. It may be observed that, for large \( \gamma \), the cavity velocity remains constant for a short initial distance and time and then it begins to promptly conform to asymptotic self-similar behaviour. As \( \gamma \) decreases, the early stage of uniform velocity increases and the slope of the asymptotic line decreases until, for \( \gamma \approx 5/3 \), the cavity speed becomes essentially constant. For \( \gamma \lesssim 5/3 \), the cavity moves to the centre with almost uniform velocity \(-2c_0/(\gamma - 1)\) during its entire course. In fact, there exists a value of \( \gamma = \gamma_1, 1.5 < \gamma_1 < 5/3 \), below which the flow never approaches self-similar behaviour and the cavity moves inward with constant velocity right up to the point of collapse. Thomas et al. (1986) compared their conclusions regarding the asymptotic nature of the cavity collapse with those from the stability analysis of these flows by Lazarus (1982) and observed that there are no (one-dimensionally) stable asymptotic solutions for \( 3/2 < \gamma < 5/3 \); there exist degenerate stable asymptotic solutions with \( \dot{R} = \text{constant} \) for \( \gamma \lesssim 3/2 \). These solutions were numerically simulated by Thomas et al. (1986).
5.4 Large time behaviour of solutions of compressible flow equations with damping

In an interesting study, Liu (1996) showed how the solutions of compressible flow equations with damping, which have a hyperbolic character, tend for large time to those of a nonlinear parabolic equation. For this purpose, he made use of a special class of solutions of each of these systems. Barenblatt (1953) had constructed a special class of solutions for the porous flow equations. Making use of this class of solutions, Liu (1996) justified, at least in a limited sense, the so-called Darcy’s law which is used to describe compressible flow in a porous medium. The physical situation described by these flows is rather special and includes a vacuum front where $\rho = 0$. It is assumed that no shocks are formed in the flow.

The vector form of the isentropic compressible flow equations with damping may be written as

$$\rho_t + \nabla \cdot (\rho \bar{u}) = 0,$$  \hspace{1cm} (5.4.1)

$$\rho \bar{u}_t + \nabla \cdot \rho (\bar{u} \otimes \bar{u}) + \nabla p(\rho) + \alpha \rho \bar{u} = 0,$$  \hspace{1cm} (5.4.2)

where $\rho$ and $\rho$ are related by the polytropic law

Fig. 5.3 $-\dot{R}$ versus $R/R_0$ with undisturbed pressure $\rho_0 = 10^6$ dyn cm$^{-2}$, density $\rho_0 = 1.6 \times 10^{-4}$ gm cm$^{-3}$ and $\gamma = \infty, 7, 4, 2.4, 5/3$. (Thomas et al. 1986. Copyright © 1986 American Institute of Physics. Reprinted with permission. All rights reserved.)
\[ p(\rho) = kp^\gamma, \quad k > 0, \quad \gamma > 1. \] (5.4.3)

\( \alpha \) in (5.4.2) is positive and denotes the (constant) coefficient of friction. Using Darcy’s law
\[ \nabla p(\rho) = -\alpha \rho \bar{u}, \] (5.4.4)
equation (5.4.2) becomes the porous media equation
\[ \rho_t = \alpha^{-1} \Delta p(\rho). \] (5.4.5)

It was shown by Liu (1996) that a certain class of solutions of equations (5.4.1)–(5.4.3) tends, for large time, to the corresponding solutions of the porous medium equation (5.4.5). Darcy’s law for the porous medium is thus shown to hold, at least in this asymptotic sense. We later summarise more general results in this context due to Hsiao and Liu (1992). Introducing the speed of the sound via
\[ c^2 = p'(\rho) = k \gamma p^\gamma. \] (5.4.6)
into (5.4.1) and (5.4.2) we get
\[ \left( c^2 \right)_t + \nabla (c^2) \cdot \bar{u} + (\gamma - 1)c^2 \nabla \cdot \bar{u} = 0, \] (5.4.7)
\[ \bar{u}_t + (\bar{u} \cdot \nabla) \bar{u} + (\gamma - 1)^{-1} \nabla (c^2) = -\alpha \bar{u}. \] (5.4.8)

To motivate the form of the special solution in the neighbourhood of the vacuum \( \rho = c = 0 \), one may observe that the trajectory of the vacuum front,
\[ \Gamma \equiv \{ (\bar{x}, t) : \rho(\bar{x}, t) \geq 0 \} \cap \{ (\bar{x}, t) : \rho(\bar{x}, t) = 0 \}, \] (5.4.9)
is a particle line along which
\[ \frac{d\bar{x}}{dt} = \bar{u}(\bar{x}(t), t). \] (5.4.10)

Equations (5.4.8) in the direction (5.4.10) may be written as
\[ \frac{d\bar{u}}{dt} + \alpha \bar{u} = - (\gamma - 1)^{-1} \nabla (c^2). \] (5.4.11)

Because we assume that \( d\bar{u}/dt \) is finite, (5.4.11) suggests that we may write
\[ c^2(\bar{x}, t) = \eta(\bar{x}, t) |\bar{x}(t) - \bar{x}|, \] (5.4.12)
where the function \( \eta(\bar{x}, t) \) is differentiable right up to the vacuum front \( \Gamma \) (see (5.4.9)). Equation (5.4.12) implies that \( c(\bar{x}, t) \approx |\bar{x}(t) - \bar{x}|^{1/2} \) and therefore the characteristic speeds of the system (5.4.7) and (5.4.8) are not Lipschitz continuous near the vacuum front. Using (5.4.12) we may write
\[ \rho(\bar{x}, t) \approx \frac{1}{\gamma} |\bar{x}(t) - \bar{x}|^{(\gamma - 1)}, \]
\[ p(\bar{x}, t) \approx \frac{1}{\gamma} |\bar{x}(t) - \bar{x}|^{(\gamma - 1)}. \] (5.4.13)

It turns out that the singularities of the system (5.4.7) and (5.4.8) are the same as were first observed by Barenblatt (1953) for the porous medium equation (5.4.5). Restricting himself to the plane and spherically symmetric flows with \( n = 1 \) and \( n = 3 \), respectively, Liu (1996) specialised (5.4.7) and (5.4.8) as

\[ (c^2)_t + u(c^2)_x + (\gamma - 1)c^2 u_x + \frac{n-1}{x} (\gamma - 1)c^2 u = 0, \] (5.4.14)
\[ u_t + uu_x + \frac{1}{\gamma - 1} (c^2)_x + au = 0, \] (5.4.15)

\[ x = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}, \quad \bar{u} = (\bar{x}/x) u \] and sought their solutions in the form (see Sachdev 2004)

\[ \rho(x, t) = 0, \quad |x| > \left( \frac{e(t)}{b(t)} \right)^{1/2}, \] (5.4.16)
\[ c^2(x,t) = e(t) - b(t)x^2, \] (5.4.17)
\[ u(x,t) = a(t)x, \] (5.4.18)

where the particle velocity is linear in \( x \) and the sound speed square is quadratic. Substituting (5.4.17) and (5.4.18) into (5.4.14) and (5.4.15) and equating coefficients of \( x^i, i = 0, 1, 2 \) to zero we obtain the following system of ODEs for the functions \( e(t), b(t), \) and \( a(t) \).

\[ e' + n(\gamma - 1)ea = 0, \] (5.4.19)
\[ b' + (n\gamma - n + 2)ab = 0, \] (5.4.20)
\[ a' + a^2 + \alpha a - \frac{2}{\gamma - 1}b = 0. \] (5.4.21)

A phase plane analysis of (5.4.20) and (5.4.21) shows that the functions \( a(t) \) and \( b(t) \) and hence \( e(t) \) exist for all time. Thus we find that the solutions of (5.4.14) and (5.4.15) of the form (5.4.17) and (5.4.18) with given \( a(0), b(0), \) and \( e(0) \) exist for all \( t > 0 \). There also exist travelling wave solutions of (5.4.14) and (5.4.15) with \( n = 1 \) of the simple form

\[ c^2(x,t) = D(e(t) - x), \quad x < e(t), \] (5.4.22)
\[ u(x,t) = a(t), \] (5.4.23)

where \( D \) is a constant. Here the functions \( a(t) \) and \( e(t) \) are governed by
\[
a' + \alpha a = \frac{D}{\gamma - 1}, \quad e' = a. \tag{5.4.24}
\]

Equations (5.4.24) admit the exact solution
\[
a(t) = a(0)e^{-\alpha t} + \frac{D}{\alpha(\gamma - 1)} (1 - e^{-\alpha t}), \tag{5.4.25}
\]
\[
e(t) = e(0) + \left( \frac{a(0)}{\alpha} - \frac{D}{\alpha^2(\gamma - 1)} \right) (1 - e^{-\alpha t}) + \frac{D}{\alpha(\gamma - 1)} t. \tag{5.4.26}
\]

This class of solutions tends, as \( t \to \infty \), to the simple travelling waveform
\[
c^2(x, t) = (\gamma - 1)u_0\alpha(u_0 t - x),
\]
\[
u(x, t) = u_0, \tag{5.4.27}
\]
where \( x < u_0 t \) and \( u_0 = D/(\alpha(\gamma - 1)) > 0 \). It turns out that they are also the travelling wave solutions of the porous medium equation (5.4.5).

We may now obtain the corresponding solution of the basic system under Darcy’s law. Thus, (5.4.4) in one dimension implies that
\[
\frac{1}{\gamma - 1}(c^2)_x + \alpha u = 0. \tag{5.4.28}
\]

The ansatz
\[
c^2(x, t) = \bar{e}(t) - \bar{b}(t)x^2, \tag{5.4.29}
\]
and
\[
u(x, t) = \bar{a}(t)x \tag{5.4.30}
\]
in (5.4.14) and (5.4.28) and so on lead to the system of ODEs
\[
\bar{e}' + n(\gamma - 1)\bar{e}\bar{a} = 0, \tag{5.4.31}
\]
\[
\bar{b}' + (n\gamma - n + 2)\bar{a}\bar{b} = 0, \tag{5.4.32}
\]
\[
\alpha\bar{a} = \frac{2\bar{b}}{\gamma - 1}. \tag{5.4.33}
\]
Equation (5.4.33) follows from Darcy’s law (5.4.28) and is, therefore, called Darcy’s line. The solution of the system (5.4.31)–(5.4.33) leads to Barenblatt’s form (5.4.29) and (5.4.30) with
\[
\bar{a}(t) = \frac{1}{n\gamma - n + 2}t^{-1}, \tag{5.4.34}
\]
\[
\bar{b}(t) = \frac{(\gamma - 1)\alpha}{2(n\gamma - n + 2)}t^{-1}, \tag{5.4.35}
\]
\[
\bar{e}(t) = e_0t^{-n(\gamma - 1)/(n\gamma - n + 2)}, \tag{5.4.36}
\]
where we assume that the solution passes through the point \( x = 0 \) at \( t = 0 \); thus \( \tilde{a}(0) = \tilde{b}(0) = \infty \). The (positive) constant \( e_0 \) may be related to the total mass

\[
m = \Omega_{n-1} \int_{0}^{\sqrt{\frac{\bar{e}(t)/\bar{b}(t)}{\bar{e}(t)/\bar{b}(t)}}} \rho(x,t)x^{n-1}dx
\]

\[
= \Omega_{n-1}(k\gamma)^{-1/(\gamma-1)} \int_{0}^{\sqrt{\frac{\bar{e}(t)/\bar{b}(t)}{\bar{e}(t)/\bar{b}(t)}}} \left( c^2(x,t) \right)^{1/(\gamma-1)}x^{n-1}dx
\]

\[
= \Omega_{n-1}(k\gamma)^{-1/(\gamma-1)} \int_{0}^{\sqrt{\frac{\bar{e}(t)/\bar{b}(t)}{\bar{e}(t)/\bar{b}(t)}}} \left( \bar{e}(t) - \bar{b}(t)x^2 \right)^{1/(\gamma-1)}x^{n-1}dx
\]

\[
= \Omega_{n-1}(k\gamma)^{-1/(\gamma-1)} \left( \frac{2(n\gamma-n+2)}{(\gamma-1)\alpha} \right)^{n/2} e_0 (n\gamma-n+2)/(2(\gamma-1))
\]

\[\times \int_{0}^{1} (1-y^2)^{1/(\gamma-1)}y^{n-1}dy, \quad (5.4.37)\]

where we have used (5.4.29) to determine the limit in the integral in (5.4.37) and \( \Omega_{n-1} \) equals 1 and \( 4\pi \) for \( n = 1 \) and \( n = 3 \), respectively.

To prove formally the asymptotic nature of the Darcy solution we need to define the following trajectories;

\[
\Gamma_1 : b = \frac{\gamma-1}{2} \left( a^2 + \alpha a \right), \quad (5.4.38)
\]

\[
\Gamma_2 : b = \frac{\gamma-1}{2} \alpha a; \quad (5.4.39)
\]

see (5.4.21). It follows easily from these equations that

\[
b' < 0, \quad a' > 0 \quad \text{between } b\text{-axis and } \Gamma_1, \quad (5.4.40)
\]

\[
b' < 0, \quad a' = 0 \quad \text{on } \Gamma_1, \quad (5.4.41)
\]

\[
b' < 0, \quad a' < 0 \quad \text{between } a\text{-axis and } \Gamma_1. \quad (5.4.42)
\]

It follows from (5.4.20), (5.4.21), and (5.4.39) that

\[
\frac{db}{da} = \frac{1}{2} \frac{(n\gamma-n+2)(\gamma-1)\alpha}{(\gamma-1)\alpha} \quad (5.4.43)
\]

on \( \Gamma_2 \). From these statements it may be checked that \( a(t) \) and \( b(t) \to 0 \) as \( t \to \infty \). We have already seen that all trajectories of Barenblatt’s solution move along the line \( \Gamma_2 \) (see (5.4.34), (5.4.35) and (5.4.39)). One may verify from (5.4.20) and (5.4.21) that all trajectories of this system are transversal to \( \Gamma_2 \) at a constant angle \( \theta \) given by

\[
\tan \theta = \frac{2(n\gamma-n+1)(\gamma-1)\alpha}{(n\gamma-n+2)(\gamma-1)^2\alpha^2 + 4}. \quad (5.4.44)
\]

Now we formally show that the trajectories of the system (5.4.20) and (5.4.21) obey the asymptotic law
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\[
\frac{b(t)}{a(t)} \to \frac{\gamma - 1}{2} \alpha \text{ as } t \to \infty; \tag{5.4.45}
\]

that is, they tend to the Darcy line \( \Gamma_2 \) as \( t \to \infty \). From (5.4.20) and (5.4.21) we have

\[
\left( \frac{b}{a} \right)' = \frac{b' a - db}{a^2} = -(n\gamma - n + 1)b + \frac{\alpha b}{a} - 2 \left( \frac{b}{a} \right)^2. \tag{5.4.46}
\]

Now, for a small \( \varepsilon > 0 \), consider the curve

\[
\Gamma_\varepsilon : -(n\gamma - n + 1)b^2 + \alpha b a - 2 \frac{b^2}{\gamma - 1} = \varepsilon a^2 \tag{5.4.47}
\]

(cf. (5.4.46)). The curve \( \Gamma_\varepsilon \) lies below \( \Gamma_2 \) and has the slope

\[
\frac{\gamma - 1}{2} \alpha + \frac{\gamma - 1}{4} \left( \alpha^2 - \frac{8}{\gamma - 1} \varepsilon \right)^{1/2} - \alpha > \frac{\gamma - 1}{2} \alpha - \frac{2\varepsilon}{\alpha} \tag{5.4.48}
\]

at the origin \((0,0)\). This slope tends to \((\gamma - 1)/2\alpha\), the slope of \( \Gamma_2 \), as \( \varepsilon \to 0 \). It is also clear from (5.4.46) that, below \( \Gamma_\varepsilon \), \( (b/a)' > \varepsilon \). Therefore, any trajectory can remain below \( \Gamma_\varepsilon \) for a finite time only. Because \( \varepsilon \) is arbitrary, we may choose \( \varepsilon = 0 \). Thus all trajectories approach \( \Gamma_2 \) as \( t \to \infty \).

Finally, we show that the solutions (5.4.17)--(5.4.21) of (5.4.14) and (5.4.15) with total mass \( m \) given by (5.4.37) tend to the special solutions (5.4.29) and (5.4.30) together with (5.4.34)--(5.4.36) of the porous medium equations (5.4.4) and (5.4.5):

\[
(a, b, e)(t) = (\bar{a}, \bar{b}, \bar{e})(t) + O(1) \ln \frac{t}{t} \tag{5.4.49}
\]

as \( t \to \infty \). Here, the bound \( O(1) \) is independent of \( t \geq 1 \) but varies with the trajectories of (5.4.19)--(5.4.21). Using (5.4.45) in (5.4.20) we find that

\[
b' + \frac{2(n\gamma - n + 2)}{\alpha(\gamma - 1)} (1 + o(1)) b^2 = 0, \tag{5.4.50}
\]

where the term \( o(1) \) tends to zero as \( t \to \infty \). It follows from (5.4.50) that

\[
b(t) = D(t)t^{-1}, \quad b'(t) = O(1)t^{-2} \tag{5.4.51}
\]

for some function \( D(t) \) which is positive and bounded away from zero. Introducing

\[
f = a - \frac{2}{\alpha(\gamma - 1)} b, \tag{5.4.52}
\]

we may write

\[
f' + \alpha f + \frac{2}{\alpha(\gamma - 1)} b' = -a^2 = O(1)t^{-2}, \tag{5.4.53}
\]

where we have used (5.4.45) and (5.4.51). Thus, we have
\[ f(t) = a(t) - \frac{2}{\alpha(\gamma - 1)} b(t) = O(1)t^{-2} = O(1)b(t)t^{-1}. \]  
(5.4.54)

It follows that

\[ |a(t)| + |b(t)| = O(1)t^{-1} \]  
(5.4.55)

and

\[ \left| a(t) - \frac{2}{\alpha(\gamma - 1)} b(t) \right| = O(1)b(t)t^{-1} \]  
(5.4.56)

as \( t \to \infty \).

Using the above estimates we may compare the solution \( b(t) \) of (5.4.20) and \( \bar{b}(t) \) of (5.4.32):

\[ b' + \frac{2(n\gamma - n + 2)}{\alpha(\gamma - 1)} (1 + O(1)t^{-1}) b^2 = 0 \]  
(5.4.57)

or

\[ b(t) = \left[ b(t_0)^{-1} + \frac{2(n\gamma - n + 2)}{\alpha(\gamma - 1)} \int_{t_0}^t (1 + O(1)s^{-1}) ds \right]^{-1} \]

\[ = \bar{b}(t) \left( 1 + O(1)\frac{\ln t}{t} \right). \]  
(5.4.58)

Substituting (5.4.56) and (5.4.58) into (5.4.19), we get

\[ e' + n(\gamma - 1)e \bar{a} \left( 1 + O(1)\frac{\ln t}{t} \right) = 0 \]  
(5.4.59)

which, on integration, gives

\[ e(t) = e(t_0)e^{-\int_{t_0}^t n(\gamma - 1)\bar{a}(s)ds} e^{-\int_{t_0}^t O(1)\left(\ln s/s^2\right)ds} \]

\[ = A\bar{e}(t) \left( 1 + O(1) \int_{t_0}^t \frac{\ln s}{s^2} ds \right) \]

\[ = A\bar{e}(t) \left( 1 + O(1)\frac{\ln t}{t} \right). \]  
(5.4.60)

for some constant \( A \). Because (5.4.1)–(5.4.3) as well as the porous medium equation (5.4.5) satisfy the same conservation of mass law

\[ m = \int_{-\infty}^{\infty} \rho(x,t)dx, \]  
(5.4.61)

it follows from (5.4.37), (5.4.58), and (5.4.60) that \( A = 1 \). Thus, the proof of (5.4.49) is complete.

In a related study, Hsiao and Liu (1992) considered the system (5.4.1) and (5.4.2) with \( n = 1 \), expressed in Lagrangian coordinates, namely,

\[ v_t - u_x = 0, \]  
(5.4.62)
and approximated it according to Darcy’s law. Thus, we have

\[ v_t = -\frac{1}{\alpha} \frac{\partial}{\partial x}(p(v))_{xx}, \]  
(5.4.64)

\[ (p(v))_x = -\alpha u. \]  
(5.4.65)

Hsiao and Liu (1992) again attempted to prove the time asymptotic equivalence of the systems (5.4.62), (5.4.63) and (5.4.64), (5.4.65) by referring to their solutions subject to a class of initial conditions with the same behaviour at \( x = \pm \infty \).

Specifically, equations (5.4.62) and (5.4.63) were first solved subject to initial conditions which satisfy the conditions

\[ (u, v)(x, 0) \to (u_{\pm}, v_{\pm}) \text{ as } x \to \pm \infty, \]  
(5.4.66)

The solution \( \bar{v}(x, t) \) of (5.4.64) must also satisfy the end conditions

\[ \bar{v}(\pm \infty, t) = v_{\pm}. \]  
(5.4.67)

The corresponding values for \( \bar{u}(\pm \infty, t) \) may be obtained from \( -(1/\alpha) (p(\bar{v}))_x \) (see (5.4.65)). Hsiao and Liu (1992) made use of the self-similar solution of (5.4.64) of the form

\[ v^*(x, t) = \phi \left( \frac{x}{\sqrt{t}} \right) \equiv \phi(\xi), \]  
(5.4.68)

which must also satisfy the conditions \( \phi(\pm \infty) = v_{\pm} \). This problem is known to have a unique solution, which is also strictly monotonic (Van Duyn and Peletier 1977).

First it was shown that the solutions of system (5.4.62) and (5.4.63) as well as those of (5.4.64) and (5.4.65) with the same limiting values of the initial conditions at \( x \to \pm \infty \) satisfy the asymptotic behaviour

\[ ||v(x, t) - \bar{v}(x + x_0, t)||_{L_2(x)} + ||v(x, t) - \bar{v}(x + x_0, t)||_{L_{\infty}(x)} = O(1)t^{-1/2} \]  
(5.4.69)
as \( t \to \infty \), where the translation \( x_0 \) in the solution of the porous medium equation is uniquely given by

\[ \int_{-\infty}^{\infty} [v(x, 0) - \bar{v}(x + x_0, 0)] dx = \frac{u_+ - u_-}{-\alpha}. \]  
(5.4.70)

The corresponding solution \( u \) tends to \( \bar{u} \) in the following sense. Defining any smooth function \( m_0(x) \) with compact support and requiring that \( \int_{-\infty}^{\infty} m_0(x) dx = 1 \), one may write

\[ m(x, t) = -\frac{u_+ - u_-}{\alpha} m_0(x) e^{-\alpha t}. \]  
(5.4.71)

Then it was shown by Hsiao and Liu (1992) that \( u(x, t) \) tends to \( \bar{u}(x, t) \) in the following sense;
\[ \parallel (u - \bar{u} - \hat{u})(x, t) \parallel_{L_2(x)} + \parallel (u - \bar{u} - \hat{u})(x, t) \parallel_{L_\infty(x)} = O(1)t^{-1/2} \]  
(5.4.72)

as \( t \to \infty \), where

\[ \hat{u}(x, t) = u_- e^{-\alpha t} + \int_{-\infty}^x m_t(\eta, t) d\eta. \]  
(5.4.73)

For the special case when the end conditions \( v_+ \) and \( v_- \) coincide, one may choose \( \bar{v} \) to be a multiple of the heat kernel, namely,

\[ \bar{v} = v_- + \frac{1}{\sqrt{4\pi t}} e^{-(x-p(\eta_0)t)^2/4t} \int_{-\infty}^x v(y, 0) dy. \]  
(5.4.74)

If, furthermore, \( u_+ = u_- \), then (5.4.71) shows that \( m \equiv 0 \). Moreover, if \((u_+, v_+ = (u_-, v_-) = (0, 0)\) and

\[ \int_{-\infty}^\infty v(y, 0) dy = 0, \]  
(5.4.75)

then \( \bar{u} = \bar{v} = 0, m = 0 \) and the results of Hsiao and Liu (1992) reduce to those of Matzumura (1978). The analysis of Hsiao and Liu (1992) involves energy methods and uses the decay estimates for the self-similar solution of the parabolic equation (5.4.64). The main contribution of the present study is again to show how the damping term in the basic hyperbolic system (5.4.62) and (5.4.63) produces diffusive effects as \( t \) becomes large.

In an interesting study, Gallay and Raugel (1998) studied the large time behaviour of small solutions of the damped nonlinear wave equation

\[ \epsilon u_{\tau\tau} + u_{\tau} = (a(\xi)u_\xi)_{\xi} + \mathcal{N}(u, u_\xi, u_\tau), \quad \xi \in \mathbb{R}, \quad \tau \geq 0, \quad \epsilon > 0. \]  
(5.4.76)

Here \( \epsilon > 0 \) need not be small. They assumed that (i) \( a(\xi) \to a_\pm > 0 \) as \( \xi \to \pm \infty \) and (ii) \( \mathcal{N} \to 0 \) sufficiently fast as \( u \to 0 \). They showed that the large time asymptotic expansion in powers of \( \tau^{-1/2} \) (up to second order) of the solution of (5.4.76) is given by a linear parabolic equation. This parabolic equation depends on \( a_+ \) and \( a_- \) only. We also refer to Nishihara (1996, 1997), Gallay and Raugel (2000), and Gallay and Wayne (2002) for some more related studies on large time asymptotics.

### 5.5 Large time behaviour of solutions of unsteady boundary layer equations for an incompressible fluid

We studied gas dynamic equations with damping for an isentropic flow in Section 5.4 and arrived at their asymptotic behaviour which was shown to be governed by a nonlinear diffusive equation. The damping effect in these hyperbolic equations over long time manifests itself as diffusion. In the present section and the next we take up the study of boundary layer equations in two steps. First we show, following closely the work of Oleinik (1966a), how under appropriate conditions the horizontal particle velocity for the unsteady flow in the boundary layer tends to that for
a steady flow as time tends to infinity. Later, in Section 5.6, we study the asymptotic behaviour of steady boundary layer equations, subject to a class of initial and boundary conditions and demonstrate that their solution tends to a self-similar solution governed by a solution of the Falkner–Skan equation, a third-order nonlinear ODE subject to ’reduced’ boundary conditions, as the horizontal distance becomes large. In the second asymptotic study (Section 5.6) it is the large distance behaviour (Serrin 1967) with which we are concerned. Thus, we study the asymptotic behaviour of unsteady boundary layer equations in two steps: first for large time and then for large horizontal distance. The latter is governed by a self-similar solution of the steady system. How such a solution actually evolves remains to be investigated numerically.

First we pose the initial boundary value problem and boundary value problem for the unsteady and steady boundary layer equations, respectively.

The unsteady two-dimensional viscous incompressible flow is governed by the system

\begin{align}
    u_t + uu_x + vu_y &= -\frac{1}{\rho} p_x + \nu u_{yy}, \\
    u_x + v_y &= 0,
\end{align}

which holds in the domain \( D := \{0 \leq t < \infty, \ 0 \leq x \leq x_0, \ 0 \leq y < \infty\} \). The initial and boundary conditions for this flow are taken to be

\begin{align}
    u|_{t=0} &= u_0(x, y), \\
    u|_{y=0} &= 0, \ v|_{y=0} = v_0(t, x), \ u|_{x=0} = u_1(t, y), \ \lim_{y \to \infty} u(t, x, y) = U(t, x). \quad (5.5.4)
\end{align}

The pressure \( p(t, x) \) and the ‘free stream velocity’ \( U(t, x) \) are related by Bernoulli’s law

\[ -\frac{1}{\rho} p_x = U_t + U u_x. \quad (5.5.5) \]

It is assumed that the functions \( p(t, x), \ U(t, x), \) and \( v_0(t, x) \) tend, respectively, to \( p^\infty(x), \ U^\infty(x), \) and \( v_0^\infty(x) \) uniformly in \( x \) as \( t \to \infty \). Similarly, the horizontal velocity \( u|_{x=0} = u_1(t, y) \) is equal to \( u_1^\infty(y) \) for \( t > t_1 \geq 0 \). It is also assumed that the system (5.5.1)–(5.5.5), subject to the limiting behaviour of the boundary and initial conditions stated above, has a solution for which \( u > 0 \) for \( 0 \leq y < \infty \), and \( u(t, x, y) \) and \( u_t \) have continuous and bounded first-order derivatives with respect to \( t, x, \) and \( y \) in \( D \). Besides, \( u_{yyy} \) and \( v_y \) must exist and satisfy the condition

\[ \left[ u_{yyy} u_y - (u_{yy})^2 \right] (u_y)^{-3} < K \quad (5.5.6) \]

in \( D \), where \( K \) is some constant. It was shown by Oleinik (1963a, 1966b) that these assumptions hold physically provided \( x_0 \) in the definition of the domain \( D \) is chosen to be sufficiently small. We refer to Oleinik (1966b) for proof of the existence and uniqueness of the solution of the system (5.5.1)–(5.5.5)).
Oleinik (1966a) then posed the boundary value problem for the steady form of (5.5.1)–(5.5.2), namely,

\[ u u_x + v u_y = -\frac{1}{\rho} p_x^\infty + v u_{yy}, \]  
(5.5.7)

\[ u_x + v_y = 0, \]  
(5.5.8)

which hold in the region \( D^\infty := \{ 0 \leq x \leq x_0, 0 \leq y < \infty \} \). The superscript ‘\( \infty \)’ denotes the limiting form of the relevant entity as \( t \to \infty \). The initial and boundary conditions relevant to the system (5.5.7) and (5.5.8) are

\[ u|_{y=0} = 0, \quad v|_{y=0} = v_0^\infty(x), \quad u|_{x=0} = u_1^\infty(y), \quad \lim_{y \to \infty} u(x,y) = U^\infty(x). \]  
(5.5.9)

The conditions (5.5.9) are assumed to be compatible with those for the unsteady case in the limit \( t \to \infty \). It is assumed that the system (5.5.7)–(5.5.9) has a solution \( u^\infty(x,y) \) and \( v^\infty(x,y) \) for which \( u^\infty_y > 0, 0 \leq y < \infty \) and the functions \( u^\infty(x,y) \) and \( u^\infty_y(x,y) \) have continuous and bounded derivatives of first-order with respect to \( x \) and \( y \) in \( D^\infty \). It is further assumed that the derivatives \( u^\infty_{yyy} \) and \( v^\infty_y \) exist. Indeed, existence of the solution of (5.5.7)–(5.5.9) was proved by Oleinik (1963a) in the region \( D^\infty \) for some \( x_0 > 0 \) subject to the conditions that the limiting functions \( p^\infty(x), v^\infty_0(x), u^\infty_1(y), \) and \( U^\infty(x) \) satisfy certain smoothness requirements. The functions involved must also assume the same value at \((0,0)\) consistently and satisfy the conditions \( u^\infty_1(y) > 0 \) for \( y > 0 \) and \( U^\infty(x) > 0 \) for \( x \geq 0 \). The solution \( u^\infty(x,y) \) and \( v^\infty(x,y) \) of the problem thus formulated was shown to satisfy the conditions \( u^\infty_y > 0 \) for \( y \geq 0 \) if \( \partial u^\infty_1 / \partial y > 0 \), when \( y \geq 0 \).

The main result of Oleinik (1966a) is that the horizontal velocity \( u^\infty(x,y) \) satisfying the system (5.5.7)–(5.5.9) is the large time limit of the unsteady counterpart \( u(t,x,y) \) governed by (5.5.1)–(5.5.5):

\[ \lim_{t \to \infty} u(t,x,y) = u^\infty(x,y) \]  
(5.5.10)

for all \( x,y \) in \( D^\infty \). For this purpose, the initial/boundary conditions for the two systems must be assumed to be compatible as described in the following. The analysis for the above result was carried out in terms of new variables which effectively eliminate \( v \) from the system (5.5.1) and (5.5.2). Thus, introducing

\[ \tau = t, \quad \xi = x, \quad \eta = u(t,x,y) \]  
(5.5.11)

as the new independent variables and \( w = u_y \), as the new dependent variable, the system (5.5.1)–(5.5.2) changes to a single second-order PDE for \( w = w(\tau, \xi, \eta), \)

\[ v w^2 w_{\eta\eta} - w_\tau - \eta w_\xi - \frac{1}{\rho} p_\tau w_\eta = 0, \]  
(5.5.12)

which holds in \( \Omega = \{ 0 \leq \tau < \infty, 0 \leq \xi \leq x_0, 0 \leq \eta \leq U(\tau, \xi) \} \). The conditions (5.5.3) and (5.5.4) now become
\[ w|_{\tau=0} = (u_0)_y = w_0(\xi, \eta), \quad w|_{\xi=0} = (u_1)_y = w_1(\tau, \eta), \quad w|_{\eta=U(\tau, \xi)} = 0 \quad (5.5.13) \]

and (5.5.5) is replaced by the compatibility condition
\[ (v w w_{\eta} - \frac{1}{\rho} p_x - v_0 w)|_{\eta=0} = 0. \quad (5.5.14) \]

For the steady system (5.5.7) and (5.5.8), we again introduce
\[ \xi = x, \quad \eta = u(x, y) \quad (5.5.15) \]
as the independent variables and \( w = \partial u / \partial y \) as the dependent variable. We then obtain a single second-order PDE for \( w \),
\[ \nu w^2 w_{\eta\eta} - \nu w \xi + \frac{1}{\rho} (p^\infty)_x w \eta = 0 \quad (5.5.16) \]
(cf. (5.5.12)) which holds in the region \( \Omega^\infty := \{0 \leq \xi \leq x_0, 0 \leq \eta \leq U^\infty(\xi)\} \). The boundary conditions
\[ w|_{\xi=0} = \frac{\partial u^\infty}{\partial y} \equiv w_1^\infty(\eta), \quad w|_{\eta=U^\infty(\xi)} = 0, \quad (5.5.17) \]
\[ (v w w_{\eta} - \frac{1}{\rho} (p^\infty)_x - v_0^\infty w)|_{\eta=0} = 0 \quad (5.5.18) \]
(cf. (5.5.14)) apply on the boundary of the region \( \Omega^\infty \). The solution of (5.5.16) subject to the conditions (5.5.17) and (5.5.18) is referred to as \( w^\infty(\xi, \eta) \) in consonance with the notation introduced earlier.

We define the difference function
\[ V(\tau, \xi, \eta) = w(\tau, \xi, \eta) - w^\infty(\xi, \eta) \quad (5.5.19) \]
over the region \( \Omega_1 \), which is the intersection of the region \( \Omega \) with the cylinder \( \{0 \leq \tau < \infty, 0 \leq \xi \leq x_0, 0 \leq \eta \leq U^\infty(\xi)\} \). The functions \( w \) and \( w^\infty \) are governed by the systems (5.5.12)–(5.5.14) and (5.5.16)–(5.5.18), respectively. Using (5.5.12) and (5.5.16), we get the following equation governing \( V \),
\[ \nu (w^\infty)^2 V_{\eta\eta} - V_{\tau} - \eta V_{\xi} + \frac{1}{\rho} (p^\infty)_x V_{\eta} + \nu (w + w^\infty) w_{\eta} V = \Phi(\tau, \xi, \eta), \quad (5.5.20) \]
where
\[ \Phi(\tau, \xi, \eta) = \frac{1}{\rho} (p^\infty_x - p_x) w_{\eta}. \quad (5.5.21) \]
Referring to (5.5.13), (5.5.14) and (5.5.17), (5.5.18), we infer that \( V(\tau, \xi, \eta) \) must satisfy the following conditions.
\[ V|_{\tau=0} = w_0(\xi, \eta) - w^\infty(\xi, \eta), \quad V|_{\xi=0} = w_1(\tau, \eta) - w^\infty_1(\eta), \quad (5.5.22) \]
\[ (vw^\infty V_\eta + (vw_\eta - v_0^\infty) V) |_{\eta=0} = \Psi(\tau, \xi), \quad (5.5.23) \]

where
\[ \Psi(\tau, \xi) = \left[ \left( \frac{1}{\rho} p_x - \frac{1}{\rho} p_\xi^\infty \right) + (v_0 - v_0^\infty) w \right] |_{\eta=0}. \quad (5.5.24) \]

Because we have assumed that \( \frac{\partial p(t, x)}{\partial x} \to \frac{\partial p^\infty(x)}{\partial x} \) and \( v_0(t, x) \to v_0^\infty(x) \) as \( t \to \infty \)
uniformly in \( x \) and \( w_\eta \) are bounded in \( \Omega \), it follows from (5.5.21) and (5.5.24) that
\[ |\Phi(\tau, \xi, \eta)| < \varepsilon \]
and
\[ |\Psi(\tau, \xi)| < \varepsilon \]
for \( \tau > \tau_1 \), a sufficiently large number; here \( \varepsilon \) is a (small) arbitrary positive number. Moreover,
\[ V|_{\xi=0} = \frac{\partial u_1}{\partial y} - \frac{\partial u_1^\infty}{\partial y} = 0 \quad (5.5.25) \]
as \( \tau > \tau_1 \) becomes sufficiently large.

To show the convergence of the unsteady solution to the steady solution for large \( \tau \), Oleinik (1966a) introduced the function \( V_1 \) by writing
\[ V = e^{\beta \xi} \phi(\alpha \eta)V_1(\tau, \xi, \eta), \quad (5.5.26) \]
where \( \alpha, \beta > 0 \) are sufficiently large numbers chosen later; the function \( \phi(s), s \geq 0 \) is defined such that \( \phi(s) = 3 - e^s, 0 \leq s \leq 1/2 \) and \( 1 \leq \phi(s) \leq 3 \) for all \( s \). The function \( V_1 \) is shown to tend to zero as \( \tau \to \infty \) uniformly in \( \xi \) and \( \eta \), implying that \( w(\tau, \xi, \eta) \to w^\infty(\xi, \eta) \) as \( \tau \to \infty \). Substituting \( V(\tau, \xi, \eta) \) from (5.5.26) into (5.5.20) we get the following equation for \( V_1 \),
\[ L(V_1) = \nu(w^\infty)^2 V_{1\eta \eta} - V_1 \tau - \eta V_{1\xi} + \left( \frac{1}{\rho} p_\xi^\infty + 2\nu(\alpha w^\infty)^2 \frac{\phi'}{\phi} \right) V_{1\eta} + c V_1 = \Phi \frac{e^{-\beta \xi}}{\phi}, \quad (5.5.27) \]
where
\[ c = \nu(w + w^\infty) w_{\eta \eta} - \eta \beta + \frac{\alpha}{\rho} p_\xi^\infty \frac{\phi'}{\phi} + \nu(\alpha w^\infty)^2 \frac{\phi''}{\phi}. \quad (5.5.28) \]

We easily check from the definition of the function \( \phi \) that, for \( \alpha \eta < 1/2 \), we have
\(-2 < \phi' \leq -1, \phi'' \leq -1 \), and \( 1 \leq \phi \leq 3 \). We also observe from the nature of the solution \( w^\infty(x, y) \) of the problem (5.5.7)–(5.5.9) (see Oleinik (1963a) or Serrin (1967)) that \( w^\infty(\xi, \eta) \geq a > 0 \) in \( 0 \leq \eta \leq \delta_1 \), where \( a \) is some constant and \( \delta_1 > 0 \) is sufficiently small; besides, \( w_{\eta \eta} \) is bounded. Therefore, it follows from (5.5.28) that, if we choose \( \alpha > 0 \) sufficiently large, we have
\[ v(w + w^\infty)w_\eta - \eta \beta + 2 \frac{\alpha}{\rho} |p_\infty^\infty| - \frac{1}{3} \nu a^2 \alpha^2 < -M, \quad (5.5.29) \]

where \( M > 0 \) is arbitrary. We easily check from (5.5.28) and (5.5.29) that \( c < -M \) provided that \( \alpha \eta < 1/2 \), and \( \eta < \delta_1 \). Furthermore, we may choose \( \beta > 0 \) so large that \( c < -M \) when \( \eta > \min \left( (1/2) \alpha^{-1}, \delta_1 \right) \). Now we write the conditions on \( V_1 \) in accordance with (5.5.22), (5.5.23), and (5.5.26):

\[ V_1|_{\tau=0} = (w_0(\xi, \eta) - w_\infty(\xi, \eta)) e^{-\beta \xi} \phi, \quad V_1|_{\xi=0} = \frac{1}{\phi} (w_1(\tau, \eta) - w_\infty(\eta)), \quad (5.5.30) \]

\[ l(V_1) \equiv (\nu w_\infty V_1 - c_1 V_1)_{\eta=0} = \frac{1}{2} \Psi e^{-\beta \xi}, \quad (5.5.31) \]

where

\[ c_1 \equiv \left( \frac{1}{2} \nu \alpha w_\infty - \nu w_\eta + v_0^\infty \right)_{\eta=0}. \quad (5.5.32) \]

We now require that

\[ \alpha > \frac{2}{\nu a} \left( \max |v_0^\infty| + v_\eta |v_\eta| + 1 \right); \quad (5.5.33) \]

this ensures that \( c_1 \) defined by (5.5.32) is greater than 1. Because \( w_\infty(\xi, \eta) \to 0 \) as \( U_\infty(\xi) - \eta \to 0 \) and \( w(\tau, \xi, \eta) \to 0 \) as \( U(\tau, \xi) - \eta \to 0 \) uniformly with \( \tau \), there exists \( \kappa > 0 \) sufficiently small such that \( |U(\tau, \xi) - U_\infty(\xi)| \leq \kappa \) for \( \tau \) sufficiently large; we may then write

\[ |V| = |w - w^\infty| \leq \epsilon \text{ for } \eta > U_\infty(\xi) - \kappa, \ \tau > \tau_2, \quad (5.5.34) \]

where \( \tau_2^{-1} \) and \( \epsilon \) are sufficiently small. In view of (5.5.26) and the definition of the function \( \phi \) (see below (5.5.26)), we have

\[ |V_1| = \left| Ve^{-\beta \xi} \right|_{\phi} \leq \epsilon \text{ for } \tau > \tau_2, \ \eta > U_\infty(\xi) - \kappa. \quad (5.5.35) \]

Now consider the part of the domain \( \Omega_1 \) (see below (5.5.19)) for which \( \tau \geq \sigma \). We refer to this as \( G_\sigma \).

It is now shown that in the domain \( \Omega_1 \) we have

\[ |V_1(\tau, \xi, \eta)| \leq \delta + M_1 e^{-\gamma \tau}, \quad (5.5.36) \]

where \( \delta > 0 \) is an arbitrary given number, \( \gamma > 0 \) is a constant less than \( M \), and \( M_1 > 0 \) is a constant which depends on \( \delta \) and \( \gamma \). We consider the functions \( W_\pm \) in \( G_\sigma \) defined by

\[ W_+ = \delta + M_1 e^{-\gamma \tau} + V_1, \quad W_- = \delta + M_1 e^{-\gamma \tau} - V_1, \quad (5.5.37) \]
where $\delta$ and $\gamma$ have been defined earlier; $M_1$ is chosen presently. Using (5.5.27) we check that

$$L(W_\pm) = cM_1 e^{-\gamma \tau} + \gamma M_1 e^{-\gamma \tau} + c \delta \pm \Phi e^{-\beta \xi} \frac{1}{\phi}.$$  

(5.5.38)

Because we have ensured that $c$ defined by (5.5.28) is negative and less than $-M$ and $\gamma < M$ and because $|\phi^{-1} \Phi e^{-\beta \xi}| \to 0$ as $\tau \to \infty$ uniformly in $\xi$ and $\eta$, we have

$$\pm \Phi e^{-\beta \xi} \phi^{-1} + c \delta < 0$$  

when $\delta > 0$  

(5.5.39)

provided that $\tau$ is chosen to be sufficiently large. Equation (5.5.38) now shows that $L(W_\pm)$ and $L(W_-)$ are both negative in $G_\sigma$ if $\sigma$ is chosen sufficiently large. It follows that $W_\pm$ cannot have a negative minimum in the region $G_\sigma$ or when $\xi = x_0$, and also on $\tau = \tau_3$, where $\tau_3 > \sigma$, if we consider $W_\pm$ over $0 < \tau < \tau_3$.

Now we show that $W_\pm \geq 0$ in $G_\sigma$. Here, we have (see (5.5.31))

$$l(W_\pm) = -c_1 \left( \delta + M_1 e^{-\gamma \tau} \right) \pm \frac{1}{2} \Psi e^{-\beta \xi} < 0,$$  

(5.5.40)

if we choose $\tau > \tau_4$ where $\tau_4$ is sufficiently large. The inequality in (5.5.40) follows from the fact that $c_1 > 1$ and $|\Psi e^{-\beta \xi}| \to 0$ as $\tau \to \infty$ uniformly in $\xi$. Thus, $W_\pm$ cannot have a negative minimum when $\eta = 0$ and $\tau > \tau_4$. We may choose $M_1$ large enough to ensure that both $W_\pm$ and $W_-$ are positive when $\tau = \sigma$ where $\sigma > \max(\tau_2, \tau_4)$.

Thus we have shown that, provided $\sigma$ is sufficiently large, $W_\pm = \pm V_1 + M_1 e^{-\gamma \tau} + \delta \geq 0$ over $G_\sigma$; that is,

$$|V_1| \leq \delta + M_1 e^{-\gamma \tau} \text{ in } G_\sigma.$$  

(5.5.41)

Now we may choose $M_1$ larger, if necessary, to ensure that (5.5.41) holds for all of $\Omega_1$; hence the equality (5.5.36) follows. We, therefore, conclude from (5.5.19), (5.5.26), and (5.5.41) that, because $\delta$ is arbitrary, $w(\tau, \xi, \eta) \to w^\infty(\xi, \eta)$ uniformly in $\xi$ and $\eta$ as $\tau \to \infty$.

Now, we prove that

$$\lim_{t \to \infty} u(t, x, y) = u^\infty(x, y)$$  

(5.5.42)

for all $x, y$ in $D^\infty$. From (5.5.4), (5.5.9), and (5.5.13), we have

$$|U^\infty(x) - u^\infty(x, y)| < \varepsilon,$$  

(5.5.43)
\[ |U(t,x) - u(t,x,y)| < \epsilon, \quad (5.5.44) \]

for \( y > y_1 \) and \( w(\tau, \xi, \eta) \to 0 \) when \( U(\tau, \xi) - \eta \to 0 \) uniformly with respect to \( \tau \). Therefore,

\[ |u^\infty(x,y) - u(t,x,y)| \leq |u(t,x,y) - U(t,x)| + |U(t,x) - u^\infty(x,y)| < 2\epsilon, \quad (5.5.45) \]

for \( y > y_1 \) and \( t \) sufficiently large. Thus, we obtain (5.5.42) for \( y > y_1 \). To prove (5.5.42) for \( y \leq y_1 \), we write

\[ y = \int_{0}^{u(t,x,y)} \frac{ds}{w(t,x,s)} \]
\[ y = \int_{0}^{u^\infty(x,y)} \frac{ds}{w^\infty(x,s)} \]
(see below (5.5.11)). This implies that

\[ 0 = \int_{0}^{u(t,x,y)} \frac{ds}{w(t,x,s)} - \int_{0}^{u^\infty(x,y)} \frac{ds}{w^\infty(x,s)} \]
\[ = \int_{0}^{u^\infty(x,y)} \left( \frac{1}{w} - \frac{1}{w^\infty} \right) ds + \int_{u(t,x,y)}^{u^\infty(x,y)} \frac{ds}{w(t,x,s)} \]
(5.5.47)

For \( y \leq y_1 \),

\[ U(t,x) - u(t,x,y) > \kappa_1, \quad U^\infty(x) - u^\infty(x,y) > \kappa_1, \quad (5.5.48) \]
and

\[ w(t,x,s) \geq a_1 > 0, \quad w^\infty(x,s) \geq a_1 > 0 \]
(5.5.49)

for \( s < U(t,x) - \kappa_1 \) and \( s < U^\infty(x) - \kappa_1 \), respectively. Then, by (5.5.47),

\[ u(t,x,y) - u^\infty(x,y) = w(t,x,s_1) \int_{0}^{u^\infty(x,y)} \frac{w - w^\infty}{ww^\infty} ds, \quad y \leq y_1; \]
(5.5.50)

here, \( s_1 \) lies between \( u(t,x,y) \) and \( u^\infty(x,y) \). This, in turn, implies that

\[ |u(t,x,y) - u^\infty(x,y)| \leq |w(t,x,s_1)| \left| \int_{0}^{u^\infty(x,y)} \frac{w - w^\infty}{ww^\infty} ds \right|, \]
\[ \leq |w(t,x,s_1)| \frac{1}{a^2} \int_{0}^{u^\infty(x,y)} |w - w^\infty| ds, \]
\[ \leq \delta_2 + M_2 e^{-\gamma}, \quad y \leq y_1, \quad (5.5.51) \]

where \( \delta_2 \) and \( M_2 \) are positive constants. Furthermore, inasmuch as \( \delta_2 \) is arbitrarily small, we have (5.5.42) for \( y \leq y_1 \). It follows that

\[ \lim_{t \to \infty} u(t,x,y) = u^\infty(x,y) \quad (5.5.52) \]

for all \( x \) and \( y \) in \( D^\infty \).
5.6 Asymptotic behaviour of velocity profiles in Prandtl boundary layer theory

We have shown in Section 5.5, following the work of Oleinik (1966a), how the solutions of unsteady two-dimensional boundary layer equations tend to those of the corresponding steady system. Here, we consider the asymptotic behaviour of the latter as $x \to \infty$. A most lucid exposition of asymptotics in (steady) boundary layers governed by Prandtl’s equations was given by Serrin (1967). He explained pointedly the important role played by the similarity solutions:

Perhaps the most fruitful source of information in this regard is the body of exact solutions derived under the assumption of similarity, of which the famous Blasius solution is typical. Nevertheless, in spite of the success of these particular solutions in predicting actual motions, a well known and primary problem has been present, namely, in what way are similar solutions unique or special among the totality of solutions of Prandtl’s equations? What theoretical justification can be offered for the pre-eminent role of similar solutions in boundary layer theory?

This is the question which Serrin (1967) attempted to answer in a rigorous manner. We follow his work closely in the following. Similar results for nonlinear parabolic equations have been presented in Chapter 4.

Consider a steady two-dimensional flow past a rigid wall governed by the Prandtl equations

\begin{align*}
 u_x + v_y &= 0, \quad (5.6.1) \\
 uu_x + vu_y &= UU_x + \nu u_{yy}, \quad (5.6.2)
\end{align*}

where $x$ and $y$ are coordinates in the horizontal and vertical directions: $x$ denotes the length along the wall whereas $y$, the perpendicular distance from the wall. $u$ and $v$ are velocity components in the horizontal and vertical directions, respectively. $\nu$ is the kinematic viscosity. $U = U(x) \geq 0$ represents the external streaming speed which is preassigned. It is related to the pressure $p = p(x)$ in the boundary layer via the Bernoulli relation

\begin{equation}
 \frac{dp}{dx} + UU_x = 0. \quad (5.6.3)
\end{equation}

The boundary conditions on the wall are

\begin{equation}
 u = v = 0 \quad \text{on } y = 0. \quad (5.6.4)
\end{equation}

The flow must merge with the external streaming conditions, requiring that

\begin{equation}
 u \to U(x) \quad \text{as } y \to \infty, \text{ uniformly in } x. \quad (5.6.5)
\end{equation}

One must also impose appropriate initial conditions at the leading edge:

\begin{equation}
 u(0,y) = \tilde{u}(y), \quad 0 < y < \infty. \quad (5.6.6)
\end{equation}
The function \( \tilde{u}(y) \) is assumed to be nonnegative and continuous; it tends to \( U(0) \) as \( y \to \infty \). It is also assumed that \( u \geq 0 \) in some neighbourhood of the initial line \( x = 0 \). It is known (see Nickel 1958, Oleinik 1963b) that the above initial boundary value problem for appropriate functions \( \tilde{u}(y) \) and \( U(x) \) possesses a unique solution. How does this solution behave as \( x \to \infty \)?

Serrin (1967) considered specifically the streaming flow

\[
U(x) = C(x + d)^m, \quad 0 \leq x < \infty,
\]

where \( C > 0 \) and \( d \geq 0 \) are constants.

The similarity solution for the system (5.6.1) and (5.6.2) with the streaming function (5.6.7) is given by

\[
\tilde{u}(x, y) = U(x) f'(\zeta),
\]

where \( \zeta = y/g(x), g(x) = \sqrt{v(x + d)/U(x)} \). The function \( f'(\zeta) \) depends on \( m \) and is governed by the Falkner–Skan equation

\[
f''' + \frac{m+1}{2} f'' + m(1 - f'^2) = 0.
\]

It satisfies the boundary conditions

\[
f'(0) = 0, \quad f'(\infty) = 1, \quad \text{and} \quad f(0) = 0;
\]

furthermore, \( f' \) is a monotonically increasing, concave function of \( \zeta \) for all values of \( m \) (see Coppel 1960).

The main result that was proved by Serrin (1967) demonstrates the asymptotic character of the similarity solutions governed by (5.6.9) and (5.6.10): with \( \tilde{u}(y) \) an arbitrary initial profile at \( x = 0 \), he assumed that the solution \( u(x, y) \) of the Prandtl equations (5.6.1) and (5.6.2) subject to the initial/boundary condition (5.6.4)–(5.6.6) and the streaming flow (5.6.7) has a continuous derivative \( u_y \) in \( 0 < x < \infty, 0 \leq y < \infty \). Then, Serrin (1967) showed that

\[
\left| \frac{u}{U} - f' \right| = o\left( \frac{1 + m \ln x}{x^m} \right) \quad \text{as} \quad x \to \infty, \quad \text{uniformly in} \quad y;
\]

that is, the normalised velocity component \( u/U \) tends uniformly to the (derivative of) normalised similarity solution, \( f' \), of the Falkner–Skan equation (5.6.9) subject to (5.6.10) as \( x \) tends to infinity downstream, bringing out clearly the central position of the similarity solution for this class of problems. This solution is asymptotically ‘independent’ of the motion at the leading edge \( x = 0 \).

Serrin (1967) also proved the asymptotic uniqueness of the solution; he showed that this solution is independent of the state of motion at the initial point \( x = 0 \) provided that the free stream velocity \( U \) is twice continuously differentiable and obeys the inequality
\[ C_1(x+d)^{2m-1} \leq UU_x \leq C_2(x+d)^{2n-1}, \quad 0 \leq x < \infty, \]  
(5.6.12)

where \( C_1 \) and \( C_2 \) are positive constants and \( m \) and \( n \) are exponents satisfying the condition \( m \leq n < 5m/3 \).

The main assumption in the proof is that the parameter \( m \) in (5.6.7) is greater than or equal to zero ensuring a favourable pressure gradient, \( dp/dx \leq 0 \). This follows from (5.6.3) and (5.6.7).

Because \( u \geq 0 \) in some neighbourhood of \( x = 0 \) and is continuously differentiable on \( y = 0 \) and because it is assumed that \( UU_x \geq 0 \), it follows from a theorem of Velte (1960) that \( u > 0 \) in \( 0 < x < \infty, 0 < y < \infty \) and \( u_y(x,0) > 0 \) for \( 0 < x < \infty \). Serrin (1967) also assumed the initial line to be \( x = 1 \) rather than \( x = 0 \). With this choice, he relabelled the coordinates such that the new initial position is again called \( x = 0 \). It is further assumed that \( u \) and \( u_y \) are continuous in \( 0 \leq x < \infty, 0 \leq y < \infty \) and the initial profile \( \tilde{u}(y) \) satisfies the conditions

\[ \tilde{u}(0) = 0, \quad \tilde{u}_y(0) > 0 \quad \text{and} \quad \tilde{u}(y) > 0 \quad \text{for} \quad y > 0. \]  
(5.6.13)

Inasmuch as \( u > 0 \), one may introduce the von Mises variables

\[ x = x, \quad \psi = \psi(x,y) = \int_0^y u(x,t)dt \]  
(5.6.14)

into the Prandtl system (5.6.1) and (5.6.2) and obtain

\[ (u^2)_x = \nu u(u^2)_{\psi\psi} + (U^2)_x \]  
(5.6.15)

for which the boundary and initial conditions become

\[ u = 0 \quad \text{on} \quad \psi = 0 \]  
(5.6.16)

and \( u \to U \) as \( \psi \to \infty \), uniformly in \( x \) on any finite interval \( 0 \leq x \leq A \). Moreover,

\[ u(0,\psi) = \tilde{u}(\psi) \quad (0 \leq \psi < \infty), \]  
(5.6.17)

where \( \tilde{u}(\psi) \) is the transformed initial condition (see (5.6.4)–(5.6.6)) (see Schlichting 1960).

Serrin (1967) proved his main result via several lemmas. We discuss them here informally.

Let \( u \) and \( \tilde{u} \) be two solutions of the boundary layer equations as above with free stream speeds \( U(x) \) and \( \tilde{U}(x) \) and initial conditions \( \tilde{u}(\psi) \) and \( \tilde{u}(\psi) \), respectively, such that

\[ (U^2)_x \leq (\tilde{U}^2)_x \quad \text{and} \quad \tilde{u}(\psi)^2 \leq \tilde{\tilde{u}}(\psi)^2 + a^2, \]  
(5.6.18)

where \( a > 0 \) is some constant. Further let either \( u_{yy} < 0 \) or \( \tilde{u}_{yy} < 0 \). Then, \( u(x,\psi) \leq \tilde{u}(x,\psi) + a \).

We consider first the case \( u_{yy} < 0 \). Because \( \tilde{u} \) satisfies

\[ (\tilde{u}^2)_x = \nu \tilde{u}((\tilde{u}^2)_{\psi\psi} + (\tilde{U}^2)_x \]  
(5.6.19)
(see (5.6.15)), the difference function $\phi(x, \psi) = \bar{u}^2 - u^2$ satisfies the equation

$$
\phi_x = v\bar{u}(\bar{u}^2)_{\psi\psi} - v\nu(u^2)_{\psi\psi} + (\bar{U}^2)_x - (U^2)_x \\
= v\bar{u}\phi_{\psi\psi} + \frac{v(u^2)_{\psi\psi}}{u + \bar{u}} \phi + (\bar{U}^2 - U^2)_x,
$$

(5.6.20)

where we have used the relation $(u^2)_{\psi\psi} = 2\nu_{yy}/u$ (see (5.6.1) and (5.6.14)) and have set $\alpha = 2\nu_{yy}/u(u + \bar{u})$. The function $\phi$ satisfies the boundary conditions

$$
\phi = 0 \text{ on } \psi = 0, \quad \phi \geq -a^2 \text{ on } x = 0
$$

(5.6.21)

(see (5.6.16) and (5.6.18)$_2$). Furthermore, $\phi \rightarrow \bar{U}^2 - U^2$ as $\psi \rightarrow \infty$, uniformly in $x$ on any finite interval. Because it is given that $(\bar{U}^2 - U^2)_x \geq 0$ and $\bar{U}^2(0) - U(0)^2 \geq -a^2$ (see (5.6.18) and (5.6.21)), it follows that $\bar{U}^2 - U^2 \geq -a^2$ for all $x$. Now it may be shown that $\phi(x, \psi) \geq -a^2$ for $0 \leq x < \infty$, $0 \leq \psi < \infty$. To that end, we assume the contrary, namely, that $\phi < -a^2$ at some point $(x_0, \psi_0)$. Consider $\phi(x, \psi)$ on $R : 0 < x \leq x_0, 0 < \psi < \infty$. It follows, from (5.6.21) and the asymptotic behaviour of $\phi$ as $\psi \rightarrow \infty$, that $\phi$ assumes its absolute minimum in $R$. Let this point of minimum be $(x_1, \psi_1)$. At this point, we have

$$
\phi_x \leq 0, \quad \phi_{\psi\psi} \geq 0, \quad \phi < 0
$$

(5.6.22)

and $\alpha = 2\nu_{yy}/u(u + \bar{u}) < 0$ (because $u_{yy} < 0$) and $(\bar{U}^2 - U^2)_x \geq 0$. This contradicts (5.6.20) because the signs on the two sides are different. Thus, we have proved that $\phi = \bar{u}^2 - u^2 \geq -a^2$. The case $\bar{u}_{yy} < 0$ may be treated similarly.

Next we show how the constants $C^*$ and $C^{**}$ in the free stream velocities

$$
U^*(x) = C^*(x + d)^m, \quad U^{**}(x) = C^{**}(x + d)^n
$$

(5.6.23)

may be chosen so that the corresponding solutions $u$ and $\bar{u}$ of the Prandtl equations are bounded by the similarity solutions: $u^* \leq u \leq u^{**}$ and $u^* \leq \bar{u} \leq u^{**}$, where

$$
u(x, y) = U^*(x)f'(\zeta),
$$

(5.6.24)

$$
u(x, y) = U^{**}(x)f'(\hat{\zeta}),
$$

(5.6.25)

$$
\zeta = \frac{y}{g^*(x)}, \quad g^*(x) = \sqrt{\frac{v(x + d)}{U^*(x)}},
$$

(5.6.26)

$$
\hat{\zeta} = \frac{y}{g^{**}(x)}, \quad g^{**}(x) = \sqrt{\frac{v(x + d)}{U^{**}(x)}}.
$$

(5.6.27)

$(U^*)_x = 2mC^*(x + d)^{2m-1}$ and $(U^{**})_x = 2nC^{**}(x + d)^{2n-1}$, thus it follows from (5.6.12) that we may satisfy the inequalities

$$
(U^*)_x \leq (U^2)_x \leq (U^{**})_x
$$

(5.6.28)
by choosing $C^*$ and $C^{**}$ appropriately. Assuming that $C^* < C^{**}$, we prove that

$$u^*(0, \psi) \leq \bar{u}(\psi), \quad \tilde{u}(\psi) \leq u^{**}(0, \psi); \quad (5.6.29)$$

the required result then follows from the statement involving (5.6.18) if we choose $a = 0$ therein. We prove the result for the function $\bar{u}(\psi)$. The other part follows in a similar manner. From the assumption (5.6.13) regarding the solution, we have for $x = 0$ and $y$ small, $\bar{u} \approx by, \psi \approx \frac{1}{2}by^2$ and, therefore, $\bar{u}^2 \approx 2b\psi$ where $b > 0$. In addition $\bar{u} > 0$ for $\psi > 0$ and $\bar{u}$ tends to $U(0)$ as $\psi$ tends to infinity.

From the similarity form of the solution we have

$$u^*_y(0, 0) = \frac{U^*(0)f''(0)}{g^*(0)} = \text{constant}.C^{*3/2}. \quad (5.6.30)$$

It follows that

$$u^{*2} \approx \text{constant } C^{*3/2}\psi \quad (5.6.31)$$

for small $\psi$. Because, by assumption, $u^*(0, \psi)$ tends to $C^*d^m$ as $\psi$ tends to infinity and $(u^{*2})_{\psi\psi} < 0$, it follows that $C^*$ can be chosen so small that $u^*(0, \psi) \leq \bar{u}(\psi)$. The similarity solution square $u^{**2}$ is known to be a concave, monotonically increasing function of $\psi$. Besides, for the flow $u^{**}$, we have from (5.6.14) and the definition of $u^{**}$,

$$\psi = U^{**}g^{**}\hat{f}(\hat{\zeta}). \quad (5.6.32)$$

Because $\hat{f}'$ is monotonic and $\hat{f}'(\infty) = 1$, one may use the mean value theorem to show that $\hat{f}'(\hat{\zeta}) < \hat{\zeta}$. Therefore, we have at $x = 0, \psi = 1$ the inequality

$$\hat{\zeta} > \hat{f}'(\hat{\zeta}) = \{U^{**}g^{**}\}^{-1} = \frac{\text{constant}}{C^{**1/2}}. \quad (5.6.33)$$

It follows that

$$u^{**} = U^{**}\hat{f}'(\hat{\zeta}) \geq \text{constant }.C^{**1/2}, \quad (5.6.34)$$

where we assume that $C^{**} > 1$. Therefore, $u^{**}$ at $x = 0, \psi = 1$ can be made arbitrarily large. Thus, the second inequality in (5.6.29) follows.

Now if we assume that $|\bar{u}(\psi)^2 - \bar{u}(\psi)^2| \leq a^2$ and $u_{\psi\psi} < 0$, we may prove $|u(x, \psi) - \bar{u}(x, \psi)| \leq a$ (see (5.6.18) and below). $a$, here, is positive.

The main proof for the asymptotic result consists of three parts. The first part requires the result that if $u_{\psi\psi} < 0$, then we have for $0 < x < \infty$,

$$y - \bar{y} \leq \frac{ag^*}{bU^*} \left(1 + \frac{a}{bU^*}\right) \ln \left(1 + \frac{bU^*}{a}\right) + \frac{a}{bU^*}\bar{y}, \quad (5.6.35)$$

where $b = f'(1)$ and $f$ is the Falkner–Skan function with the exponent $m$.

We recall that

$$y = \int_0^\psi \frac{d\psi}{u(x, \psi)}, \quad \bar{y} = \int_0^\psi \frac{d\psi}{\bar{u}(x, \psi)} \quad (5.6.36)$$
are, respectively, the \( y \) coordinates associated with the flows \( u \) and \( \bar{u} \) for a given value of \( \psi \).

Because \( u_{yy} < 0 \), we may use (5.6.36) and write

\[
y - \bar{y} = \int \left( \frac{1}{u} - \frac{1}{u} \right) d\psi \leq \int \left( \frac{1}{u} - \frac{1}{u + a} \right) d\psi = a \int \frac{d\psi}{u(u + a)}. \tag{5.6.37}
\]

Letting \( \psi_1 = U^* g^* f(1) \) and using the result \( u^* \leq u \) proved earlier (see above (5.6.24)), we write (5.6.37) as

\[
y - \bar{y} \leq a \int_0^{\psi_1} \frac{d\psi}{u^*(u^* + a)} + a \int_0^{\psi_1} \frac{d\psi}{u(u^* + a)}, \tag{5.6.38}
\]

where the second integral is absent if \( \psi \leq \psi_1 \). Because \( d\psi/u^* = g^* d\zeta \), we have

\[
\int_0^{\psi_1} \frac{d\psi}{u^*(u^* + a)} = \int_0^1 \frac{g^* d\zeta}{u^* + a} \leq \int_0^1 \frac{g^* d\zeta}{b\zeta U^* + a}, \tag{5.6.39}
\]

where we have used the fact that \( u^* = U^* f'(\zeta) \geq U^* f'(1) \zeta, 0 < \zeta < 1, f' \) is concave and \( f'(1) = b \). For the second integral in (5.6.38) we have

\[
\int_0^{\psi_1} \frac{d\psi}{u(u^* + a)} \leq \int_0^{\psi_1} \frac{d\psi}{u(bU^* + a)} \leq \int_0^{y} \frac{dy}{bU^* + a}, \tag{5.6.40}
\]

because \( u^* \geq U^* f'(1) \) for \( \zeta > 1 \). Combining (5.6.38)–(5.6.40), we have

\[
y - \bar{y} \leq \frac{a g^*}{bU^*} \ln \left( 1 + \frac{bU^*}{a} \right) + \frac{a}{bU^* + a} y. \tag{5.6.41}
\]

Rewriting (5.6.41), we have

\[
y \leq \frac{bU^* + a}{bU^*} \left\{ \frac{a g^*}{bU^*} \ln \left( 1 + \frac{bU^*}{a} \right) + \bar{y} \right\}.
\]

Using this inequality in (5.6.41) we get (5.6.35). In a similar manner, one may prove that if \( u_{yy} < 0 \), then for \( 0 < x < \infty \), we also have

\[
\bar{y} - y \leq \frac{ag^*}{bU^*} \left( 1 + \frac{a}{bU^*} \right) \ln \left( 1 + \frac{bU^*}{a} \right) + \frac{a}{bU^*} y. \tag{5.6.42}
\]

Next we show that, if \( u_{yy} < 0 \), then for \( 0 < x < \infty \),

\[
|u(x,y) - u(x,\bar{y})| \leq \frac{aU^{**}}{bU^*} \left\{ 1 + f''(0) \left( \frac{U^{**}}{U^*} \right)^{1/2} \left( 1 + \frac{a}{bU^*} \right) \ln \left( 1 + \frac{bU^*}{a} \right) \right\}, \tag{5.6.43}
\]

where \( f \) is the Falkner–Skan function associated with the exponent \( n \). Suppressing the dependence of \( u \) on \( x \) for convenience, we have
5.6 Asymptotic behaviour of velocity profiles in Prandtl boundary layer theory

\[ u(y) - u(\bar{y}) = \hat{u}_y(y - \bar{y}). \]  \hfill (5.6.44)

The function \( u \) has been assumed to be a (positive) concave function in \( y \) with \( u(0) = 0 \) and \( u(\infty) = U \). Therefore, \( u_y \) is a positive decreasing function of \( y \). Thus, for \( y \geq \bar{y} \), we have

\[ \hat{u}_y(y - \bar{y}) \leq u_y(\bar{y})(y - \bar{y}) \]  \hfill (5.6.45)

whereas for \( \bar{y} \geq y \),

\[ \hat{u}_y(y - \bar{y}) \leq 0. \]  \hfill (5.6.46)

In view of (5.6.35) we have for both the cases (5.6.45) and (5.6.46) the inequality

\[ \hat{u}_y(y - \bar{y}) \leq \frac{ag^* u_y(\bar{y})}{bU^*} \left(1 + \frac{a}{bU^*}\right) \ln \left(1 + \frac{bU^*}{a}\right) + \frac{a}{bU^*} \bar{y}u_y(\bar{y}). \]  \hfill (5.6.47)

Because \( u_{yy} < 0 \), we easily check that

\[ \bar{y}u_y(\bar{y}) \leq U \quad \text{and} \quad u_y(\bar{y}) \leq u_y(0). \]  \hfill (5.6.48)

Using the result that \( u^* \leq u \leq u^{**} \) proved earlier, we have

\[ u \leq u^{**}, \quad u_y(0) \leq u_y^{**}(0) = \frac{U^{**} f''(0)}{g^{**}}. \]  \hfill (5.6.49)

From (5.6.44) and (5.6.47)–(5.6.49) it follows that \( u(y) - u(\bar{y}) \) is bounded by the RHS of (5.6.43).

To obtain the reverse inequality we observe that

\[ u(\bar{y}) - u(y) = \hat{u}_y(\bar{y} - y) \leq \max(0, u_y(y)(\bar{y} - y)). \]  \hfill (5.6.50)

Now using (5.6.42), we obtain (5.6.43) as for the previous case.

In pursuit of the final result, we further show that, if \( u_{yy} < 0 \), then

\[ \left| \frac{u - \bar{u}}{U} \right| = O \left\{ \frac{a(1 + m \ln x + \ln_+ 1/a)}{x^{(5m - 3n)/2}} \right\} \quad \text{as} \; x \to \infty, \; \text{uniformly in} \; y. \]  \hfill (5.6.51)

We observe that, for any positive number \( \bar{y} \),

\[ \left| u(x, \bar{y}) - \bar{u}(x, \bar{y}) \right| \leq |u(x, \bar{y}) - u(x, y)| + |u(x, y) - \bar{u}(x, \bar{y})|. \]  \hfill (5.6.52)

Here \( y \) is such that

\[ \psi = \int_0^y u(x, t)dt = \int_0^\bar{y} \bar{u}(x, t)dt. \]

We have already shown (see above (5.6.35)) that the second term on the RHS of (5.6.52) is less than or equal to \( a \). The first term therein can be estimated by using (5.6.43). Here, one employs the simple result that \( (1 + r^{-1}) \ln(1 + r) \leq 2 \ln 2 + \ln_+ r \) for any nonnegative number \( r \); \( r \) in (5.6.43) is \( bU^*/a \). \( U^* \) and \( U^{**} \) are explicitly given by (5.6.23). The RHS of (5.6.43) is evaluated in the limit \( x \to \infty \). The
multiplication factor implied in (5.6.51) involves \( C^* \), \( C^{**} \), and the given constants \( f'(1) \) and \( f''(0) \).

The final step in proving the estimate (5.6.11) requires the inequality

\[
\| \bar{u}(x_0, \psi) - u(x_0, \psi) \| \leq \epsilon,
\]

(5.6.53)

where \( \epsilon > 0 \) and \( x = x_0 \) is a point far downstream. This inequality requires considerable technical detail and we refer the reader to the original paper of Serrin (1967). We assume this result in the following. We let \( U(x) = C(x + d)^m \); the corresponding similarity solution is denoted by \( \bar{u} \). When \( m > 0 \), this function satisfies (5.6.12) with \( n = m \) and \( C_1 = C_2 = 2mC^2 \). Now we use (5.6.51) (where the solutions \( \bar{u} \) and \( u \) interchange but this is only a notational matter). The estimate (5.6.51) with \( \bar{u} = Uf' \) becomes

\[
\frac{u}{U} - f' = O \left( \frac{a(1 + m \ln x + \ln + 1/a)}{x^m} \right) \quad \text{as} \ x \to \infty, \quad \text{uniformly in} \ y.
\]

(5.6.54)

We assume that suitably downstream at \( x = x_0 \), say, (5.6.53) holds and then take \( x_0 \) as the new initial position. \( C^* \) and \( C^{**} \) need not be changed in this process. Now we let \( a \) in (5.6.54) be equal to \( \epsilon^{1/2} \). In the limit \( x \to \infty \), we may choose \( \epsilon \) arbitrarily small by choosing \( x_0 \) sufficiently large; the estimate (5.6.54) then reduces to (5.6.11).

We may observe that, if \( m = 0 \), the free stream speeds \( U, U^*, U^{**} \) are all constants and all the previous results continue to hold. Therefore (5.6.54) and hence (5.6.11) apply in this case too.

Our main purpose here was to briefly discuss Serrin’s (1967) work to demonstrate the central position of the similarity solution as an asymptotic as \( x \to \infty \). The second asymptotic result is also important. Let \( u \) and \( \bar{u} \) be two solutions of the Prandtl system corresponding to the same streaming speed \( U(x) \) but with different initial profiles \( \bar{u}(y) \) and \( \bar{u}(y) \). Assume further that \( U(x) \) is twice continuously differentiable and satisfies the inequality

\[
C_1(x + d)^{2m-1} \leq UU_x \leq C_2(x + d)^{2n-1}, \quad 0 \leq x < \infty,
\]

(5.6.55)

where \( C_1 \) and \( C_2 \) are positive constants; the exponents \( m \) and \( n \) obey the inequality \( m \leq n < 5m/3 \). Then

\[
\frac{\bar{u} - u}{U} = o(1) \quad \text{as} \ x \to \infty, \quad \text{uniformly in} \ y.
\]

(5.6.56)

This result proves the asymptotic uniqueness of the normalised velocity profile for arbitrary conditions at the initial point \( x = 0 \). We refer the reader to Serrin (1967) for the proof of (5.6.56).

Now we summarise two interesting and related investigations which followed Serrin (1967). Peletier (1972) essentially studied the same problem as Serrin (1967). However, there were some interesting departures. He assumed the flow to be governed by (5.6.1) and (5.6.2), where, however, the exterior streaming speed was chosen to be
Asymptotic behaviour of velocity profiles in Prandtl boundary layer theory

\[ U(x) = U_0(x + 1)^m, \]  

(5.6.57)

where \( U_0 > 0, m \geq 0 \) (cf. equation (5.6.7)). The velocity \( U(x) \) and pressure \( p(x) \) in the boundary layer are related by Bernoulli’s equation

\[ p + \frac{1}{2} \rho U^2 = \text{constant} \]

or

\[ \frac{dp}{dx} = -\rho U U_x \leq 0 \]  

(5.6.58)

for the choice (5.6.57) of the free stream velocity. Thus, the adverse pressure gradient is excluded. The boundary conditions relevant to the flow are the same as in Serrin (1967), namely, (5.6.4) and the Prandtl’s stream condition

\[ u \to U \quad \text{as} \quad y \to \infty, \quad \text{uniformly in} \quad x. \]  

(5.6.59)

As in Serrin (1967), the initial station \( x = 0 \) is chosen to be located at some distance from the leading edge where

\[ u(0, y) = u_0(y), \quad 0 < y < \infty. \]  

(5.6.60)

\( u_0 \) is assumed to be a smooth function with continuous and uniformly bounded first and second derivatives. Moreover, it is required that \( u_0(0) = 0, 0 < u'_0(0) < \infty, u_0 > 0 \) if \( y > 0 \), and \( u_0 \to U(0) \) as \( y \to \infty \). The existence of the solution of this boundary value problem was proved by Nickel (1958) and Oleinik (1963b). Nickel (1958) also proved that if \( U(x) \) and \( u_0(y) \) satisfy the conditions laid down above, then \( u > 0 \) in the entire domain, and \( u_y(x, 0) > 0 \) for \( 0 < x < \infty \). Peletier (1972) found the similarity solution of the Prandtl equations, expressed in von Mises variables, namely,

\[ u_x = \frac{1}{2} v(u^2)_{\psi\psi} + u^{-1} U U_x \]  

(5.6.61)

(cf. (5.6.15)), where \( u \) is now a function of \( x \) and \( \psi \). The boundary conditions in these variables become

\[ u(x, 0) = 0 \quad \text{for} \quad 0 < x < \infty \]  

(5.6.62)

and

\[ u \to U \quad \text{as} \quad \psi \to \infty, \quad \text{uniformly in} \quad x. \]  

(5.6.63)

The initial condition at \( x = 0 \) was imposed in the form

\[ u(0, \psi) = u_0(\psi), \quad 0 < \psi < \infty. \]  

(5.6.64)

Peletier (1972) sought the similarity solution of (5.6.61) in the form

\[ u(x, \psi) = (x + 1)^m f(\eta), \quad \eta = \psi(x + 1)^{-(m+1)/2} \]  

(5.6.65)
which would satisfy the conditions (5.6.62) and (5.6.63). This, when substituted into (5.6.61), leads to the second-order ODE

$$\frac{1}{2}\nu(f^2)'' + \frac{m+1}{2} \eta f' + m \left( \frac{U_0^2}{f} - f \right) = 0,$$

(5.6.66)

where the prime denotes differentiation with respect to $\eta$. The solution of equation (5.6.66) must satisfy the conditions

$$f(0) = 0, \quad f \to U_0 \text{ as } \eta \to \infty,$$

(5.6.67)

(see (5.6.57), (5.6.62), and (5.6.63)). Peletier (1972) also obtained asymptotic behaviour of the solution of (5.6.66) and (5.6.67) in the form

$$U_0 - f(\eta) = O \left\{ \eta^{-1-2\beta} \exp \left( -\frac{m+1}{4\nu U_0^2} \eta^2 \right) \right\} \text{ as } \eta \to \infty,$$

(5.6.68)

where $\beta = 2m/(m+1)$.

Unlike Serrin (1967) and Peletier (1972) obtained an estimate for the convergence of the velocity profile in terms of the arc length along the plate and the stream function rather than the physical variables $x$ and $y$ themselves. His main theorem may be stated as follows. Let $u(x,y)$ be a solution of the boundary layer equation (5.6.1) and (5.6.2) and let $u(0,y) = u_0(y)$ be a smooth function such that $u_0(0) = 0, 0 < u_0'(0) < \infty, u_0 > 0$ if $y > 0$, which further satisfies the following conditions as $y \to \infty$.

(i) $\delta = \int_0^\infty \left| 1 - \frac{u_0}{U_0} \right| dy < \infty, \quad (5.6.69)$

(ii) $U_0 - u_0(y) = O \left\{ (y+\delta)^{-1-2\beta} \exp \left\{ \frac{1}{4\nu} \frac{(m+1)U_0}{(y+\delta)^2} \right\} \right\} \text{ as } y \to \infty.$

(5.6.70)

(In the above, if $u_0 \leq U_0$ for $0 \leq y < \infty$, $\delta$ is the displacement thickness at $x = 0$.) Let $\bar{u}(x,y)$ be the similarity profile corresponding to the stream speed $U(x)$ (see (5.6.57) and (5.6.65)). Then,

$$\int_0^\infty \eta \left| \frac{u - \bar{u}}{U} \right| d\eta \leq M(x+1)^{-(2+\mu)m-1}, \quad x \geq 0, \quad (5.6.71)$$

where the positive constants $M$ and $\mu$ depend only on $\nu, U(x)$, and the initial velocity profile $u_0(y)$. Moreover, $\mu \leq 1$; when $u_0 \leq U_0$, we may set $\mu = 1$. It was also shown that the power of $(x+1)$ in (5.6.71) is best possible for $m = 0$.

In another related study, Khusnutdinova (1970) generalised the work of Serrin (1967) to compare solutions which correspond not only to different initial conditions at $x = 0$ but also to different external streaming flows. Specifically, he considered solutions $u^1(x,y)$ and $u^2(x,y)$ of the system (5.6.1) and (5.6.2) which arise
from two different initial conditions \( u_0^i(y)(i = 1, 2) \) and external streaming conditions \( U_i(x)(i = 1, 2) \), respectively, where

\[
\lim_{x \to \infty} U_i(x) = U_\infty = \text{constant (} i = 1, 2). \tag{5.6.72}
\]

It was shown that if \( \lim_{x \to \infty} |U_1(x) - U_2(x)| = \lim_{y \to \infty} |u_0^1(y) - u_0^2(y)| = 0 \), then for \( x \to \infty \), the difference between the solutions \( u^1(x,y) \) and \( u^2(x,y) \) tends uniformly to zero with respect to \( y \), where \( y \in [0, \infty) \). One consequence of this theorem is that the solution \( u(x,y) \) of (5.6.1)–(5.6.5) in the boundary layer converges for large \( x \) to the well known Blasius solution

\[
u_1 = U_\infty f'(\eta), \quad \eta = \frac{y\sqrt{U_\infty}}{\sqrt{2\nu(x+1)}}, \tag{5.6.73}
\]

which describes flow past a plate in the longitudinal direction at velocity \( U(x) \equiv U_\infty \).

In this case, \( f(\eta) \) is governed by the boundary value problem

\[
f''' + ff'' = 0, \quad f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1. \tag{5.6.74}
\]

The function \( f \) together with its first derivative is monotonically increasing. In this regard, the following theorem was proved by Khusnutdinova (1970). Let the following inequalities hold.

\[
0 \leq u_0(y) \leq U(0), \quad u'_0(0) > 0, \quad u_0(0) = 0, \quad 0 \leq \frac{dU}{dx} \leq \frac{M_0}{(x+1)^{\gamma_0} + 1}, \quad \gamma_0 > 0.
\]

Then, as \( x \to \infty \), \( |u(x,y) - u_1(x,y)| \to 0 \), uniformly in \( y \in [0, \infty) \); here, \( u_1(x,y) = U_\infty f'(\eta) \), where \( f(\eta) \) is the solution of the boundary value problem (5.6.74).

If, in addition, the inequalities

\[
U(0)f'(y - N) \leq u_0(y), \quad y \in [N, \infty)
\]

and

\[
|u_0(y) - U(0)| \leq M_1 \exp\left(-\gamma_1 y^2\right), \quad y \in [0, \infty)
\]

hold for some constants \( N, M_1, \) and \( \gamma_1 > 0 \), then

\[
|u(x,y) - u_1(x,y)| \leq \frac{M}{(x+1)^{\gamma}}, \tag{5.6.75}
\]

where \( M \) and \( 0 < \gamma < \gamma_0 \) are some constants which depend only on the initial data of the problem. Khusnutdinova (1970) also worked with basic equations in terms of von Mises variables.

We may observe that the external flow conditions \( U(x) \) lead to different forms of ODEs, Falkner–Skan or Blasius, governing self-similar flows, and thus characterising
different asymptotic behaviour as $x$ tends to $\infty$. Before proving the asymptotic nature of these solutions, Serrin (1967) and Khusnutdinova (1970) ensured the existence and uniqueness of the systems of PDEs and ODEs that were involved subject, of course, to the relevant initial and boundary conditions.

5.7 Conclusions

In this chapter, we have discussed the asymptotic behaviour of solutions of some physical problems arising from fluid mechanics. Section 5.1 presented the introduction to the chapter. Section 5.2 was concerned with the flow due to a strong explosion at the centre of an ideal gas sphere. It was assumed that the preshock density is $\rho_0 = kr^{-\omega}$; here $r$ is the distance from the origin and $k$ and $\omega$ are positive constants. An interesting feature of this problem is that the asymptotic flow is described by the self-similar solutions of first kind for $\omega < 3$ (Sedov–Taylor solutions) and by the self-similar solutions of the second kind for $\omega_k(\gamma) < \omega < \omega_c(\gamma)$; here $\omega_k$ and $\omega_c$ depend on the adiabatic index $\gamma$ of the gas. This section followed the work of Waxman and Shvarts (1993). Section 5.3 dealt with the self-similar solutions of the second kind which describe a collapsing spherical cavity. We showed, by following Hunter (1960), that the numerical solution of the governing system of nonlinear partial differential equations with appropriate initial/initial boundary conditions converges to the relevant self-similar solutions of the second kind, for different values of $\gamma$, as the radius of the cavity tends to zero. We have also summarised the work of Thomas et al. (1986). In Section 5.4, we have presented a study of solutions of the compressible Euler equations with damping. Following Liu (1996), we constructed a family of solutions for the compressible flow with damping. As $t \to \infty$, these solutions converge to the Barenblatt solutions of the porous medium equation. This study justifies, in a limited sense, Darcy’s law for the compressible flow for large time. In Section 5.5, we have studied, following Oleinik (1966a), the boundary layer equations for an unsteady flow of incompressible fluid. Under a certain set of conditions, it was shown that the large time behaviour of the longitudinal velocity component of the unsteady flow is described by the longitudinal velocity component of the steady flow. Section 5.6 was concerned with the study of boundary layer equations for the steady two-dimensional laminar flow of an incompressible viscous fluid past a rigid wall. Following Serrin (1967) closely, we clearly brought out the importance of similarity solutions governed by the Falkner–Skan differential equation. We have shown that the asymptotic behaviour (for large $x$) of the downstream velocity profile is described by the Falkner–Skan similarity solution when the streaming speed is $U(x) = c(x + d)^m$, $m \geq 0$.

References


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