Chapter 2
Fronts in Periodic Media

Fronts or interfaces in periodic media are deterministic problems in between homogeneous media and random media. Much can be learned on how front solutions transition from monoscale simple solutions in Chapter 1 to multiple-scale solutions. Periodic homogenization and PDE techniques based on maximum principles are essential tools for constructing front solutions and analyzing their asymptotics. We shall observe the close relationship between Hamilton–Jacobi (HJ) and reaction–diffusion (RD) equations, and present the variational principles of front speeds.

2.1 Periodic Media and Homogenization

Multiscale problems are common in applications such as finding the effective conductivity of a composite material or the effective permeability for flows in porous media, where one has at least two scales, the large scale of the sample and the small scale of the embedded inclusions or pores. These two scales normally differ significantly and render the full resolution of the problem difficult. Therefore, it is of great theoretical and practical interest to find out how to upscale the collective effect of the small scale into the large scale and simplify the problem. When the small scale possesses a periodic structure, the upscale problem has a well-developed theory called homogenization. See [18] for a systematic account of the foundational works.

We give here an example of homogenization and use formal asymptotic analysis to illustrate the ideas. Consider a two-point boundary value problem of a second-order ODE with rapidly oscillating periodic coefficients,

$$(a(\varepsilon^{-1}x)u^{\varepsilon}_x)_x = f(x), \quad x \in [0,1], \tag{2.1}$$

with boundary condition $u^{\varepsilon}(0) = u^{\varepsilon}(1) = 0$. Here a is a positive smooth function with period 1 in $y \equiv \varepsilon^{-1}x$, and $f(x)$ is a bounded continuous function in x. We are going to examine the limit of u^{ε} as $\varepsilon \to 0$, where the large-scale x and small-scale $\varepsilon^{-1}x$ are separated. Since there are two separate scales in the problem, it is natural

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to search for a two-scale expansion of the solution in the form

$$u^\varepsilon \sim u_0(\varepsilon^{-1}x) + \varepsilon u_1(x, \varepsilon^{-1}x) + \varepsilon^2 u_2(x, \varepsilon^{-1}x) + \cdots, \tag{2.2}$$

where the $y = \varepsilon^{-1}x$ dependence has period 1 also. Substituting the ansatz (2.2) into (2.1), and regarding x and y as independent variables, we have (noting that the x derivative is replaced by the operator $\partial_x + \varepsilon^{-1}\partial_y$)

$$(\partial_x + \varepsilon^{-1}\partial_y)(a(y)(\partial_x + \varepsilon^{-1}\partial_y)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots)) = f. \tag{2.3}$$

At the highest order $O(\varepsilon^{-2})$, we have

$$\partial_y(a(y)\partial_y u_0) = 0, \tag{2.4}$$

which has only a y-independent periodic solution. Thus $u_0 = u_0(x)$. At the next-highest order $O(\varepsilon^{-1})$, we have

$$\partial_y(a(y)(\partial_x u_0 + \partial_y u_1)) = 0, \tag{2.5}$$

which implies

$$a(y)(\partial_x u_0 + \partial_y u_1) = c(x) \tag{2.6}$$

for some function $c(x)$. Dividing (2.6) by a and integrating the resulting equation over $y \in [0,1]$ yields

$$\frac{d}{dx}u_0 = c(x)\langle a^{-1}\rangle, \tag{2.7}$$

where $\langle \cdot \rangle$ denotes the integral or average over $y \in [0,1]$. At the next order $O(1)$, we have

$$\partial_x(a(y)(\partial_x u_0 + \partial_y u_1)) + \partial_y(a(y)(\partial_x u_1 + \partial_y u_2)) = f. \tag{2.8}$$

Averaging (2.8) over $y \in [0,1]$ gives

$$\partial_x \big\langle a(y)(\partial_x u_0 + \partial_y u_1)) \big\rangle = f,$$

which in view of (2.6) is just $dc/dx = f$. This then becomes, when we insert (2.7),

$$\frac{d}{dx}\left(a^* \frac{d}{dx}u_0\right) = f, \tag{2.9}$$

where $a^* = \langle a^{-1}\rangle^{-1}$ is the harmonic mean of a. Equation (2.9) is the homogenized equation and is the same type of equation from which we started; however, its coefficient has been changed to the harmonic mean of the original one in the rapidly oscillating variable $y = \varepsilon^{-1}x$. Now we have only to solve the large-scale equation (2.9) subject to the same boundary condition, and the small-scale effect has been built in already.

Rigorous justifications of the above formal asymptotics in any number of dimensions are presented in [18] using the energy method and in [78] using the weak convergence method; see [190] for the first homogenization result in random media

(a is a bounded positive random matrix). Equation (2.5) is posed on the periodic domain in terms of the y variable, and is called the cell problem. Only in one dimension can one solve it in closed form; as a result, we know the homogenized coefficient explicitly. In several dimensions, the corresponding elliptic boundary value problem can be homogenized, but the homogenized coefficients are not known explicitly in general.

2.2 Reaction–Diffusion Traveling Fronts in Periodic Media

Now let us consider what happens if we let the reaction–diffusion (R-D) fronts discussed in Section 2.1 pass through a medium with periodic structure. If we model the medium with a periodic coefficient, then a model equation for R-D fronts is

$$u_t = (a(x)u_x)_x + f(u), \tag{2.10}$$

where $a(x)$ is a positive 1-periodic smooth function and $f(u)$ is a nonlinear function of one of the five types. Since we expect solutions to behave like fronts, we should see them in the large-space and large-time scaling limit. That is, let us consider (2.10) under the change of variables $x \to \varepsilon^{-1}x$, $t \to \varepsilon^{-1}t$, for ε small. The rescaled equation is

$$u^\varepsilon_t = \varepsilon(a(\varepsilon^{-1}x)u^\varepsilon_x)_x + \varepsilon^{-1}f(u^\varepsilon), \tag{2.11}$$

which resembles a homogenization problem except that there is also a singular prefactor ε^{-1} in front of the nonlinear term. We realize that there are two scales present in this problem. One is the width of the front, and the other is the wavelength of the periodic medium. The first one is easy to capture if we look at the rescaled form of a traveling front in a homogeneous medium, or $U(\varepsilon^{-1}(x-ct))$. The second one can be built in as in the homogenization ansatz (2.2). Combining the two ideas, we come up with the following two-scale ansatz for R-D fronts in periodic media:

$$u^\varepsilon \sim U(\varepsilon^{-1}(x-c^*t), \varepsilon^{-1}x) + \cdots, \tag{2.12}$$

where c^*, the average wave speed, plays the role of a^* in the homogenization example shown before. Certainly, we impose periodicity in $y = \varepsilon^{-1}x$, and a 0 or 1 far-field boundary condition in $s = (x-c^*t)/\varepsilon$.

Substituting (2.12) into (2.11), we find that U as a function of (s,y) satisfies the PDE

$$(\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)U) + c^*U_s + f(U) = 0. \tag{2.13}$$

If (2.13) has a solution under the boundary conditions

$$U(s,\cdot) \text{ has period } 1, \quad U(+\infty,y) = 1, \quad U(-\infty,y) = 0, \tag{2.14}$$

the leading term of (2.12) is actually an exact solution! Recalling that the scaling was just to motivate ourselves, we see that we could have worked with the original equation (2.10) to begin with. The exact traveling front then has the functional form $U(x-ct,x)$, and it was first found and constructed in [240].

Comparing (2.2) and (2.12), we see that the two scales of (2.12) are not necessarily separate. In fact, they can be arbitrary, while in (2.2), the two scales are vastly separate. In this sense, (2.12) is a general two-scale representation. Also for this reason, we end up with a PDE cell problem to solve instead of an ODE cell problem. We will see that what makes (2.12) possible is the nonlinearity $f(U)$, and that the extreme cases when the front width is either much larger or much smaller than the wavelength of the medium are simpler.

It is easy to generalize the above form of traveling front to several spatial dimensions. Let us consider an R-D equation of the form

$$u_t = \nabla_x \cdot (a(x)\nabla_x u) + b(x) \cdot \nabla_x u + f(u), \quad u|_{t=0} = u_0(x), \tag{2.15}$$

where

(A1): $a(x) = (a_{ij}(x)), x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is a smooth positive definite matrix on $\mathbb{R}^n$, 1-periodic in each coordinate x_i;

(A2): $b(x) = (b_j(x))$ is a smooth divergence-free vector field, 1-periodic in each coordinate x_i, with mean zero.

Equations of the form (2.15) appear in the study of premixed flame propagation through turbulent (random) media [56], where u is the temperature of the combustible fluid, $b(x)$ is the prescribed turbulent incompressible (divergence-free) fluid velocity field with zero ensemble mean, $f(u)$ is the Arrhenius reaction term, and $a(x)$ is taken as a constant matrix. Since the fluid velocity b is given as we solve for the temperature u, the above problem is called passive, and the traveling fronts are called passive fronts. In [56], formal asymptotic analysis suggests that u propagates with an averaged (effective) speed, also called the turbulent flame speed [56, 203]. Turbulence refers to complex random flows involving a wide range of spatial and temporal scales. Let us first consider periodic media to achieve a good preliminary understanding of effective flame speed. In Chapter 5, we shall give a definitive answer to front speeds of (2.15) in random (turbulent) media.

Let us fix a unit vector $k \in R^N$ and look for a traveling wave (front) moving in this direction with speed $c = c(k)$. The traveling front is of the form

$$u(x,t) = U(k \cdot x - ct, x), \tag{2.16}$$

where the front speed c is an unknown constant depending on k, while U, the front profile, satisfies as a function of $s = k \cdot x - ct$ and $y = x$ the boundary conditions

$$U(-\infty, y) = 1, \quad U(+\infty, y) = 0, \quad U(s, \cdot) \text{ has period } 1. \tag{2.17}$$

Upon substitution into equation (2.15), we obtain the following traveling-front equation for $U = U(s,y)$ and c:

$$(k\partial_s + \nabla_y)(a(y)(k\partial_s + \nabla_y)U) + b(y) \cdot (k\partial_s + \nabla_y)U + cU_s + f(U) = 0. \tag{2.18}$$

The above form of traveling fronts (2.16) in periodic media and the mathematical study of (2.18) were initiated in the author's work in the early 1990s on bistable and ignition nonlinearities [240, 241, 242, 243], where existence and uniqueness are proved under suitable conditions.

A special case of (2.18) is when a is the identity, $b(y) = (b_1(y'),0)$, $y' = (y_2,\ldots,y_N)$, and $k = (1,0,\ldots,0)$. Such a vector field b is called shear flow. Then $u = U(x_1 - ct, x')$, $x' = (x_2,\ldots,x_N)$, and (2.18) reduces to

$$\Delta_{s,y'} U + (c + b_1(y'))U_s + f(U) = 0, \tag{2.19}$$

a semilinear elliptic PDE.

Equation (2.19) appeared earlier ([27] and references therein) as a model of flame propagation inside an infinite cylinder $(s,y') \in \mathbb{R} \times D$ for type-5 nonlinearity. The cylinder has a bounded cross section D, and the boundary condition on y' is zero Neumann, so the cylinder boundary is insulated for heat transfer. Existence and uniqueness of solutions to (2.19) is thoroughly studied for nonlinearities of types 1 through 5 in [29, 30, 31].

Interestingly, mathematicians were not alone in thinking about traveling fronts in periodic media. Theoretical biologists have long been interested in R-D fronts since the days of Fisher [91] and Hodgkin and Huxley [167]. An interdisciplinary problem of fundamental importance often draws attention and ideas from different scientific communities. Indeed, a different notion of traveling front in periodic media was proposed by biologists [221] in the mid 1980s. A traveling (pulsating) front is a solution $u(x,t)$ satisfying

$$\begin{aligned} u(x, t - L\cdot k/c) &= u(x+L,t), \quad \forall (x,t), \\ u(x,t) &\to 1 \text{ as } x\cdot k \to -\infty, \\ u(x,t) &\to 0 \text{ as } x\cdot k \to +\infty, \end{aligned} \tag{2.20}$$

where L is the (vector) period of the media, c the front speed. The solution repeats itself in time $L\cdot k/c$ if it is observed at two points a distance L apart in space. Clearly, $u(x,t) = U(k\cdot x - ct, x)$ is such a front. In [221], formal arguments and linearizations at the unstable state $u = 0$ are made to find approximate solutions in one spatial dimension in the case of a (type-1) KPP reaction. However, error estimates of approximations are not demonstrated.

Interestingly in the late 1970s, about six years earlier than [221], mathematicians then working in the former Soviet Union had already developed a probabilistic functional integration method [94, 100] to find the KPP minimal speeds in periodic media of any dimensions. In the mid 1980s, this line of work was published in detail in the West [95, 96]. Though the work was quickly known in the mathematics community, apparently the authors of [221] were unaware of it, partly because of the lack of communication across scientific and geographical boundaries at the time.

Likewise, [240, 241, 242, 243] were done without knowledge of [221]. The analytical form (2.16)–(2.18) turns out to be more friendly to work with than a property of the time-dependent solution (2.20).

The probabilistic method [94, 100, 96] relies on the large-deviation technique to analyze the Feynman–Kac representation of KPP solutions. It leads to a variational formula for KPP minimal front speeds, and also serves as a rigorous justification of the formal linearization analysis [221]. In the physics literature, the method of linearization at an unstable state to determine front speeds is known as the marginal stability criterion (MSC) [210]. It originated in the 1950s from the plasma community [43] and was used by physicists in studying pattern selection in the early 1980s [63, 140]. Pattern selection refers to the dynamic selection of a front among a continuum of front solutions from a class of initial data. KPP is one example of a pattern-forming system in which dynamic selection is called for. In the case of homogeneous media, the works [8, 9] established the MSC of the KPP front speed $2\sqrt{f'(0)}$ by the PDE method.

The probabilistic method [100, 96] proved that MSC also holds for KPP in inhomogeneous media. We shall discuss this method in conjunction with periodic homogenization of HJ equations in the next section. Its advantage is that it bypasses the front profile and goes straight to the front speed. Impressively, it was worked out also for random media in one spatial dimension [100, 96]. PDE methods are more robust, and can handle more general forms of equations and nonlinearities, though they are traditionally restricted to deterministic media. We shall see in Chapter 5 that combining ideas of the large-deviation and PDE methods is a way to handle equation (2.15) in the random setting in arbitrary dimensions and to solve the turbulent front speed problem [203, 194] for KPP.

2.3 Existence of Traveling Waves and Front Propagation

Let us state the existence results for bistable and ignition fronts [240, 242].

Theorem 2.1. *Let T^n be the n-dimensional unit torus and $\|\cdot\|_{H^m(T^n)}$ the Sobolev norm of functions on T^n with up to m integrable derivatives. Define $\overline{a} = \int_{T^n} a(x)\,dx$, and assume that conditions (A1) and (A2) hold.*

1. *If the nonlinearity $f(U)$ is of type 3 (bistable nonlinearity) with $\mu \in \left(0, \frac{1}{2}\right)$, there is a positive number δ_{cr} such that if $\|a(x) - \overline{a}\|_{H^m(T^n)} < \delta_{cr}$, $\|b(x)\|_{H^m(T^n)} < \delta_{cr}$, $m > n+1$, then equation* (2.18) *has a unique classical solution (U,c) such that $0 < U < 1$, $U_s < 0$ for all $(s,y) \in \mathbb{R} \times T^n$, and $c > 0$.*
2. *If the nonlinearity $f(U)$ is of type 5 (combustion nonlinearity with ignition temperature), then for all a and b, equation* (2.18) *has a unique classical front (U,c) satisfying the same properties.*

Here uniqueness means that c is uniquely determined by the coefficients (a,b) and the nonlinearity $f(U)$, and U is unique up to a constant translation in s due to the translation-invariance of equation (2.18). The threshold phenomenon in the bistable case is because the unequal potential wells of the antiderivative of $f(u)$ (which are essentially the driving force behind front motion) can have effectively the same depth due to the influence of periodic media. Front speed is zero, and equation (2.15) has a stationary front solution $u = u(x)$. A similar situation occurs in the homogeneous case in which the intermediate zero of $f(u)$ is equal to $\frac{1}{2}$.

As in homogeneous media, type $(1,2,4)$ front speeds occupy an interval $[c_*,\infty)$, or the speed spectrum is a continuum. More precisely, we have [20] the following theorem.

Theorem 2.2. *If reaction nonlinearity f is of type $(1,2,4)$, there exists $c_* > 0$ such that no solution exists to (2.17)–(2.18) if $c < c_*$, and a monotone decreasing (in s) solution exists to (2.17) if $c \geq c_*$.*

Variational formulas of front speeds of type $(1,3,5)$ will be discussed later.

The next problem is to show that under certain conditions on the initial data, the time-dependent solutions behave like these special traveling-front solutions. Let us first state front propagation results for the bistable and ignition reaction [243].

Theorem 2.3 (Front Propagation). *Consider the initial value problem for equation (2.15) with initial data $0 \leq u_0(x) \leq 1$. Let f be of type 3 with $\mu \in \left(0, \frac{1}{2}\right)$ or of type 5 with $f'(1) < 0$. Assume in the context of type-3 nonlinearity that a traveling wave solution $U(k\cdot x - c(k)t, x)$ exists for every unit vector $k \in \mathbb{R}^n$. Let $s \in \mathbb{R}$ and let the plane orthogonal to k be $S = \{y \in \mathbb{R}^n | y = x - (k\cdot x)k, \quad \forall x \in \mathbb{R}^n\}$.*

I. *Suppose the initial date are frontlike: $u_0(x) \to 0$ sufficiently fast as $k_0 \cdot x \to -\infty$, and $u_0(x) \to 1$ sufficiently fast as $k_0 \cdot x \to -\infty$, uniformly in $S(k_0)$, for some $k_0 \in \mathbb{R}^n$. Then*
$$\lim_{t\to\infty} u(t, sk_0t) = \begin{cases} 1, & s > c(k_0), \\ 0, & s < c(k_0). \end{cases}$$

II. *Suppose the initial data are pulselike: for some unit vector k, $u_0(x) \to 0$ sufficiently fast as $k_0 \cdot x \to -\infty$; $u_0(x) > \mu + \eta$, $|k\cdot x| < L$, for some positive constants η and L (θ replacing μ for f of type 5). Then there is a positive number $L_0(\eta) > 0$ such that if $L \geq L_0$, then*

$$\lim_{t\to\infty} u(t, skt) = \begin{cases} 1, & c(k) < s < -c(-k), \\ 0, & s < c(k) \text{ or } s > -c(-k). \end{cases}$$

The existence, uniqueness, and propagation results above are all based on maximum principles. The idea is to bound from above and below the exact solutions by simplified comparison functions, then extract asymptotic information. Let us explain the main ingredients below.

Consider equation (2.18) with a type-5 nonlinearity. Our first observation is that the three linear terms there do not form a strongly elliptic operator (such as the Laplacian $\Delta_{s,y}$), since the second derivatives are along directions

$$(k_i, 0, \ldots, 0, y_i, 0, \ldots, 0) \in \mathbb{R}^{n+1}, \quad i = 1, \ldots, n,$$

which do not cover all $n+1$ directions. The other derivative along direction

$$(1,0,\ldots,0) \in \mathbb{R}^{n+1}$$

is the s derivative of U. Hence if c is not equal to zero, we have a parabolic operator (similar to the heat operator $\partial_t - \Delta_x$). This may sound like trouble, since for the standard heat equation, we cannot pose a boundary value problem in t.

However, what saves us is that the s direction of the infinite cylinder is not characteristic, since it is not orthogonal to all the directions $(k_i, 0, \ldots, 0, y_i, 0, \ldots, 0)$. The other observation is that (2.18) is translation-invariant in s. The loss of ellipticity is absent in the shear flow case, or equation (2.19).

Now, do we still have a strong maximum principle for the linear operator in (2.18),

$$Lu = (\nabla_y + k\partial_s)(a(y)(\nabla_y + k\partial_s)u) + b(y)^T \cdot (\nabla_y + k\partial_s)u + cu_s, \tag{2.21}$$

even though it is not strongly elliptic?

As long as $c \neq 0$, the answer is yes, thanks to the parabolic maximum principle and the periodicity in y. Periodicity helps us to overcome the degeneracy! For classical maximum principles, we refer to [198, 226].

Now let us take $c = -1$ for convenience and prove the following result.

Proposition 2.4. *Let u be a classical solution of the differential inequality $Lu \leq 0$ ($Lu \geq 0$) on $\mathbb{R} \times T^n$. If u achieves its minimum (maximum) at (s_0, y_0) with s_0 finite, then $u \equiv$ constant.*

Proof. We first treat the special case $n = 1, k = 1$, in which case we have

$$Lu = (\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)u) + b(y)(\partial_s + \partial_y)u - u_s.$$

For the time being, unfold T into $\mathbb{R}$ and regard L as an operator on $\mathbb{R}^2$. If we make the change of variables

$$s' = \frac{1}{\sqrt{2}}(s - y), \quad y' = \frac{1}{\sqrt{2}}(s + y),$$

then

$$\partial_s = \frac{1}{\sqrt{2}}(\partial_{s'} + \partial_{y'}), \quad \partial_y = \frac{1}{\sqrt{2}}(-\partial_{s'} + \partial_{y'}), \quad \partial_s + \partial_y = \sqrt{2}\partial_{y'}.$$

In terms of (s', y'), Lu becomes

$$Lu = 2(au_{y'})_{y'} - \frac{1}{\sqrt{2}}u_{s'} + \left(\sqrt{2}b - \frac{1}{\sqrt{2}}\right)u_{y'}.$$

Here L is a standard parabolic operator in (s', y'), elliptic in y', and parabolic in s'. By the strong maximum principle for parabolic operators, we see that if u attains its minimum at some finite point (s'_0, y'_0), then

$$u \equiv \text{constant if } s' \leq s'_0,$$

or

$$u \equiv \text{constant if } s - y \leq s_0 - y_0.$$

By the periodicity of u in y, we see that $u \equiv$ constant for all s and y. If $n \geq 2$, we can always subject y to an orthogonal transform, i.e., $y = Qy'$, and then Lu becomes

$$\begin{aligned} Lu &= (k\partial_s + Q^T \nabla_{y'})^T a(k\partial_s + Q^T \nabla_{y'})u + b^T \cdot (k\partial_s + Q^T \nabla_{y'})u - u_s \\ &= (Qk\partial_s + \nabla_{y'})^T QaQ^T (Qk\partial_s + \nabla_{y'})u + b^T \cdot Q^T (Qk\partial_s + \nabla_{y'})u - u_s. \end{aligned}$$

Choosing Q such that $Qk = e_1 = (1,0,\ldots,0)$ and setting $a_1 = QaQ^T$ and $b_1 = Qb$, we have

$$Lu = (e_1\partial_s + \nabla_{y'})^T a_1 (e_1\partial_s + \nabla_{y'})u + b_1^T \cdot (e_1\partial_s + \nabla_{y'})u - u_s.$$

If we make the change of variables

$$s' = \frac{1}{\sqrt{2}}(s - y_1'), \quad z_1 = \frac{1}{\sqrt{2}}(s + y_1'), \quad z_i = \frac{1}{\sqrt{2}} y_i', \;\; i \geq 2,$$

then just as in the case $n = 1$, we have

$$Lu = 2\nabla_z^T (a_1 \nabla_z u) - \frac{1}{\sqrt{2}} u_{s'} + \sqrt{2} b_1^T \cdot \nabla_z u - \frac{1}{\sqrt{2}} u_{z_1}.$$

By the strong maximum principle for parabolic operators, if u attains its minimum at some finite point $P_0 = (s_0', z_0)$, then

$$u = \text{constant if } s' \leq s_0',$$

or

$$u = \text{constant if } \; s - y_1' \leq s_0 - y_{1,0}'.$$

In terms of (s,y), this asserts that u is a constant under some hyperplane that is not orthogonal to the s-axis. The periodicity of u in y implies that $u \equiv$ constant for all s and y. The proof is complete. □

Let us outline the two steps of the construction for existence of type-5 solutions based on a degree-theoretic approach. In step one, we consider a family of elliptically regularized problems ($\varepsilon > 0$, $\tau \in [0,1]$),

$$\varepsilon U_{ss} + L_\tau U + \tau f(U) = 0, \quad (s,y) \in \Omega_a = [-a,a] \times T^n, \tag{2.22}$$

subject to the boundary conditions $U(-a,y) = 1$, $U(+a,y) = 0$. The operator L_τ is L with a replaced by $\langle a \rangle (1-\tau) + \tau a$ and b replaced by τb, with $\langle \cdot \rangle$ being the period average.

To remove the translation-invariance of solutions, we must also impose a normalization condition: $\max_{y \in T^n} U(0,y) = \theta$. By the elliptic maximum principle, we know that U is bounded between 0 and 1 and that $U_s < 0$. Elliptic regularity also tells us that the maximum of ∇U is bounded independently of a and τ. The parameter τ links the linear problem ($\tau = 0$) with the problem of interest $\tau = 1$.

Consider the space $E = C^1(\Omega_a) \times R$. For $(v,c) \in E$, $\tau \in [0,1]$, let $u = \varphi_\tau(v,c)$ be the unique solution of the elliptic boundary value problem

$$\varepsilon u_{ss} + L_\tau u + \tau f(v) = 0$$

under the same 0 and 1 boundary conditions. Define

$$h_\tau(v,c) = \max_{\substack{y \in T^n \\ s=0}} \varphi_\tau(v,c).$$

Then the solution of (2.22) satisfies

$$u = \varphi_1(u,c), \quad h_1(u,c) = \theta. \tag{2.23}$$

Define $F_\tau(u,c) = (\varphi_\tau(u,c), c - h_\tau(u,c) + \theta)$, $\tau \in [0,1]$. Now the existence of the solution is the same as the fixed-point problem

$$F_1(u,c) = (u,c).$$

Notice that the mapping $(\tau,(u,c)) \to F_\tau(u,c)$ from $[0,1] \times E$ to E is continuous and compact. Due to the a priori bounds on the solutions and their derivatives, the Leray–Schauder degree of the mapping $\mathrm{Id} - F$ is well defined on a bounded closed set of the form

$$D \equiv \{(u,c) \in E, \|u\|_{C^1(R_a)} \le K, \; |c| \le K\},$$

where K some constant larger than the bounds of the solutions. This is because the zeros of $\mathrm{Id} - F$ cannot occur on the boundary of the set D. The degree is a measure of the number of zeros counting multiplicity, and is invariant under a change of $\tau \in [0,1]$; see [254] for details. If the degree is nonzero, then we have a fixed point. This is easily checked when $\tau = 0$, since (2.23) is explicitly solvable, and we find that the degree is equal to one.

In step two, we pass to the limit $a \to \infty$ first and then to the limit $\varepsilon \to 0$. To this end, the main technical work is to bound the wave speed c away from 0 and ∞ independently of both parameters. This can be achieved with the help of comparison principles of wave speeds for the $a \to \infty$ limit, see [241], and the identity $c = -\int_{\mathbb{R} \times T^n} f(U)$ for the $\varepsilon \to 0$ limit; see [242].

Thanks to the normalization condition and $U_s \le 0$, we have $U \le \theta$ if $s \ge 0$. Hence we have a linear equation for U on $s \ge 0$. We can now look for a special decay solution of the form $\bar{U} = e^{\mu s}\psi(y)$ with $\psi(y) > 0$ and $\mu < 0$. This decay solution has a continuous limit as $\varepsilon \to 0$ along a subsequence, and $\limsup_{\varepsilon \to 0} \mu < 0$. It follows that the limiting solution must decay to zero as $s \to +\infty$. As $s \to -\infty$, monotonicity implies $U(s,y) \to U_-$. It is not hard to show that U_- satisfies the elliptic equation (dropping s derivatives from (2.18)

$$\nabla_y \cdot (a(y)\nabla_y U) + b(y) \cdot \nabla_y U + f(U) = 0 \tag{2.24}$$

under periodic boundary conditions. Since $f(U) \geq 0$, the maximum principle implies that (2.24) has only constant nonnegative solutions. Thus U_- equals either θ or 1. In the former case, $U \leq \theta$, and hence $f(U) \equiv 0$, for any (s,y). So $LU = 0$, for all (s,y), and thus U attains its maximum θ at a finite point $(0,y^\star)$ as a result of imposing a normalization condition at $s = 0$.

By the strong maximum principle property of the operator L in Proposition 2.4, U must be identically equal to a constant, which is impossible since it has a zero limit at $s = +\infty$. We have constructed a desired traveling-front solution with the property $U_s < 0$ (strict inequality again follows from the strong maximum principle of L).

The other bonus of the strong maximum principle of L is that the sliding domain method [30, 145] applies to show that traveling-front solutions to (2.18) must be unique. The uniqueness means that there is only one value of the wave speed c for any given coefficients (a,b) and nonlinearity f of type 5. Moreover, the profile U is unique up to a constant translation in s, and is strictly monotone in s.

The basic argument to show monotonicity is as follows. First, we compare $U(s,y)$ and its translate $U_\lambda = U(s-\lambda,y)$. For large λ, U_λ is larger than U for those points (s,y) in a bounded cylinder. The bounded cylinder is large enough that $U(s,y)$ is close to either 0 or 1 outside of it. Then $w_\lambda \equiv U(s-\lambda,y) - U(s,y)$ satisfies the differential inequality $Lw_\lambda \leq 0$ outside of the finite cylinder. The strong maximum principle for L implies that $w_\lambda > 0$ holds at any point. Then we decrease λ to the infimum value λ_0 at which U_λ is no less than U. Now $w_{\lambda_0} \geq 0$.

Again, the strong maximum principle implies that at λ_0, U and U_λ must be identical, which is possible only if $\lambda_0 = 0$, due to the front boundary conditions. We conclude that U is strictly monotone and actually has positive derivative anywhere (by invoking the minimum principle on the derivative). A similar argument can be carried out for any two profiles to show that they agree up to a constant translation, and for the uniqueness of c, see [241].

The other nonnegative f of type $(1,2,4,5)$ can be approximated by a sequence of ignition nonlinear functions f_{θ_n}, where $\theta_n \downarrow 0$. Let χ_θ $(\theta < \frac{1}{2})$ be a smooth compactly supported function such that $\chi_\theta(u) = 0$ if $u \leq \theta$, and $\chi_\theta(u) = 1$ if $u \geq 2\theta$. Defining $f_\theta = \chi_\theta f$ will do. The speed c_θ is monotone in θ. One may then pass the approximate type-5 solutions to the limit and verify that the limit remains a front solution. The minimal speed c_* is equal to $\lim_{n\to\infty} c_{\theta_n}$. One then uses the corresponding solution U_* to prove the existence of other solutions for $c > c^*$, and the converging sequence $(U_{\theta_n}, c_{\theta_n})$ to exclude solutions for $c < c_*$; see [20] for complete proofs. The KPP minimal speed is related to the existence of a positive solution of the form $e^{\lambda s}\psi(y)$ to the linearized equation at $U = 0$, which will be presented later in a more general context.

For type-3 nonlinearity, new difficulties come up in step two. Because f changes sign, there can be many nontrivial periodic solutions of (2.24). As we know, traveling fronts may not exist for all $a(y)$ due to the existence of steady states. The convenient method for establishing traveling waves in type 3 (the bistable case) is to use the method of continuation [242, 244] and treat a family of problems in which $a(y)$ is replaced by $(1-\delta)\langle a \rangle + \delta a(y)$.

We start with δ small and obtain solutions by perturbing the known one-dimensional front. The linearized operator has a simple eigenvalue at zero, and the rest of the spectrum is isolated away from zero. The monotonicity of the perturbed solutions guarantees that the same spectral property of the linearized operator remains, and so the perturbation continues on δ. Since each perturbative step relies on the contraction mapping principle, there is no difficulty as $|s| \to \infty$. Of course, the same difficulty arises if we want to show that the continuation goes to any value of $\delta \in [0,1]$, which we know is false in general.

The continuation method is convenient in that it deals with the problem on the infinite domain, where estimates of solutions are usually simpler. However, it relies on good spectral properties of the linearized operators. It works for nonlinearity of type 3, also type 5 if $f'(1) < 0$, as well as spatially periodic conservation laws [244]. For type 5, the assumption $f'(1) < 0$ can be removed by further approximation of nonlinearity. To summarize, the degree-theoretic method and the continuation method with the help of maximum principles guarantee the existence of traveling-front solutions as stated.

Let us sketch the proof of statement (I) of Theorem 2.3 in the case of f of type 5, and refer to [243] for the complete proof. The proof is similar for type 3. The idea is to construct subsolutions (supersolutions) using the parabolic maximum principle [198, 226] and the traveling wave solutions. The long-time asymptotics of the subsolutions (supersolutions) rely on the decay property of solutions of the variable-coefficient linear parabolic equations of the form

$$u_t = \nabla \cdot (a(x)\nabla u) + b(x) \cdot \nabla u, \quad \nabla \cdot b(x) = 0. \tag{2.25}$$

The fundamental solution (Green's function) of (2.25), in turn, has pointwise lower and upper bounds in terms of heat kernels [169, 84, 186].

First we note that due to the fast convergence of u_0 to 1 as $k \cdot x \to \infty$, there are a number $\xi_0 > 0$ large enough and a positive spatially decaying function $q_0 = q_0(k \cdot x) < (1-\theta)/2$ such that

$$U(k \cdot x - \xi_0, x) - q_0(k \cdot x) \le u_0(x)$$

on $\mathbb{R}^n$. Now consider the function

$$u_l \equiv U(k \cdot x - c(k)t - \xi_1(t), x) - q_1(t,x),$$

where ξ_1 and q_1 will be chosen to satisfy

$$\xi_1'(t) > 0, \quad \xi_1(t) > 0, \quad \xi_1(t) = o(t), \quad t \to \infty.$$

We calculate

$$\begin{aligned} N[u_l] &= u_{l,t} - \nabla_x \cdot (a(x)\nabla_x u_l) - b(x) \cdot \nabla_x u_l - f(u_l) \\ &= -\xi_1'(t) U_s - q_{1,t} + \nabla_x \cdot (a(x)\nabla_x q_1) + b(x) \cdot \nabla_x q_1 + f(U) - f(U - q_1). \end{aligned} \tag{2.26}$$

There exists $\delta \in (0,\theta)$ sufficiently small that if $q \in \left[0, \frac{1-\theta}{2}\right]$ and $U \in [1-\delta, 1]$, then

$$f(U) \leq f(U-q).$$

Since $0 \leq q \leq q_0 < \frac{1-\theta}{2}$, we have for $U \in [1-\delta, 1]$,

$$N[u_l] \leq -\xi_1'(t)U_s - q_{1,t} + \nabla_x \cdot (a(x)\nabla_x q_1) + b(x) \cdot \nabla_x q_1. \tag{2.27}$$

If $U \in [0,\delta]$, then $f(U) = f(U-q_1) = 0$, so (2.27) holds with an equality sign. If $U \in (\delta, 1-\delta)$, then there exists $\beta > 0$ such that $U_s \geq \beta$ and $|f(U) - f(U-q_1)| \leq Kq_1$ for some $K > 0$. It follows that

$$N[u_l] \leq -\xi_1'\beta - q_{1,t} + \nabla_x \cdot (a(x)\nabla_x q_1) + b(x) \cdot \nabla_x q_1 + Kq_1. \tag{2.28}$$

Let us choose q_1 to satisfy the equation

$$q_{1,t} = \nabla_x \cdot (a(x)\nabla_x q_1) + b(x) \cdot \nabla_x q_1, \quad q_1|_{t=0} = q_0(k \cdot x). \tag{2.29}$$

To make u_l a subsolution or $N[u_l] \leq 0$, we just need to impose the condition

$$-\xi_1'\beta + Kq_1 \leq 0, \text{ or } -\xi_1'\beta + K\|q_1\|_{L^\infty(\mathbb{R}^n)} = 0,$$

or

$$\xi_1' = \frac{K\|q_1\|_{L^\infty(\mathbb{R}^n)}}{\beta} > 0, \tag{2.30}$$

with $\xi_1(0) = \xi_0 > 0$. By our early comments on the fundamental solution of (2.29), $\|q_1\|_{L^\infty} = o(1)$ as $t \to \infty$. Therefore $\xi_1(t) = o(t)$. We have shown that u_l is a subsolution, $u_l \leq u$. A supersolution can be constructed in a similar way. We conclude that statement (I) holds.

2.4 KPP Fronts and Periodic Homogenization of HJ Equations

Consider the KPP front $u(x,t) = U(k\cdot x - c_*(k)t, x)$. Then under the hyperbolic scaling $t \to \varepsilon^{-1}t$, $x \to \varepsilon^{-1}x$, the scaled solution $u^\varepsilon(x,t) = U((k \cdot x - c_*(k)t)/\varepsilon, x/\varepsilon)$ converges to a step function traveling at speed $c^*(k)$ in the direction k. This way of finding the speed by scaling limit can be done directly on the PDE. To fix ideas, let us consider the homogeneous medium in one spatial dimension. The scaled equation is

$$u_t^\varepsilon = \frac{\varepsilon}{2}u_{xx}^\varepsilon + \varepsilon^{-1}f(u^\varepsilon), \tag{2.31}$$

where we have modified the diffusion constant to $\frac{1}{2}$ for convenience of stochastic representation. The initial condition is the indicator function $1_{\overline{G_0}}(x)$, where G_0 is an open interval. Our goal is to recover $c_* = \sqrt{2f'(0)}$ from (2.31).

Let $c(u) = u^{-1} f(u)$. Then equation (2.31) can be regarded as a heat equation with a time-dependent potential c. The solution has a well-known stochastic representation formula and the Feynman–Kac formula [96, Chapter 2], [55, Chapter 3]

$$u^{\varepsilon}(x,t) = E_x g(X_t^{\varepsilon}) \exp\left\{\varepsilon^{-1} \int_0^t c(u(t-s, X_s^{\varepsilon}))\, ds\right\}, \tag{2.32}$$

where $X_t^{\varepsilon} = x + \sqrt{\varepsilon} W_t$, where W_t is the standard Wiener process. Since $0 < u^{\varepsilon} \leq 1$, it follows from the KPP assumption of f that

$$u^{\varepsilon}(x,t) \leq E_x g(X_t^{\varepsilon}) \exp\left\{\varepsilon^{-1} \int_0^t c(0)\, ds\right\} = e^{f'(0)t/\varepsilon} P(X_t^{\varepsilon} \in \overline{G_0}). \tag{2.33}$$

If we denote the distribution of $\sqrt{\varepsilon} W_t$ by P_{ε}, then it follows from the properties of the standard Wiener process that $P_{\varepsilon} \overset{\text{law}}{\longrightarrow} \delta_x$, where δ_x is the measure with unit mass concentrated at the function identically equal to x. The covariance of $X^{\varepsilon} - x$ equals $\varepsilon \min(s,t)$. There is, however, an exponentially small probability that X^{ε} may escape from being close to x. These are called rare events. The large-deviation theory [229] studies the asymptotics of such small probabilities. A family of probability measures P_{ε} on a complete separable metric space X is said to obey the large-deviation principle (LDP) with a rate function $I(\cdot)$ if there exists a function $I(\cdot) : X \longrightarrow [0,\infty]$ satisfying:

1. $0 \leq I(x) \leq \infty$, $\forall x \in X$.
2. $I(\cdot)$ is lower semicontinuous.
3. For each $l < \infty$, the set $\{x : I(x) \leq l\}$ is compact in X.
4. For each closed set $C \subset X$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log P_{\varepsilon}(C) \leq -\inf_{x \in C} I(x). \tag{2.34}$$

5. For each open set $G \subset X$,

$$\liminf_{\varepsilon \to 0} \varepsilon \log P_{\varepsilon}(G) \geq -\inf_{x \in G} I(x). \tag{2.35}$$

If $x = 0$, the scaled Wiener process $X^{\varepsilon}(s)$, $s \in [0,1]$, satisfies LDP with the rate function $I = I(g)$ defined for any continuous function g on $[0,1]$ with $g(0) = 0$ as

$$I(g) = \frac{1}{2} \int_0^1 (g'(s))^2\, ds \tag{2.36}$$

if $g(s)$ is absolutely continuous with L^2 derivative, otherwise $I(g) = \infty$. See [229, Section 5] for a proof. In view of (2.36), we have

$$\limsup_{\varepsilon \to 0} \varepsilon \log P(X_t^{\varepsilon} \in \overline{G_0}) = \limsup_{\varepsilon \to 0} \varepsilon \log P_{\varepsilon}(\overline{G_0}) = -\inf_{\substack{\varphi_0 = x \\ \varphi_t \in \overline{G_0}}} \int_0^t |\dot{\varphi}(s)|^2\, ds, \tag{2.37}$$

which is a minimal action (cost) equal to $-d^2(x,G_0)/2t$, where d is the distance function. It follows from (2.33) that

$$\lim_{\varepsilon\to 0} \varepsilon \log u^\varepsilon(x,t) \le f'(0)t - \frac{d^2(x,G_0)}{2t} \equiv V. \tag{2.38}$$

Clearly,

$$\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = 0 \quad \forall (x,t) \in N \equiv \{(x,t) : V(x,t) < 0\}. \tag{2.39}$$

The function $V(x,t)$ is continuous, and the convergence is uniform on compact subsets. Setting $V(x,t) = 0$ gives the front equation $d(x,G_0) = \sqrt{2f'(0)}t$ and the desired front speed $c_* = \sqrt{2f'(0)}$. One verifies by direct calculation that the function $V(x,t)$ satisfies the HJ equation

$$\psi_t - \psi_x^2/2 - f'(0) = 0, \tag{2.40}$$

and initial data $\psi(x,0) = 0$ if $x \in \overline{G_0}$, $\psi(x,0) = -\infty$ otherwise.

It remains to show that $u^\varepsilon \to 1$ if $V(x,t) > 0$, or that $u^\varepsilon(x,t) \ge 1-\lambda$, on any compact subset of $P = \{(x,t) : V(x,t) > 0\}$ for any small positive number λ. We need a more general Feynman–Kac formula with stopping times installed [96]:

$$\begin{aligned} u^\varepsilon(x,t) &= E_{t,x} u^\varepsilon(t_\tau, X^\varepsilon_\tau) \exp\left\{\varepsilon^{-1}\int_0^\tau c(u^\varepsilon(t-s,X^\varepsilon_s))\,ds\right\} \\ &= E_{t,x} 1_{\tau=\tau_1} u^\varepsilon(t_{\tau_1}, X^\varepsilon_{\tau_1}) \exp\left\{\varepsilon^{-1}\int_0^{\tau_1} c(u^\varepsilon(t-s,X^\varepsilon_s))\,ds\right\} \\ &\quad + E_{t,x} 1_{\tau=\tau_2} u^\varepsilon(t_{\tau_2}, X^\varepsilon_{\tau_2}) \exp\left\{\varepsilon^{-1}\int_0^{\tau_2} c(u^\varepsilon(t-s,X^\varepsilon_s))\,ds\right\}, \end{aligned} \tag{2.41}$$

where τ_1, τ_2, and τ are given by

$$\begin{aligned} \tau_1 &= \inf\{s : u^\varepsilon(t-s,X^\varepsilon_s) \ge 1-\lambda\}, \\ \tau_2 &= \inf\{s : V(t-s,X^\varepsilon_s) = 0\}, \\ \tau &= \min(\tau_1,\tau_2). \end{aligned}$$

A stopping time is a random variable whose value depends only on the information known up to this value of time, also called the hitting time (the first time that some event occurs). The minimum of two stopping times is also a stopping time. See [72, Section 3.1] for more examples. The expectation takes a double subscript to mean that it acts on the vector trajectory $(t-s, X^\varepsilon_s)$. Since $c \ge 0$, the first term on the right-hand side of (2.41) is bounded from below by

$$(1-\lambda)E^\varepsilon_{t,x} 1_{\tau=\tau_1} = (1-\lambda)P^\varepsilon_{t,x}(\tau=\tau_1).$$

For the second expectation term in (2.41), we need to control u^ε on $V=0$ so that it is not too small and the exponential of the integral can balance it out. Then

$$u^\varepsilon \ge (1-\lambda)P^\varepsilon_{t,x}(\tau=\tau_1) + P^\varepsilon_{t,x}(\tau=\tau_2) \ge 1-\lambda,$$

and we would be done.

Note that over $s \in [0, \tau_2]$, we have $u^\varepsilon \in (0, 1-\lambda]$, so $c(u^\varepsilon) \geq \min_{u \in [0,\lambda]} c(u) \equiv c_\lambda > 0$, and the exponential term indeed provides growth of order $O(\exp\{hc_\lambda/\varepsilon\})$, where $h \leq \tau_2$ is a positive number almost surely independent of ε. This is because it takes a positive amount of time for $((t-s), X_s^\varepsilon)$ to leave (t,x) where $V > 0$ to reach a point where $V = 0$. A lower bound of u^ε on the interface $V = 0$ of order $O(\exp\{-\delta/\varepsilon\})$ for small δ suffices. This lower bound requires $V(x,t)$ to satisfy a condition that for $(x,t) \in N$,

$$V(x,t) = \sup\left\{f'(0)t - \int_0^t |\dot{\varphi}(s)|^2 ds : \varphi_0 = x, \varphi_t \in \overline{G_0}, (t-s, \varphi_s) \in N, s \in (0,t)\right\}, \tag{2.42}$$

for any $t > 0$. Condition (2.42) says that $V(x,t)$ is the supremum of the action functional over the paths in the region of $V < 0$. The lower bound of u^ε on $V = 0$ is obtained by conditioning the stochastic path X^ε in formula (2.32) near the optimal path in region $V < 0$. The probability of the conditioning is controlled by the action function in (2.42), whose value over the optimal path is close to zero and so can be bounded from below by $-\delta$. The lower bound on u^ε on $V = 0$ of the form $O(\exp\{-\delta/\varepsilon\})$ holds.

Condition (2.42) is valid for the function V in (2.38). Provided that (2.42) continues to hold, the above argument extends to slowly varying media in higher dimensions, for example when $f = f(x,u) > 0$ for $u \in (0,1)$, $f(u,x) < 0$ for $u < 0$ and $u > 1$, $f_u(x,0) = \sup_{0<u\leq 1} u^{-1} f(u,x)$. See [96] for complete results. The large-deviation method motivates the ansatz

$$u^\varepsilon \sim \exp\left\{\frac{-I(x,t)}{\varepsilon}\right\}, \tag{2.43}$$

and the PDE approach [81, 82, 83] based on the logarithmic change of variable $v^\varepsilon = -\varepsilon \ln u^\varepsilon$. Let $f(u) = u(1-u)$. Then the function v^ε satisfies the equation

$$v_t^\varepsilon = \frac{\varepsilon}{2} v_{xx}^\varepsilon - \frac{1}{2}\left|v_x^\varepsilon\right|^2 + \exp\left\{-\frac{v^\varepsilon}{\varepsilon}\right\} - f'(0),$$

where

$$v^\varepsilon(x,0) = 0, \quad x \in G_0; \quad v^\varepsilon(x,t) \to +\infty, \quad \text{as } t \downarrow 0^+, \quad x \in G_0^c. \tag{2.44}$$

The next step is to pass to the limit $\varepsilon \to 0$ for v^ε. Comparison functions and maximum principles imply that the supremum norm and the Hölder norms (with exponent $\alpha \in (0,1)$) of v^ε are bounded in any space–time compact set. Hence v^ε has a uniformly convergent subsequence with limiting function v.

The function v satisfies the variational inequality

$$\min\left[v_t + |v_x|^2/2 + f'(0), v\right] = 0, \quad x \in \mathbb{R}, \ t > 0. \tag{2.45}$$

This is understood as follows. Fix $T > 0$. If $v \geq 0$, then

$$v_t + |v_x|^2/2 + f'(0) \geq 0, \quad (x,t) \in \mathbb{R} \times (0,T], \tag{2.46}$$

and on the set $\{v > 0\} \cap \mathbb{R} \times (0,T]$,

$$v_t + |v_x|^2/2 + f'(0) = 0, \tag{2.47}$$

both in the viscosity sense. The viscosity sense in (2.46) means that for each smooth function φ, if $u - \varphi$ has a local minimum at $(x_0, t_0) \in \mathbb{R} \times (0,T]$, then

$$\varphi_t(x_0,t_0) + \frac{1}{2}|\varphi_x(x_0,t_0)|^2 + f'(0) \geq 0. \tag{2.48}$$

In (2.47), we have in addition that if $v - \varphi$ has a local maximum at $(x_1,t_1) \in \mathbb{R}^n \times (0,T]$ and if $v(x_1,t_1) > 0$, then

$$\varphi_t(x_1,t_1) + \frac{1}{2}|\varphi_x(x_1,t_1)|^2 + f'(0) \leq 0. \tag{2.49}$$

To see (2.45), we know that $v \geq 0$ by the maximum principle. Equation (2.44) implies the inequality

$$v_t^\varepsilon - \frac{\varepsilon}{2} v_{xx}^\varepsilon + \frac{1}{2}|v_x^\varepsilon|^2 + f'(0) \geq 0,$$

which yields (2.46) as $\varepsilon \to 0$ in the viscosity sense. Also on any compact subset of $\{v > 0\}$, $b\exp\{-\varepsilon^{-1}v^\varepsilon\} \to 0$; hence we have (2.47). The solution to equation (2.47) differs from that of (2.40) by a sign, and the solution to (2.45) can be written as $v = \max(-V, 0)$. In more general slowly varying media, the solution v in the variational inequality (2.45) with initial condition $v = 0$ on G_0, $v = +\infty$ on G_0^c admits a representation in terms of a two-player, zero-sum differential game with stopping times [83, 92], which resembles the action functional in (2.42).

We see from the above analysis that the HJ equation (2.47) or (2.40) carries the information on the KPP front speed. Let us exploit this KPP–HJ connection further and consider KPP fronts with minimal speed in periodic media by studying the equation

$$u_t = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i,x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + f(x,u), \tag{2.50}$$

with KPP nonlinearity $f(x,u)$ and initial data $g(x)$ of compact support G_0. The problem is solved in [100, 96] by the large-deviation method and a path integral representation of solutions as we illustrated above.

Again, the nonlinearity $f(x,u)$ can be approximated by $c(x) = f_u(x,0)$ times u, so that the implicit solution formula becomes explicit, and the large-deviation method yields the long-time front speed. Here we state the result [96].

Theorem 2.5. *Let $z \in \mathbb{R}^n$. Define the operator*

$$L_z = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(y)(\partial_{y_i} - z_i)(\partial_{y_j} - z_j) + \sum_{i=1}^{n} b_i(\partial_{y_i} - z_i) + c(y) \tag{2.51}$$

on 1*-periodic functions in* $y \in T^n$*, the n-dimensional unit torus. Let* $\lambda = \lambda(z)$ *be the principal eigenvalue of* L_z*, which can be shown to be convex and differentiable in* z*. Let* $H(y)$ *be the Legendre transform of* λ*,*

$$H(y) = \sup_{z\in\mathbb{R}^n}[(y,z) - \lambda(z)],$$

$y \in \mathbb{R}^n$*. The function* $H(y)$ *is also convex and differentiable. Then for any closed* $F \subseteq \{y : H(y) > 0\}$*, we have* $\lim_{t\to\infty} u(t,ty) = 0$ *uniformly in* $y \in F$*. For any compact* $K \subseteq \{y : H(y) < 0\}$*, we have* $\lim_{t\to\infty} u(t,ty) = 1$ *uniformly in* $y \in K$*.*

It follows that the asymptotic front speed $v = v(e)$ along the unit direction e satisfies $H(ve) = 0$. If $\min_{\mathbb{R}^n} \lambda(z) > 0$, then the H equation can be solved to yield the *KPP speed variational formula*

$$v = v(e) = \inf_{(e,z)>0} \frac{\lambda(z)}{(e,z)}. \tag{2.52}$$

In fact, $\lambda(z)$ grows quadratically in z, and so the supremum in the definition of $H(y)$ is achieved. There exists z^* such that

$$0 = H(ve) = v(e,z^*) - \lambda(z^*),$$

and $(e,z^*) > 0$ due to $\lambda(z^*) > 0$. It follows that $v = \lambda(z^*)/(e,z^*) > 0$ and $\lambda(z)(v(e,z^*) - \lambda(z^*)) = 0 \geq \lambda(z^*)(v(e,z) - \lambda(z))$, implying

$$\frac{\lambda(z)}{(e,z)} \geq \frac{\lambda(z^*)}{(e,z^*)}.$$

This implies formula (2.52). The assumption $\min_{\mathbb{R}^n} \lambda(z) > 0$ holds if the operator L is self-adjoint or of the form $L = \nabla\cdot(a(x)\nabla\cdot) + b(x)\cdot\nabla\cdot$, where b is a mean-zero incompressible velocity.

Instead of going through the large-deviation method, let us follow the spirit of the logarithmic transform in the PDE approach and derive the same result. First consider equation (2.50) under the scaling $x \to \varepsilon^{-1}x$, $t \to \varepsilon^{-1}t$. The rescaled equation reads

$$u^\varepsilon_t = \frac{1}{2}\varepsilon \sum_{i,j=1}^{n} a_{ij}(\varepsilon^{-1}x)u^\varepsilon_{x_i,x_j} + \sum_{i=1}^{n} b_i(\varepsilon^{-1}x)u^\varepsilon_{x_i} + \varepsilon^{-1}f(\varepsilon^{-1}x, u^\varepsilon), \tag{2.53}$$

for which we make the change of variable

$$u^\varepsilon = \exp\{\varepsilon^{-1}v^\varepsilon\}. \tag{2.54}$$

Then v^ε satisfies the equation

$$v_t^\varepsilon = \frac{\varepsilon}{2}\sum_{i,j=1}^{n} a_{ij}(\varepsilon^{-1}x)v^\varepsilon_{x_i,x_j} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(\varepsilon^{-1}x)v^\varepsilon_{x_i}v^\varepsilon_{x_j} + \sum_{i=1}^{n} b_i(\varepsilon^{-1}x)v^\varepsilon_{x_i} + \frac{f(\varepsilon^{-1}x,u^\varepsilon)}{u^\varepsilon}. \tag{2.55}$$

The last term is bounded from above by $c(\varepsilon^{-1}x) = f_u(\varepsilon^{-1}x,0)$, which also happens to be the right approximation of the nonlinearity for small values of u^ε. For locating the front or the region where u^ε is near zero, one can replace the nonlinear term by its linearization at u^ε equal to zero as we approach the front from the interior where $v^\varepsilon < 0$. Then equation (2.55) becomes the periodic homogenization problem of a viscous Hamilton–Jacobi equation.

The periodic homogenization of the inviscid Hamilton–Jacobi equation was first studied in [148]. Let v^ε be a solution of

$$v_t^\varepsilon + H(\nabla v^\varepsilon, \varepsilon^{-1}x) = 0, \quad x \in \mathbb{R}^n \times (0,+\infty), \tag{2.56}$$

with initial data $v^\varepsilon(x,0) = v_0$, where H is periodic in the second variable, say with period 1. Under the conditions that H is locally Lipschitz in all variables, $H(p,x) \to +\infty$ as $|p| \to +\infty$ uniformly in $x \in \mathbb{R}^n$, u_0 is bounded and uniformly continuous, and $\nabla v_0 \in L^\infty(\mathbb{R}^n)$; the solution v^ε converges uniformly on compact sets to the viscosity solution v of the homogenized Hamilton–Jacobi equation

$$v_t + \overline{H}(\nabla v) = 0, \quad x \in \mathbb{R}^n \times (0,+\infty), \tag{2.57}$$

where the homogenized Hamiltonian is defined through solving the cell problem stated below.

Theorem 2.6. *For each $p \in \mathbb{R}^n$, there exits a unique real number $\overline{H}(p)$ such that the equation $H(p+\nabla w, y) = \overline{H}(p)$ has a 1-periodic viscosity solution $w = w(y)$.*

The solution v^ε has the two-scale expansion

$$v^\varepsilon \sim v_0(x,t) + \varepsilon v_1(x, \varepsilon^{-1}x, t) + \cdots, \tag{2.58}$$

implying to leading order upon substitution in (2.56) that

$$v_{0,t} + H(\nabla_x v_0(x,t) + \nabla_y v_1(x,y,t)) = 0, \tag{2.59}$$

which leads to the cell problem in Theorem 2.6, where $y = x/\varepsilon$ is the variable, (x,t) are parameters. Equation (2.59) is a nonlinear eigenvalue problem, producing the cell problem in terms of the variable y, and the homogenized equation (2.57) in the variables (x,t).

The ansatz (2.58) is utilized in the convergence proof of [148]. For generalizations to fully nonlinear first- and second-order equations, see [79], where a weak convergence method called the perturbed test function method is employed. Such a method incorporates the above ansatz in the structures of the test functions instead, and can handle equations of first and second order in a unified way.

The homogenized Hamiltonian $\overline{H}$ is convex if H is in p, but it may lose strict convexity. One example [148] is that the homogenized Hamiltonian $\overline{H}$ of the strictly convex classical Hamiltonian $H(p,x) = p^2/2 + V(x)$ is flat near $p = 0$. In fact, let $V \le 0$ and $\max V = 0$. The cell problem reads

$$\frac{1}{2}(p+w_y)^2 + V(y) = \bar{H}, \quad y \in T^1,$$

which is solvable and gives $\bar{H} \ge 0$ such that

$$\begin{aligned} \bar{H} &= 0 \quad \text{if } |p| \le \left\langle \sqrt{-2V} \right\rangle, \\ |p| &= \left\langle \sqrt{2\bar{H} - 2V(y)} \right\rangle \quad \text{if } |p| > \left\langle \sqrt{-2V} \right\rangle, \end{aligned} \tag{2.60}$$

where $\langle \cdot \rangle$ denotes the average over one period.

Now we return to equation (2.55) with $c(\varepsilon^{-1}x)$ in place of the last nonlinear term. Using the above homogenization ansatz (2.58), it is straightforward to derive the cell problem

$$\overline{H} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(y) w_{y_i y_j} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(y)(p_i + w_{y_i})(p_j + w_{y_j}) + \sum_{i=1}^{n} b_i (p_i + w_{y_i}) + c(y), \tag{2.61}$$

where we solve for a periodic function w and a real constant $\overline{H}$ for given p. The homogenized equation is $v_t - \overline{H}(\nabla v) = 0$. The cell problem (2.61) can be transformed into a linear eigenvalue problem with $\overline{H}$ the principal eigenvalue. To see this, let $\bar{w} = e^w > 0$. Then (2.61) in terms of $\bar{w}$ reads

$$\begin{aligned} \overline{H}\bar{w} = {} & \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\bar{w}_{y_i y_j} + \sum_{i,j=1}^{n} a_{ij} p_i \bar{w}_{y_j} + \sum_{i,j=1}^{n} b_i (p_i \bar{w} + \bar{w}_{y_i}) \\ & + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} p_i p_j \bar{w} + c(y)\bar{w}. \end{aligned} \tag{2.62}$$

The right-hand-side operator in (2.62) is just L_{-p}, in view of (2.51). Hence $\overline{H}(-z) = \lambda(z)$.

To derive the front speed formula (2.52), consider the Hamilton–Jacobi equation

$$v_t - \overline{H}(\nabla v) = 0$$

with initial condition

$$v_0(x) = \begin{cases} 0 & \text{if } x \in G_0, \\ -\infty & \text{otherwise.} \end{cases}$$

The Hopf formula is

$$v(x,t) = -\inf_{y \in G_0} \overline{H}^* \left(\frac{y-x}{t} \right), \tag{2.63}$$

where $\overline{H}^*$ is the Legendre transform of $\overline{H}$. The function $H(y)$ in the large-deviation approach is related to $\overline{H}^*$ by

$$H(y) = \sup_{-z\in\mathbb{R}^n} [(y,-z) - \lambda(-z)] = \sup_{z\in\mathbb{R}^n} [(-y,z) - \overline{H}(z)] = \overline{H}^*(-y).$$

The points (x,t) where $v < 0$ or $\lim_{\varepsilon\to 0} u^\varepsilon = 0$ then satisfy

$$\overline{H}^*\left(\frac{y-x}{t}\right) > 0, \ \ \forall y \in G_0.$$

Since G_0 is compact, we can take both x and t large compared with the size of G_0. Then we drop y to get the condition

$$\overline{H}^*\left(-\frac{x}{t}\right) = H\left(\frac{x}{t}\right) > 0,$$

implying that the front speed $v(e)$ along direction e satisfies $H(v(e)e) = 0$.

Putting the homogenization ansatz (2.58) into (2.54) shows that for KPP fronts in periodic media, the solution u^ε behaves like

$$\begin{aligned} u^\varepsilon(t,x) &= \exp\{-I(t,x,\varepsilon)/\varepsilon\} + \cdots, \\ I(t,x,\varepsilon) &= I_0(t,x) + \varepsilon I_1(t,x,x/\varepsilon) + \cdots, \end{aligned} \tag{2.64}$$

where I can be regarded as a phase function as in a geometric optics (Wentzel–Kramers–Brillouin (WKB)) ansatz [237]. However, for fronts of type 3 and type 5, the ansatz for u^ε in the same scaling ($x \to \varepsilon^{-1}x$, $t \to \varepsilon^{-1}t$) is

$$\begin{aligned} u^\varepsilon(t,x) &= U(\varphi(t,x,\varepsilon)/\varepsilon, x/\varepsilon) + \cdots, \\ \varphi(t,x,\varepsilon) &= \varphi_0(t,x) + \varepsilon\varphi_1(t,x) + \cdots, \end{aligned} \tag{2.65}$$

where $\varphi(t,x,\varepsilon)$ is the phase variable. Plugging (2.65) into (2.50), we have

$$\begin{aligned} &\frac{1}{2}(\nabla_x\varphi_0\partial_s + \nabla_y)(a(y)(\nabla_x\varphi_0\partial_s + \nabla_y)U) + b(y)\cdot(\nabla_x\varphi_0\partial_s + \nabla_y)U \\ &\quad - \varphi_{0,t}U_s + f(U) = 0, \end{aligned} \tag{2.66}$$

where $U = U(s,y)$, $s = \varphi(t,x,\varepsilon)/\varepsilon$, $y = x/\varepsilon$. We see that (2.66) is just the traveling-front equation (2.18) with $k = \nabla_x\varphi_0$, and $c(k) = -\varphi_{0,t}$. Relating them gives the Hamilton–Jacobi equation

$$\varphi_{0,t} + c(\nabla_x\varphi_0) = 0 \tag{2.67}$$

for the general front evolution. By uniqueness of c in the case of type $(3,5)$, it is seen from (2.18) that $c = c(k)$ is homogeneous of degree 1 in k, so $c(\nabla_x\varphi_0) = |\nabla_x\varphi_0|c(\nu_n)$, where $\nu_n = \nabla_x\varphi_0/|\nabla_x\varphi_0|$ is the normal direction of the level set of φ_0. The effective Hamiltonian of type-$(3,5)$ nonlinearities is anisotropic and has linear growth in $|p|$. In contrast, the effective Hamiltonian $\bar{H}$ of KPP in (2.62) is quadratic in p.

Interestingly, there are mechanical analogies of quadratically and linearly growing Hamiltonians. The Hamiltonian of classical mechanics $H = |p|^2/2 + V(x)$ is quadratic in $|p|$. The Lagrangian of a special relativistic particle of mass m in a scalar potential [103, 139] is

$$L(q,x) = -mc^2\sqrt{1-|q|^2/c^2} - V(x), \quad |q| \in [0,c],$$

where c is the speed of light. The corresponding Hamiltonian is

$$H(p,x) = mc^2\sqrt{1+|p|^2/c^2} + V(x),$$

which has linear growth in $|p|$. Bistable and ignition-type fronts belong to the family of special relativity, while KPP fronts are Newtonian.

2.5 Fronts in Multiscale Media

The study of traveling fronts in heterogeneous media has been an active area of research in recent years. One may find that other equations also have periodically varying traveling waves (2.16), and one may also consider more complicated media arising in applications, such as space–time-periodic media, time- or space-almost-periodic media, or more general and complex media. We shall present some of these extensions here. An interesting trend is that extended front equations become more and more degenerate if we continue to construct their time dependence explicitly (e.g., constant-speed motion), and that one must use more general dynamic variables to capture these fronts. More complicated media make more complicated fronts.

Recall the solute transport equation (1.1) in the introduction of Chapter 1. In one spatial dimension, v is a constant, and equation (1.1) simplifies after a rescaling of constants to

$$(u + k(x)u^p)_t = (D(x)u_x)_x - u_x, \tag{2.68}$$

where we also make D spatially dependent. We consider the boundary conditions $u(-\infty,t) = u_l$, $u(+\infty,t) = u_r = 0$, $0 < u_l$, representing constant input of solute from the left end of a solute-free soil column. Solutions of (2.68) under such boundary conditions give rise to front solutions.

If k and D are constants, then by making the change of variable $v = u + ku^p$, we can write (2.68) as a standard conservation law:

$$v_t + (f(v) - (g(v))_x)_x = 0, \qquad x \in \mathbb{R}. \tag{2.69}$$

Front solutions $v = v(x - ct)$ are solvable in closed form, and $c = (f(u_l) - f(u_r))/(u_l - u_r)$ is the so-called Rankine–Hugoniot relation.

Let us now consider periodic media by supposing $k(x)$ and $D(x)$ to be 1-periodic regular functions. In periodic media, just as in reaction–diffusion equations, traveling fronts take the form $u = U(x - ct, x)$, which turn out to exist also for conservative equations such as (2.68) and are asymptotically stable [244, 245]:

Theorem 2.7. *Let $k(x)$ and $D(x)$ be smooth positive functions with period 1. If $u_r = 0 < u_l$, then equation (2.68) admits a Hölder continuous traveling wave solution of the form $u = U(x - st, x) \equiv U(\xi, y)$, $\xi = x - st$, $y = x$, $U(-\infty, y) = u_l$, $U(+\infty, y) = 0$, and $U(\xi, \cdot)$ has period 1. Such solutions are unique up to constant translations in ξ, and have wave speeds*

$$s = \frac{u_l}{u_l + \langle k \rangle f(u_l)} > 0, \tag{2.70}$$

with $\langle k \rangle$ the periodic mean. The wave profile U satisfies

$$0 \le U < u_l \ \forall(\xi, y); \ \ U(\xi_1, y) \le U(\xi_2, y) \ \forall \xi_1 \ge \xi_2, \forall y; \ \ U_\xi < 0 \ \ \text{if} \ \ U(\xi, y) > 0.$$

Assume that the initial condition $u_0(x)$ satisfies

$$0 \le u_0(x) \le u_l, \ u_0 \in L^1(\mathbb{R}^+); \ u_0^p \in L^1(\mathbb{R}^+), \ u_0 - u_l \in L^1(\mathbb{R}^-), \ u_0^p - u_l^p \in L^1(\mathbb{R}^-).$$

Let also $m(u, x) = u + k(x) u^p$. Then there exists a unique number x_0 such that

$$\int_{\mathbb{R}} m(u_0(x), x) - m(U(x + x_0, x), x) \, dx = 0 \tag{2.71}$$

and such that

$$\lim_{t \to \infty} \| u(t, x) - U(x - st + x_0, x) \|_1 = 0. \tag{2.72}$$

The construction of traveling waves uses the continuation method, and the existence result holds also in several spatial dimensions [244]. The Hölder continuity of solutions is a consequence of u^p being nondifferentiable at $u = 0$. The explicit effective wave speed (2.70) is due to the fact that equation (2.68) is conservative. Only the mean value of k contributes to the speed; the rest of the information in k influences the wave profile.

The stability proof extends that of [188] and uses L^1 contraction of dynamics, as well as a space–time translation invariance of the traveling fronts in the moving frame coordinate. For fronts in another conservative equation (the Richards equation of water infiltration) with more complicated dependence of wave speeds on the periodic media, see [88].

Space–time-dependent media (flows) arise in combustion [10, 65]. KPP fronts in a periodic flow field with space–time-separated scales are studied in [152], which considers the temperature field of a reacting passive scalar,

$$T_t^\varepsilon + V(x, t, \varepsilon^{-\alpha} x, \varepsilon^{-\alpha} t) \cdot \nabla T^\varepsilon = \varepsilon \kappa \Delta T^\varepsilon + \varepsilon^{-1} f(T^\varepsilon), \tag{2.73}$$

with compactly supported (in G_0) nonnegative initial data, and $\alpha \in (0, 1]$. The velocity V is bounded and Lipschitz continuous and has periodic dependence on the fast-oscillating scales $y \equiv \varepsilon^{-\alpha} x$, $\tau \equiv \varepsilon^{-\alpha} t$. The small parameter ε measures the ratio of the front thickness and large scale (dependence on (x, t)) of the velocity field, say of order $O(1)$. The effective Hamiltonian $H(p,x,t)$ is defined as a solution of the following cell problem: for each $(p,x,t) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,+\infty)$ there are a unique number $H(p,x,t)$ and a $w(y,\tau) \in C^{0,1}(\mathbb{R}^n \times (0,+\infty))$ periodic in both y and τ such that

$$w_\tau - a(\alpha)\kappa\Delta w - \kappa|p+\nabla w|^2 + V(x,t,y,\tau)\cdot(p+\nabla w) = -H(p,x,t), \qquad (2.74)$$

where $a(\alpha) = 0$ if $\alpha \in (0,1)$, $a(\alpha = 1) = 1$.

The case $\alpha = 1$ can be derived using an exponential change of variable and a Hamilton–Jacobi equation as in the last section except that due to the time dependence, the w_τ term is added. The condition $a(\alpha) = 0$ in the case $\alpha \in (0,1)$ implies the loss of viscosity in the cell problem (2.74), which can be understood as follows. Ignore the slow variable (x,t) for now and change the scaling to $x = \varepsilon^{-1+\alpha}x'$. Then the velocity V is $V(\varepsilon^{-1}x', \varepsilon^{-1}t')$, and the diffusion coefficient becomes $\varepsilon^{3-2\alpha}\kappa \ll \varepsilon\kappa$. Hence the diffusion term is too small to be seen at the order of the cell problem.

The function H is locally Lipschitz continuous, convex in p, and grows quadratically in $|p|$ as $|p| \to +\infty$ uniformly in (x,t). The asymptotics of T^ε as $\varepsilon \to 0$ are given by the following theorem.

Theorem 2.8. *Let T^ε be a solution of* (2.73) *under the above assumptions. Then as $\varepsilon \to 0$, $T^\varepsilon \to 0$ locally uniformly in $\{(x,t) : Z < 0\}$ and $T^\varepsilon \to 1$ locally uniformly in the interior of $\{(x,t) : Z = 0\}$, where $Z \in C(\mathbb{R}^n \times [0,+\infty)$ is the unique viscosity solution of the variational inequality*

$$\max(Z_t - H(\nabla Z, x, t) - f'(0), Z) = 0, \quad (x,t) \times \mathbb{R}^n \times (0,+\infty),$$

with initial data $Z(x,0) = 0$ in G_0 and $Z(x,0) = -\infty$ otherwise. The set $\Gamma_t = \partial\{x \in \mathbb{R}^n : Z(x,t) < 0\}$ can be regarded as a front.

Given a space–time-periodic incompressible flow field, the "cell problem" for KPP front speed in the limit $t \to +\infty$ is always viscous. To show this, let us consider

$$u_t = \Delta u + b(x,t)\cdot\nabla u + f(u), \qquad (2.75)$$

where $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, and f is of KPP type. The N components of the vector field $b(x,t) := (b^1(x,t), b^2(x,t), \ldots, b^N(x,t))$ are smooth and spatially divergence-free, are periodic of period 1 in both x and t, and have mean zero over the period cell $Q \times (0,1)$, where Q is the unit cube in $\mathbb{R}^N$. Then the KPP large-time minimal front speed in direction k (denoted by $c^*(k)$) is identified by a front propagation (front spreading) theorem similar to Theorem 2.3. It is given by the variational formula [173]

$$c^*(k) = \inf_{\lambda>0} \mu(\lambda,k)/\lambda,$$

where $\mu(\lambda,k)$ is the principal eigenvalue of the periodic–parabolic operator [117]

$$L^\lambda\Phi := \Delta_x\Phi + (b - 2\lambda k)\cdot\nabla_x\Phi + \left(\lambda^2 - \lambda b\cdot k + f'(0)\right)\Phi - \Phi_t, \qquad (2.76)$$

defined on spatially–temporally periodic functions $\Phi(x,t)$. The eigenvalue problem (2.76) is related to (2.74) at $\alpha = 1$ by a logarithmic transform. In the case $\alpha = 1$, taking $\varepsilon \to 0$ is the same as $t \to \infty$.

Moreover, there is a traveling-front solution of the form $u = U(k\cdot x - c^*(k)t, x, t) \equiv U(s,x,t)$, locally integrable in (s,x,t), periodic in (x,t), $U(\pm\infty, x,t) = 0/1$, with the continuous directional derivatives

$$U_\tau - c^* U_s, \quad k^i U_s + U_{y^i}, \;\; i = 1,\ldots,N, \quad \text{and} \quad (k\partial_s + \nabla_y)^2 U$$

and satisfying the traveling-front equation

$$U_\tau - c^* U_s = (k\partial_s + \nabla_y)^2 U + b\cdot(k\partial_s + \nabla_y)U + f(U). \tag{2.77}$$

Note that (2.77) is an extension of (2.18), and is more degenerate in the sense that there are not enough derivatives in (2.77) to ensure continuity (smoothness) of U. The function U has one more dependent variable than u, which does not happen in spatially periodic media. The value c^* is minimal in that no solutions exist if c^* is replaced by a number $c < c^*$. Recently, it was proved in [168] that for almost all $\eta \in \mathbb{R}$, we have that $U(k\cdot x - c^*(k)t + \eta, x, t)$ satisfies (2.75). Similar existence results [173, 168] hold for type $(2,4,5)$. We refer to [168] for further results on existence of KPP fronts at $c > c^*$ and continuous KPP fronts.

In view of (2.74) and (2.76), the front speed obtained from the limit $t \to \infty$ at a fixed $\varepsilon > 0$ and that from $\varepsilon \downarrow 0$ at a fixed time interval $[0,T]$ may not agree in general. A numerical study of the difference due to finite ε (finite front thickness) is carried out in [161]. At any finite $\varepsilon > 0$, the cell problem of a front is always viscous, while it is not in the limit $\varepsilon \downarrow 0$ when $\alpha < 1$. This shows the subtlety of front speed upscaling in heterogeneous media. The front speed would be the same from either limit in homogeneous media.

Fronts of the form (2.16) persist in space-almost-periodic media [168]. Front solutions of the form $u = U(k\cdot x - ct, t)$ in time-periodic media have been studied in [5] (bistable f) and [97] (nonnegative f). Bistable fronts of the form $u = U(k\cdot x - ct, t)$ have been found in time-almost-periodic media. Interestingly, there are also fronts of the form $u = U(k\cdot x + \int_0^t c(s)\,ds)$ where $c(s)$ is almost periodic. The latter fronts are not reducible to the former. KPP fronts in space-periodic and time-almost-periodic media were studied recently [120], where generalized speed intervals were proved to exist, and they reduce to singletons in the case of time-periodic media. A more general form of fronts has been introduced recently [21, 159, 219], and proved to exist in various settings [219, 163, 174]. For fronts in periodic media in the context of discrete models, see [236, 106] and references therein. For fronts and homogenization in the context of free-boundary limits and models, see [46, 47, 129, 130, 248, 249]. For fronts in periodically perforated media and fragmented environments among other applications, see [113, 25]. For KPP speeds c_* under various parameter asymptotic limits (diffusion–reaction rates, periods), see [225]. For pulselike waves of reaction–diffusion systems in heterogenous media, see [171, 252].

2.6 Variational Principles, Speed Bounds, and Asymptotics

The KPP variational principle (2.52) reduces the front speed problem to the analysis and estimation of the principal eigenvalues of linear advection–diffusion operators, where many classical methods apply. Let us consider (2.75) in space dimension two ($N = 2$), and scale the velocity field $b(x,t)$ to δb. If δ is small, a perturbation analysis of eigenvalues yields the quadratic enhancement law $c_* = c_0 + O(\delta^2)$, where c_0 is the KPP front speed in homogeneous media. The quadratic correction is explicit in the case of shear flow [191, 177]. In the case of spatial shear, write $b = (0, b_2(x_1))$, $b_2 = \tilde{b}_{x_1}$. The function $\tilde{b}$ has mean equal to zero and serves as the velocity potential. Then the front speed along the x_2 direction has the expansion [191]

$$c_* = c_0 \left(1 + \frac{1}{2} \|\tilde{b}\|_2^2 \, \delta^2 + \text{higher-order terms} \right), \quad \delta \ll 1. \tag{2.78}$$

The energy (half of L^2 norm square) of $\tilde{b}$ (velocity potential) is the amount of enhancement to leading order. In the case of bistable nonlinearity, one may perform a perturbation analysis of the traveling-front equation (a nonlinear eigenvalue problem) [191]. The interesting finding is that the correction term in (2.78) is the same. By monotonicity of c_* in terms of f, the value of c_* from other types of f must behave the same. This is the first indication that for front speed c_*, the type of nonlinearity does not matter as much as the flow. In other words, there is *universality of c_* in terms of nonlinearity*.

For time-periodic shear flow, let

$$b_2 = b_2(x_1,t) = \sum_{m\neq 0, l\neq 0} b_{m,l} e^{imx_1 + i\omega l t}.$$

Then

$$c_* = c_0 \left(1 + \frac{1}{2} \left(\sum_{\substack{m>0 \\ l>0}} |b_{m,l}|^2 \frac{2m^2}{m^4 + l^2\omega^2} \right) \delta^2 + \text{higher-order terms} \right). \tag{2.79}$$

We see that the enhancement decreases with increasing frequency of temporal oscillations (or as ω increases). The speed slowdown due to temporal oscillations is called the speed-bending phenomenon in the combustion literature, and is studied in various models [10, 128, 65]. It persists in random flows as well [65, 179, 182], which we shall discuss more in Chapter 5. Again formula (2.79) holds for all nonlinearities.

In the large $\delta \gg 1$ regime, consider again spatial shear flow. The eigenvalue analysis of KPP front speeds [19] shows that $c_*(\delta)/\delta$ is monotone decreasing in $\delta \gg 1$ and converges to a positive limit. The limiting value or the linear growth rate depends on $\tilde{b}$ in an implicit way, and it has a variational formula [115]:

$$\lim_{\delta\to\infty} c_*(\delta)/\delta = \sup_{\psi\in D_1} \int_\Omega \tilde{b}(x_1)\,\psi^2(x_1)\,dx_1, \tag{2.80}$$

where

$$D_1 = \left\{\psi \in H^1(\Omega) : \|\nabla\psi\|_2^2 \le f'(0), \|\psi\|_2 = 1\right\},$$

and Ω is the periodic domain of variable x_1. If $\tilde{b}$ has a flat piece near its maximal point in Ω, a test function in D_1 can be supported near the maximal point, and the limit equals $\max_\Omega \tilde{b} = \|\tilde{b}\|_\infty$. If the reaction is fast ($f(u)$ replaced by $rf(u)$, $r \gg 1$) or the diffusion constant (equal to one in (2.75)) is made small, the constraint $\|\nabla\psi\|_2^2 \le f'(0)$ is easy to satisfy: again a test function may be localized near the maximal point of $\tilde{b}$, and so the speed growth rate is close to $\max_\Omega \tilde{b}$ [12, 24]. In general,

$$c_* = O(\delta), \quad \delta \gg 1, \tag{2.81}$$

for other nonnegative nonlinearities. Here again we see universal behavior. The linear law (2.81) holds also for time-periodic shear flows [177], and is numerically observed for bistable f as well. It is true for more general percolating flows that contain at least two infinitely long channels of flow trajectories [59, 132]. The open channels (streamlines) in the flow are like multiple lanes on the freeway to help the transport process and speed up the reaction front. The growth exponent is less than one (sublinear growth) for flows with enough closed streamlines. For example,

$$c_*(\delta) = O(\delta^{1/4}), \quad \delta \gg 1, \tag{2.82}$$

holds [12, 184] for the KPP front and cellular flow:

$$b = (-\phi_{x_2}, \phi_{x_1}), \quad \phi = \cos(\pi x_1)\cos(\pi x_2). \tag{2.83}$$

The proof of the $\frac{1}{4}$ scaling for the KPP front [184] uses the speed variational formula (2.52), boundary layer analysis of cellular flows, and properties of convection-enhanced diffusion [86]. For ignition nonlinearity, (2.82) is supported by numerical simulations [231]. Moreover, analytical bounds $O(\delta^{1/5}) \le c_* \le O(\delta^{1/4})$ hold [132]. A useful criterion for distinguishing linear and sublinear speed growth is in terms of first integrals of the flow fields [24]. A first integral for a periodic vector field b is a nonzero periodic solution w of the equation $b \cdot \nabla w = 0$. An extension of (2.80) to KPP speed c_* in a mean-zero divergence-free vector field $b(x)$ is [256]

$$\lim_{\delta\to\infty} c_*(\delta,k)/\delta = \sup_{w\in D_I} \int_{T^N} (b\cdot k)\,w^2(x)\,dx,$$

where

$$D_I = \left\{w \in H^1(T^N) : b\cdot\nabla w = 0, \|\nabla w\|_2^2 \le f'(0), \|w\|_2 = 1\right\}, \tag{2.84}$$

where k is the direction of front propagation and T^N the N-dimensional unit torus (a unit cube in $\mathbb{R}^N$ with opposite faces identified).

It follows that an upper bound is $\|b \cdot k\|_\infty$, and that the limit in (2.84) is nonzero if $\int_{T^N} (b \cdot e) w_0 \, dx \neq 0$ for some first integral w_0. This is the case for shear flows.

In general, suppose such a w_0 exists. Then $w = (1 + \varepsilon w_0)/\|1 + \varepsilon w_0\|_2 \in D_I$ for ε small enough, and the limiting value is to leading order $\varepsilon \int_{T^N} (b \cdot e) w_0 \, dx$, which is positive if ε is chosen to have the sign of the integral. If $\int_{T^N} (b \cdot k) w^2 \, dx \leq 0$ for all first integrals, then $c_*(\delta, k) = o(\delta)$. The cellular flow (2.83) is an example for which $\int_{T^N} (b \cdot k) w^2 \, dx = 0$ for all k and first integral w.

The front asymptotic enhancement in the sense of $\lim_{\delta \to \infty} c_*(\delta) = \infty$ by periodic incompressible flow has been shown recently [255] to depend on the geometry of the flow and not on nonlinearity f. In particular, front asymptotic enhancement occurs for KPP if and only if it does so for ignition f, and so the phenomenon is universal among all nonnegative reactions.

A variant of (2.52) holds for partially periodic media where solutions in part of the variables are periodic and in the other part are subject to zero Neumann boundary conditions [20]. There are min–max variational principles of front speeds [109, 116, 232] for non-KPP f. In particular, let us state the one for the unique front speed from shear flow and bistable/ignition f that has been used in analysis of random front speeds [176].

Consider the cylindrical domain $x = (x_1, \tilde{x}) \in D = \mathbb{R} \times \Omega$, where Ω is a bounded domain in $\mathbb{R}^{N-1}$, and shear flow $b = (b_1(\tilde{x}), 0)$. The front moves along x_1, $u = U(x_1 + c_* t, \tilde{x}))$, satisfying zero Neumann boundary condition at $\mathbb{R}^1 \times \partial\Omega$. The initial datum u_0 belongs to the set I_s. The set I_s for bistable f consists of bounded continuous functions with limits one and zero at $x_1 \sim \pm\infty$ respectively. For ignition f, one requires also that u_0 decay to zero exponentially at $-\infty$. For $u_0 \in I_s$, $u(x,t)$ converges to a traveling front at large times [205]. Define the functional as in [116]:

$$\psi(v) = \psi(v(x)) \equiv \frac{Lv + f(v)}{\partial_{x_1} v} \equiv \frac{\Delta v + b_1(\tilde{x}) \partial_{x_1} v + f(v)}{\partial_{x_1} v}. \tag{2.85}$$

The min–max variational formula [116] for c_* is

$$\sup_{v \in K} \inf_{x \in D} \psi(v(x)) = c(\delta) = \inf_{v \in K} \sup_{x \in D} \psi(v(x)), \tag{2.86}$$

where K is the set of admissible functions,

$$K = \left\{ v \in C^2(D) \mid \partial_{x_1} v > 0, 0 < v(x) < 1, v \in I_s \right\}.$$

The proof uses asymptotic stability of traveling fronts and min–max front speed formulations of [232]. Likewise, similar min–max formulas have been derived and studied for the homogenized Hamiltonian $\bar{H}$ of HJ in periodic media [57, 104]. In [104], $\bar{H}$ is computed based on the formula

$$\bar{H}(p) = \inf_{\phi(y) \in C^1(T^N)} \sup_y H(p + \nabla_y \phi(y), y). \tag{2.87}$$

2.7 Exercises

1. Show that the principal eigenvalue $\lambda(z)$ of the operator L_z in (2.51) is positive for all $z \in \mathbb{R}^n$ if $(a_i j)$ is the identity matrix and $b_j(y)$ is a mean-zero and divergence-free vector field.

2. Prove by the maximum principle that the homogenized Hamiltonian of KPP nonlinearity $\bar{H} = \bar{H}(p)$ defined in (2.62) grows like $O(|p|^2)$ for large $|p|$.

3. Show that for cellular flow (2.83), the integral $\int_{T^N} (b \cdot k) w^2 \, dx$ is zero for all unit vectors $k \in \mathbb{R}^2$ and first integral w $(b \cdot \nabla w = 0)$.

4. Derive the quadratic speed-enhancement formula (2.78) for bistable and ignition fronts with the min–max formula (2.86) in a mean-zero 1-periodic shear flow $b = \delta(b_1(\tilde{x}), 0)$. The fronts move in the x_1 direction, $\tilde{x} \in \mathbb{R}^{n-1}$, $n \geq 2$. In the small-δ regime, define the test function as a perturbation of the traveling-front profile in homogeneous media of the form

$$v(x) = U(\xi) + \delta^2 w(\xi, \tilde{x}), \tag{2.88}$$

where

$$\xi = (1 + \alpha\delta^2) x_1 + \delta\chi, \tag{2.89}$$

with α a constant to be determined and $\chi = \chi(\tilde{x})$ the mean-zero periodic solution of

$$-\Delta_{\tilde{x}} \chi = b_1 .$$

Choose α properly so that w is uniformly bounded with decay at $x_1 \sim \pm\infty$ (and so v is admissible) and gives the quadratic speed correction in (2.78).

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