2

Early History of the Pell Equation

2.1 The Cattle Problem of Archimedes

This chapter is devoted to various aspects of the history of the Pell equation before the work of Lagrange. As this topic has already been dealt with in some detail by Konen,[1] Whitford,[2], and Dickson,[3] our discussion here will be brief. We will concentrate on providing a more modern historical perspective and a somewhat different presentation of this material than that given in these earlier works.

In 1773, the poet and literary critic Gotthold Ephraim Lessing (1729–1781) published[4] a Greek epigram which he had edited from an Arabic manuscript in the Herzog-August Library in Wolfenbüttel in northern Germany. The text of this epigram consists of a heading, followed by a poem of 44 lines made up of 22 elegiac distichs, a scholium giving a (false) solution, and a lengthy analysis of the problem by Chr. Leiste. There has been some controversy[5] concerning the exact translation of the heading, but it seems that Fraser's version,[6] given belown, is about as accurate as can be expected.

> A problem which Archimedes set in epigrammatic form and sent to those interested in these matters in Alexandria, in the letter addressed to Eratosthenes of Cyrene.

The most frequently cited translation of the problem itself is that of Thomas.[7]

> If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, the third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe

> further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls went to pasture together. Now the dappled in four parts[8] were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shall thou be numbered among the wise. But come, understand also all these conditions regarding the cows of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

Recently, a charming translation[9] by Hillion and Lenstra has appeared, which possesses much of the light-hearted spirit of the original.

This problem is referred to in a scholium to Plato's Charmides as being called the Cattle Problem by Archimedes. It may also have been mentioned in some work of Cicero.[10] Since Krumbiegel's[11] criticism of this work in 1880, it has been customary to regard the problem, now called the Cattle Problem, as most likely having originated with Archimedes (c. 287–212 BC), but the poem itself as a Hellenistic fabrication. However, Fraser[12] has argued very convincingly that we should also accept Archimedes as the author of the poetical form of the problem, and there seeems to be no good reason to dispute this judgement.

The problem is to find the numbers W, X, Y, and Z of the white, black, dappled, and yellow bulls, respectively and the numbers w, x, y, and z of the cows of corresponding colours. We can now write the equations which these quantities satisfy as

$$W = \left(\frac{1}{2} + \frac{1}{3}\right) X + Z \ , \tag{2.1}$$

$$X = \left(\frac{1}{4} + \frac{1}{5}\right) Y + Z \; , \tag{2.2}$$

$$Y = \left(\frac{1}{6} + \frac{1}{7}\right) W + Z \; , \tag{2.3}$$

$$w = \left(\frac{1}{3} + \frac{1}{4}\right) (X + x) \; , \tag{2.4}$$

$$x = \left(\frac{1}{4} + \frac{1}{5}\right) (Y + y) \; , \tag{2.5}$$

$$y = \left(\frac{1}{5} + \frac{1}{6}\right) (Z + z) \; , \tag{2.6}$$

$$z = \left(\frac{1}{6} + \frac{1}{7}\right) (W + w) \; , \tag{2.7}$$

$$W + X = \square \; , \tag{2.8}$$

$$Y + Z = \triangle \; . \tag{2.9}$$

Leiste[13] found integral solutions of (2.1), (2.2), and (2.3) as

$$Y = 1580m \; , \quad Z = 891m \; , \quad W = 2226m \; , \quad X = 1602m \; , \tag{2.10}$$

where m is an integer parameter. (This is simply linear algebra.) He then went on to find solutions to (2.1)–(2.7) for the unknowns that were all 20 times larger than they might be. However,[14] if we multiply (2.4) by 4800, (2.5) by 2800, (2.6) by 1260, and (2.7) by 462 and add, we get

$$4657w = 2800X + 1260Y + 462Z + 143W \; .$$

By using (2.10), we find that $m = 4657n$, for an integer parameter n. From this and (2.4)–(2.7), we find that

$$\begin{aligned} W &= 10366482n \; , & X &= 7460514n \; , \\ Y &= 7358060n \; , & Z &= 4149387n \; , \\ w &= 7206360n \; , & x &= 4893246n \; , \\ y &= 3515820n \; , & z &= 5439213n \; . \end{aligned} \tag{2.11}$$

Since the coefficients of n have greatest common divisor 1, (2.11) represents all of the possible solutions of (2.1)–(2.7). As mentioned earlier, Leiste gave a solution with $n = 20$ and the scholium,[15] with no explanation, gives a solution for $n = 80$. Neither of these satisfies (2.8) or (2.9).

It remains to consider (2.8) and (2.9). Since $W + X$ must be a square and

$$W + X = 4 \times 957 \times 4657n \; ,$$

we must have $n = 957 \cdot 4657U^2 = 4456749U^2$. Also, $Y + Z = V(V+1)/2$ means that

$$T^2 = 8(Y + Z) + 1 = DU^2 + 1 ,$$

where $T = 2V + 1$ and

$$D = 410286423278424 . \tag{2.12}$$

Thus, in order to solve the Cattle Problem, we must solve the Pell equation (1.7) of Chapter 1 with D given by (2.12). We will discuss in the next chapter (Example 3.10) how this problem can be solved.

There has been some dispute[16] about the exact wording of the Cattle Problem, but no significant changes to it have been met with acceptance by modern scholars. Some doubt has been expressed concerning whether the second part of the problem actually reduces to a Pell equation. This has to do with whether to interpret the text of the problem as asking for $W + X$ to be an integral square or whether the bulls when packed together should fill a square. As a bull is longer than it is broad, the latter reading would simply ask that $W + X$ should be a rectangular number. This problem is called Wurm's problem,[17] as it was solved by him to produce a solution where

$$W + X = 1409076 \cdot 1485583 .$$

This suggests, then, that the ratio of the length to the breadth of the bulls would be 1485583/1409076, which is rather close to 1. As the authors of this book come from the cattle-producing province of Alberta, we are able to attest that we have seen many bulls, but never a bull with these proportions, and it is unlikely that the bulls in Sicily ever had such proportions either. Indeed, as Dijksterhuis[18] has noted, this apparently simplifying assumption is nothing of the sort, because if we assume that the ratio of the length to the breadth of a bull is λ, a rational number, then the condition that $W + X$ be a square becomes $W + X$ is λ times a square, and the supposed simplification of the problem is lost. Moreover, as Archimedes was far too good a mathematician not to include in his statement of the problem all of the values needed to solve it, and no value for λ is provided, we must assume that his intent was that the second part of the problem should reduce to what we now call a Pell equation. Although it is only implicit, as far as is currently known, the Cattle Problem represents the earliest mention in history of a Pell equation.

We are left with a number of questions concerning this remarkable work. For example, what caused Archimedes to devise it in the first place? Hultsch[19] has provided a very clever explanation for this. Apollonius of Perga (c. 262–c. 190 BC) in his *Easy Delivery* produced a better approximation to π than that of Archimedes in his earlier *Measurement of the Circle*, and it seems that part of Apollonius's motivation for doing this was to exhibit his superior skill in this sort of numerical manipulation. Certainly, he must have performed more difficult multiplications than those mentioned in the *Measurement of the Circle*. Another work of Apollonius concerning the multiplication of large numbers, preserved within the *Synogoge* or *Collection* of Pappus (c. 290–c. 350

AD), although inspired by Archimedes's *Sand-reckoner*, also seems to imply some criticism of Archimedes' methods. Thus, it does not seem unreasonable for Archimedes to have responded by issuing the Cattle Problem as a challenge to Apollonius and others; for, as we shall see in §3.3, solving the second part of it involves the manipulation of enormous numbers. This supposition is to some extent supported by the *Cattle Problem's* lightly satirical tone, which is particularly evident in the mockery displayed in the last lines of the epigram, which Fraser[20] translates as:

> If thou findest out these things, and layest them to mind, giving all the measures of the numbers, go victorious in glory and know in truth that thou hast been judged consummate in this wisdom at least.

From a mathematician's perspective, the tone of this provides us with the best reason to reject the Wurm hypothesis mentioned earlier: his solution is just too simple to derive. Of course, as Dijksterhuis[21] rightly points out, it is impossible to verify these suppositions, but it is interesting that, as Apollonius spent most of his career in Alexandria, he might very likely have been there during the time that the letter containing the problem was sent to Eratosthenes (276–194 BC). Knorr[22] has made the interesting suggestion that Eratosthenes had composed the first part of the problem and that Archimedes had responded by sending it back to him with the addition of the second, more difficult part. This suggestion, however, does not seem to have found much support among scholars, as most seem to accept Krumbiegel's earlier judgement[23] that "there is no ground whatever in the poem for... a division of authorship."

The problem appears to owe some of its inspiration to Homer; for, in Book XII, lines 127–139, of the *Odyssey*, the poet wrote[24]:

> Your next landfall will be the island of Thrinacie, where the Sun-god pastures his large herds and well-fed sheep. There are seven herds of cattle and as many flocks of beautiful sheep, with fifty head in each.

The Greek word "thrinacian" means three-cornered and was used to designate the three-cornered Island of Sicily,[25] where Archimedes lived. Notice that there also seems to be a computation problem in Homer's lines. Any educated Greek of the time would have recognized this Homeric allusion in the *Cattle Problem*.

There is also another important question concerning this problem: Could Archimedes himself solve it? Given our discussion of its solution in §3.3, the answer must be no. Although the basic idea of how to go about solving it had been demonstrated by Amthor[26] as early as 1880, it was not until the advent of modern computing devices that it was possible to compute the enormous numbers representing the size of the various herds. Indeed, as late as 1964, Beiler[27] could write concerning this problem that "stupendous feats of calculation have been performed and the answers have not yet been completely computed nor is it likely that they ever will be." The more important question, as noted by Vardi,[28] is: Did Archimedes know that it had a solution? As we will see in the next section, this could be the case, but we will likely never know for certain. One thing, however, must be borne in mind. In our modern society, with its very sophisticated mathematics and computers, it is easy to lose sight of what a remarkable piece of work this is. Given its date of composition and the state of mathematics (as far as we currently understand it) at this time, it must be regarded as a work of considerable genius. Who else, but Archimedes, could have posed it? Moreover, the poem with its lighter side also contributes something to our understanding of this extraordinary man. In this regard, we can do no better than to conclude this section with a quote of Fraser.[29]

> The poem... helps us to gain a picture of Archimedes as one who, for all his extraordinary pre-eminence in his abstract and theoretical world, possessed a warm and lively human sympathy, and this side of his character is worthy of emphasis no less than the superlative tributes to his mathematical genius.

2.2 Further Contributions of the Greeks

The first explicit mention of a Pell equation seems to occur in the work of Theon of Smyrna. (c. 130 AD)[30] If we put $s_1 = 1$ and $d_1 = 1$ and compute

$$s_{n+1} = s_n + d_n, \quad d_{n+1} = 2s_n + d_n \quad (n = 1, 2, 3, \dots) ,$$

then

$$d_n^2 - 2s_n^2 = (-1)^n . \tag{2.13}$$

Of course, Theon does not use the modern notation that we are employing here, nor did he provide a proof of (2.13), being content instead to simply verify it for the first few cases. Some further light was shed on these observations much later by the neoplantonist philosopher Proclus[31] (412–485 AD). He referred to an identity, which in our notation would be expressed as

$$(2x + y)^2 + y^2 = 2x^2 + 2(x + y)^2 \tag{2.14}$$

and appears to appeal to Proposition 10 in Book II of Euclid's *Elements* for a proof. If we rewrite the identity, we get

$$(2x + y)^2 - 2(x + y)^2 = -(y^2 - 2x^2) ,$$

which does provide a proof of (2.13), although Proclus does not say this. Most mathematical historians agree that both Theon and Proclus appear to be drawing on a much earlier Pythagorean source for this material. What is remarkable about these side and diagonal numbers is that they suggest that the Pythagoreans used the values of d_n/s_n as a means of producing ever better rational approximations of $\sqrt{2}$. As the early Greek mathematicians were interested in the problem of irrationality, it is possible that the existence of this infinite sequence approaching, but never reaching, the value of $\sqrt{2}$ might have been used in producing an early (but incorrect) proof of the irrationality of this quantity.

In fact, it is possible to use (2.13) to produce a correct proof of the irrationality of $\sqrt{2}$. For, if we assume that $\sqrt{2}$ is rational, then $\sqrt{2} = a/b$ for some $a, b \in \mathbb{Z}^{>0}$. Hence, we can rewrite (2.13) as

$$bd_n + as_n = \frac{b^2}{|bd_n - as_n|} .$$

Since $bd_n - as_n \neq 0$ (otherwise (2.13) could not hold), we have $|bd_n - as_n| \geq 1$ and

$$0 < bd_n + as_n < b^2 . \tag{2.15}$$

As d_n and s_n increase beyond any limit, we see that (2.15) is impossible for all $n \in \mathbb{Z}^{>0}$. While this proof seems very simple to us, it is by no means likely that the Pythagoreans would have discovered it.

Thus, it appears that the early Greeks knew how to produce solutions of (1.7) when $D = 2$. It is difficult to say with any certainty that they extended the idea of side and diagonal numbers any further, but if[32] we put $D = 3$ and define $s_1 = 1$ and $d_1 = 2$,

$$s_{n+1} = s_n + d_n, \quad d_{n+1} = 3s_n + d_n \quad (n = 1, 2, 3, \dots) ,$$

we get

$$\frac{d_n}{s_n} = \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \frac{265}{153}, \frac{362}{209}, \frac{989}{571}, \frac{1351}{780}, \dots \tag{2.16}$$

as $n = 1, 2, 3, \dots, 11, \dots$. These are exactly the convergents in the simple continued fraction expansion (see §3.2) of $\sqrt{3}$. Furthermore, in the *Measurement of the Circle*, Archimedes[33] introduces with no explanation the inequality

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780} . \tag{2.17}$$

Note that both of the bounds used in this occur in (2.16). However, there are several other methods[34] by which Archimedes might have discovered (2.17). What does seem to be clear is that the Greeks were in possession of some techniques that allowed them to find good rational approximations to $\sqrt{n}$ (and other irrationals) for certain integral values of n. As will be demonstrated in Chapter 3, simple continued fractions can be used to produce the best rational approximations to a given irrational. Could the Greeks have been aware, at least on some level, of these objects? The answer is yes. In Proposition 2 of Book X of Euclid's *Elements* we have:

> If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.

The process Euclid (c. 325–265 BC) is describing here is called anthyphairesis, and it has become the subject of considerable scrutiny by modern historians of early Greek mathematics. Anything like a full discussion of this is well beyond the scope of this book, and we refer the interested reader to the excellent books of Knorr[35] and Fowler[36] for a fascinating treatment of this subject. We will be content here with a few simple observations.

Euclid's understanding of a magnitude is what we might call a line segment and is distinct from what he understood by a number (integer). If we have two line segments A and B, we will write $A < B$ to denote that the line segment A is shorter than the line segment B. Now, suppose we are given two line segments L_0 and L_1, where $L_1 < L_0$. We apply the anthyphairesis process to L_1 and L_0; that is, we subtract L_1 from L_0 a certain number of times, say q_0 times, until we get a remaining line segment $L_2 < L_1$. We then repeat the procedure with L_2 and L_1, etc. We will get the following sequence of equations, where the q values are all positive integers:

$$\begin{aligned} L_0 &= q_0 L_1 + L_2 \quad (L_2 < L_1)\ , \\ L_1 &= q_1 L_2 + L_3 \quad (L_3 < L_2)\ , \\ &\ \ \vdots \\ L_i &= q_i L_{i+1} + L_{i+2} \quad (L_{i+2} < L_{i+1})\ , \\ &\ \ \vdots \end{aligned}$$

If this process does not terminate (no L_n ever "measures" L_{n-1}; i.e., no length of any L_{n+1} is ever 0), then L_0 and L_1 are not commensurable or, in more modern parlance, L_0/L_1 is irrational. If we examine this process from a modern perspective and put

$$\phi_i = \frac{L_i}{L_{i+1}} \quad (i = 0, 1, 2, \dots)\ ,$$

then

$$0 < \phi_i - q_i = \frac{L_{i+2}}{L_{i+1}} < 1\ .$$

Thus, $q_i = \lfloor \phi_i \rfloor$ and

$$\phi_{i+1} = (\phi_i - q_i)^{-1} > 1 \quad (i = 0, 1, 2, \dots)\ ; \tag{2.18}$$

that is, the anthyphairesis of L_0/L_1 is given by

$$\frac{L_0}{L_1} = \phi_0 = [q_0, q_1, q_2, \dots, q_i, \dots]\ ,$$

the simple continued fraction expansion of ϕ_0 (see §3.2). We call the q_i ($i = 1, 2, \dots$) the *partial quotients* in this representation.

We know that there were several instances in which the Greeks might have employed this process, both geometrically and arithmetically.[37] This is corroborated by early references to the ancient's (5th-4th century BC) understanding that magnitudes are in proportion to each other if they have the same anthyphairesis (same sequence of partial quotients). Indeed, this seems to have formed the basis of their concept of proportion. Concerning this, Knorr[38] states:

> We can conceive of only one reason for the ancients' invention of the anthyphairetic definition of proportion: to extend the formal numerical definition so that proportions of incommensurable magnitudes may be included.

We also know that the early Greek mathematicians were very interested in the problem of incommensurability; in particular, they seem to have spent a lot of effort in demonstrating the possible incommensurability of line segments whose ratio is $\sqrt{n}/\sqrt{m}$, where m and n are positive integers,[39] and they could construct geometrically such line segments. It is not unreasonable to assume in their earliest investigations into this that they might have employed the anthyphairetic process to such line segments. This certainly seems to be what is behind parts of Books II, X, and XIII of the *Elements*. The main problem in doing this, as Fowler[40] has observed, would be the difficulty that they would face in determining the partial quotients that would be needed to express the anthyphairesis of $\sqrt{n}/\sqrt{m}$. This is simply because their arithmetic procedures would not permit the easy manipulation of the decimal numbers that would result. Fowler[41] has provided a possible and plausible solution to this problem by making use of concepts that would be known to the ancients. The basis of his procedure is what he calls the Parmenides Proposition (PP), which we give below as Proposition 2.1. A form of this result appears in Plato's *Parmenides* and was very likely known to the Greeks of Plato's time (427–347 BC). Certainly, it appears in the much later *Collection* of Pappus and could easily be derived from results[42] in Books VII or V of the *Elements*. We give this proposition next.

Proposition 2.1 (The Parmenides Proposition). *Let $A, B, C, D \in \mathbb{Z}^{>0}$. If $A/B < C/D$, then*

$$\frac{A}{B} < \frac{A+C}{B+D} < \frac{C}{D}\,.$$

Now, suppose ϕ is any real number and

$$\frac{A}{B} < \phi < \frac{C}{D}\,,$$

where $A, B, C, D \in \mathbb{Z}^{>0}$. We have $\phi B - A > 0$ and $C - \phi D > 0$; hence, $(\phi B - A)/(C - \phi D) > 0$, and, consequently, there exist positive integers p and p' such that $p > (\phi B - A)/(C - \phi D)$ and $p' > (C - \phi D)/(\phi B - A)$. This means that

$$\frac{pC + A}{pD + B} > \phi \text{ and } \frac{p'A + C}{p'B + D} < \phi\,.$$

These observations lead us to the following simple algorithm, proposed by Fowler, for finding rational approximations to ϕ.

Algorithm 2.1:

Input: Suppose ϕ, A, B, C, D are defined as above, and

$$\frac{A}{B} < \phi < \frac{C}{D}\,.$$

1: Compute $R = (A+C)/(B+D)$. We now have two cases.
2: **case 1**: $R > \phi$
3: Apply PP repreatedly to find a q so that

$$\frac{A}{B} < \frac{(q+1)A+C}{(q+1)B+D} < \phi < \frac{qA+C}{qB+D} < \frac{C}{D}\,.$$

4: Return q, $C' = qA + C$, and $D' = qB + D$. Note that

$$\frac{A}{B} < \phi < \frac{C'}{D'} < \frac{C}{D}\,.$$

5: **end case**
6: **case 2**: $R < \phi$
7: Apply PP repeatedly to find a q so that

$$\frac{A}{B} < \frac{A+qC}{B+qD} < \phi < \frac{A+(q+1)C}{B+(q+1)D} < \frac{C}{D}\,.$$

8: Return q, $A' = A + qC$, and $B' = B + qD$. Note that

$$\frac{A}{B} < \frac{A'}{B'} < \phi < \frac{C}{D}\,.$$

9: **end case**

When this algorithm is applied repeatedly, the cases will strictly alternate; that is, if a given iteration falls under Case 1, then the next iteration will fall under Case 2, and vice versa.

In Case 1 we can compute q directly from

$$q = \left\lfloor \frac{C - \phi D}{\phi B - A} \right\rfloor$$

and in Case 2 from

$$q = \left\lfloor \frac{\phi B - A}{C - \phi D} \right\rfloor\,.$$

Suppose we consider the simple case of $\phi = \sqrt{n}/1 = \sqrt{n}$ for some non-square positive integer n. We begin with

$$\frac{\lfloor\phi\rfloor}{1} < \phi < \frac{\lfloor\phi\rfloor + 1}{1} \; . \tag{2.19}$$

If $\phi_0 = \phi$, $q_0 = \lfloor\phi_0\rfloor$, and $R = \lfloor\phi\rfloor + 1/2 < \phi$, then $q_1 = 1/(\phi - q_0) = 1$; but if $R > \phi$, then we can apply Algorithm 2.1 to (2.19) to obtain

$$\frac{\lfloor\phi\rfloor}{1} < \phi < \frac{q\lfloor\phi\rfloor + \lfloor\phi\rfloor + 1}{q+1} \; ,$$

where

$$q = \left\lfloor \frac{\lfloor\phi\rfloor + 1 - \phi}{\phi - \lfloor\phi\rfloor} \right\rfloor = q_1 - 1 \; .$$

Thus, if $R < \phi$, we already have, by (2.19),

$$\frac{q_0}{1} < \phi < \frac{q_1 q_0 + 1}{q_1} \; ,$$

and if $R > \phi$, we get

$$\frac{q_0}{1} < \phi < \frac{q_1 q_0 + 1}{q_1}$$

after the application of Algorithm 2.1 to (2.19).

To proceed further with our analysis we will need a result which is proved in §3.1. If we put $A_{-2} = 0$, $A_{-1} = 1$, $B_{-2} = 1$, and $B_{-1} = 0$ and define subsequent values for A_i and B_i by the recursive formulas (3.4), then by (3.9) we have

$$\phi_{i+1} = \frac{A_{i-1} - \phi B_{i-1}}{\phi B_i - A_i} \; . \tag{2.20}$$

By our previous remarks we may assume that we have, after a possible application of Algorithm 2.1 to (2.19),

$$\frac{A_0}{B_0} < \sqrt{n} < \frac{A_1}{B_1} \; .$$

Also, it is easy to see that if for some $i \geq 1$,

$$\frac{A_{i-1}}{B_{i-1}} < \sqrt{n} < \frac{A_i}{B_i} \; , \tag{2.21}$$

then since $\phi_{i+1} > 1$, we must have $R = (A_i + A_{i-1})/(B_i + B_{i-1}) < \sqrt{n}$. Thus, on applying Algorithm 2.1 to (2.21), we get

$$q = \left\lfloor \frac{A_{i-1} - \phi B_{i-1}}{\phi B_i - A_i} \right\rfloor = \lfloor\phi_{i+1}\rfloor = q_{i+1}$$

by (2.18) and (2.20). Also,

$$\frac{A_{i+1}}{B_{i+1}} < \sqrt{n} < \frac{A_i}{B_i} \; .$$

Similarly, if

$$\frac{A_i}{B_i} < \sqrt{n} < \frac{A_{i-1}}{B_{i-1}} ,$$

we find after the application of Algorithm 2.1 (here $R > \phi = \sqrt{n}$) that $q_{i+1} = q$ and

$$\frac{A_i}{B_i} < \sqrt{n} < \frac{A_{i+1}}{B_{i+1}} .$$

By induction (a process of deduction not likely known to the early Greek mathematicians), this procedure of repeated application of Algorithm 2.1 will produce the anthyphairesis of $\sqrt{n}/1 = [q_0, q_1, q_2, \ldots]$.

Algorithm 2.1 is evidently a very simple process that anyone with knowledge of the PP could, for example, apply successively to $\sqrt{n}/1$ in the manner that we have described above. It is highly unlikely, of course, that the Greeks of the time would have been able to prove formally that this procedure would produce the anthyphairesis of $\sqrt{n}/1$ as we have done here, but they could easily have computed the successive convergents A_i/B_i to $\sqrt{n}$ and discovered their anthyphairesis to be $[q_0, q_1, q_2, \ldots, q_i]$ $(i = 0, 1, 2, \ldots)$. As they would have known by construction of the convergents that the value of $\sqrt{n}$ is always bounded above and below by two successive convergents, they would likely conclude (correctly) that the anthyphairesis $\sqrt{n}/1$ is $[q_0, q_1, q_2, \ldots]$. For small values of n, they would notice the periodic structure of $[q_0, q_1, q_2, \ldots]$ and perhaps, as Fowler[43] suggests, be able to prove geometrically that their conjectured anthyphairesis is correct. The difficulty of checking the inequalities that occur in Algorithm 2.1 would be much diminished because $\phi = \sqrt{n}$; hence, all that would be needed in each case is the determination of whether or not some rational number a/b exceeded $\sqrt{n}$. This, of course, is possible simply by checking the value of the integer $a^2 - nb^2$. During the process of checking these values, the Greeks would have discovered that if this process is carried out far enough for a given n, they would get $A_i^2 - nB_i^2 = 1$ (see §3.3) for perhaps several values of i, and thereby find solutions to the Pell equation for $D = n$. While this would not have been their original objective, they would nevertheless have been struck by the discovery, just as the Pythagoreans were in the case of $n = 2$.

Of course, this is conjectural, and it is possible to develop other plausible processes whereby the ancients might have been able to find good rational approximations to $\sqrt{n}$, but it fits very well with what we have been able to deduce from the few tantalizing grains of information that have survived time's winnowing. Certainly, the Greeks must have been able to perform some calculations like these, at least for small values of n. For example, if Archimedes had applied this to $\sqrt{27}$ (a better choice than $\sqrt{3}$ for his purpose[44]), he would have found that the first few convergents are $5/1, 26/5, 265/51$, and $1351/260$ and that

$$\frac{265}{51} < \sqrt{27} < \frac{1351}{260} ,$$

which, on dividing by 3, yields (2.17). Indeed, $1351^2 - 3 \cdot 780^2 = 1$. Thus, it is reasonable to infer that an expert calculator like Archimedes had some knowledge about how to solve the Pell equation for small values of D, at least. Possibly these investigations prompted him to believe that the Pell equation is always solvable, but that when D is large, this is a very difficult problem. This would explain his thinking in setting the Cattle Problem.

One other place where the Pell equation is explicitly mentioned by the Greeks is in Diophantus' *Arithmetica*. In Sections 9 and 11 of Book V, he solved (1.7) for $D = 26$ and $D = 30$, respectively. While this might cause us to think that the later Greeks had found a technique for solving the Pell equation, it is important to realize that the method given would, in general, only find rational solutions to the Pell equation, not integral ones. Diophantus also showed in a lemma in Section 14 of Book VI how one could find, given rationals x and y and integers D and r, a second rational solution to $x^2 - Dy^2 = r^2$. The concentration in the *Arithmetica* on techniques that only produce rational solutions to Diophantine equations strongly suggests that the later Greeks were either not able or not interested in producing integral solutions. Tannery[45] suggested that possibly Diophantus might have considered such problems, particularly the Pell equation, in the then lost seven books of the *Arithmetica*; however, although more recent research[46] has revealed some of these lost books, there is still no evidence that Diophantus ever considered the problem of finding only integral solutions. This, then, represents the very unsatisfactory state of our knowledge concerning the ancients' contributions to the study of the Pell equation.

2.3 The Indian Mathematicians

The situation is much different when we consider the achievements of the Indian mathematicians of the early to late middle ages.[47] As early as the 5th century AD, Aryabhata I (b. 476 AD) had developed a method for solving the linear Diophantine equation

$$ax - by = c \tag{2.22}$$

for integers x and y, given positive integers a, b, and c. Aryabhata's original problem was to find an integer n which on being divided by a given integer a leaves a given remainder of r_1 and on division by a given b leaves a remainder r_2. On putting $c = |r_1 - r_2|$, this problem reduces to making either $(ax+c)/b$ or $(by + c)/a$ a positive integer according to whether $r_1 > r_2$ or $r_2 > r_1$. Aryabhata then goes on to describe a solution technique, called the *kuttaka* (pulverizer), which is a variant of the now standard method of solving this problem by making use of the continued fraction expansion of a/b (see §3.2). It is often assumed by number theorists that the Greeks must have found a method of solving (2.22). Indeed, no less of an authority than Thomas Heath[48] seems to have believed this.

> Thus, the solution of the equation $ax-by=c$, given by Aryabhata... is an easy development from Euclid's method of finding the greatest common measure or proving by that process that two numbers have no common factor (Eucl. VII. 1, 2, X. 2, 3), and it would be strange if the Greeks had not taken this step.

It would not be strange, however, if the Greeks had no interest in the problem. We have seen that the earlier Greeks were concerned with finding rational approximations to irrationals, but the problem of finding a rational approximation to a rational like a/b, would likely not have been regarded as a problem at all. The later Greeks seemed to be interested only in rational solutions of Diophantine equations, and this explains why Diophantus never dealt with (2.22). In any event, what is true is that we have no evidence at all that any of the Greek mathematicians made the slightest contribution to the problem of solving (2.22) for integers x and y.

In 628, Brahmagupta (598–670) was the first to discover our identity (1.4); that is, if

$$A^2 - DB^2 = Q \tag{2.23}$$

and

$$P^2 - DR^2 = S\ , \tag{2.24}$$

then

$$(AP + DBR)^2 - D(AR + BP)^2 = QS\ . \tag{2.25}$$

Today we call this process of multiplying two quadratic forms to yield a third quadratic form *composition*, but the Indian mathematicians referred to it as *samasa*.

If we have $Q = S = \pm 2$, $A = P$, and $B = R$, then $T = (A^2 + DB^2)/2 = A^2 - (\pm 1)$, $U = AB$ is a solution of (1.7). Brahmagupta discovered this result together with those in Table 1.1 and this enabled him to solve the Pell equation whenever he had any solution (A, B) of

$$A^2 - DB^2 = -1\ , \pm 2\ , \pm 4\ . \tag{2.26}$$

However, he could do more than this: He developed an *ad hoc* way of solving the Pell equation. For example,[49] consider the equation $x^2 - 92y^2 = 1$, about which Brahmagupta declared, "[a person solving this problem] within a year [is] a mathematician." He first notes that $10^2 - 92 = 8$ and then composes this with itself to obtain $192^2 - 92 \cdot 20^2 = 64$. After dividing this equation by 64, he gets $24^2 - 92(5/2)^2 = 1$, and on composing this latter equation with itself, he obtains $1151^2 - 92 \cdot 120^2 = 1$. Brahmagupta also realized that by using this composition principle he could produce many more solutions to the Pell equation, once he had one solution.

However, the crowning achievement of Indian mathematics with respect to the Pell equation was the development of the cyclic method for solving it. The technique, described by Bhaskara II (1114–1185) in 1150 AD, and its history are well described by Selenius[50] and the interested reader should consult this work for further details and references. We will only sketch, with additional information, a variant (there are several) of the algorithm here.

We will assume that $Q, A, B \in \mathbb{Z}$ and that $(A, B) = 1$ in (2.23); this means that $(B, Q) = 1$. As the technique for solving (2.22) was known, the step of finding an integer P such that $Q \mid BP + A$ could be easily achieved by the kuttaka process. It follows that since $(B, Q) = 1$, we must have $Q \mid P^2 - D$ and $Q \mid AP + DB$. By putting $R = 1$ in (2.24), we see from (2.25) that

$$\left(\frac{AP + DB}{Q}\right)^2 - D\left(\frac{A + BP}{Q}\right)^2 = \frac{P^2 - D}{Q} . \tag{2.27}$$

From this simple observation we can develop the cyclic method for solving the Pell equation.

Given integers n, A_{n-1}, B_{n-1}, Q_n, and P_n where $(A_{n-1}, B_{n-1}) = 1$ such that

$$\left|A_{n-1}^2 - DB_{n-1}^2\right| = Q_n ,$$

find by the kuttaka process a positive[51] integer P_{n+1} such that $|P_{n+1}^2 - D|$ is minimal and $Q_n \mid (P_{n+1}B_{n-1} + A_{n-1})$. Put $Q_{n+1} = |P_{n+1}^2 - D|/Q_n$,

$$A_n = \frac{A_{n-1}P_{n+1} + DB_{n-1}}{Q_n} , \quad B_n = \frac{B_{n-1}P_{n+1} + A_{n-1}}{Q_n} . \tag{2.28}$$

By (2.27) we get

$$\left|A_n^2 - DB_n^2\right| = Q_{n+1} , \tag{2.29}$$

and $(A_n, B_n) = 1$. The latter result follows easily by observing that $|A_nB_{n-1} - B_nA_{n-1}| = 1$. The method terminates when, for some n, $Q_{n+1} = 1, 2, 4$ because, as we have explained above, Brahmagupta had already shown how to solve the Pell equation once any solution of (2.26) is known.

Consider the example of $D = 67$. We begin with $n = 0$, $A_{-1} = 1$, $B_{-1} = 0$, $Q_0 = 1$, and $P_0 = 0$. We now summarize in Table 2.1 the solution of the Pell equation by this process, called the *cakravala* (the circle or cyclic method) by the Indians.

Table 2.1. Cakravala for $D = 67$

n	P_n	Q_n	A_{n-1}	B_{n-1}	$P_{n+1} \pmod{Q_n}$
0	0	1	1	0	1
1	8	3	8	1	1
2	7	6	41	5	5
3	5	7	90	11	2
4	9	2	221	27	

Since $221^2 - 67 \cdot 27^2 = -2$, we get $T = 221^2 + 1 = 48842$, $U = 27 \cdot 221 = 5967$ as a solution of the Pell equation $T^2 - 67U^2 = 1$. Concerning this technique, Hankel[52] stated, "It is beyond all praise; it is certainly the finest thing that was achieved in the theory of numbers before Lagrange." Unfortunately, the Indians did not provide a proof that the cyclic method would always work. They were content, it seems, in the empirical knowledge that it always seemed to do so, and they used it to solve the Pell equation for $D = 61, 67, 97, 103$. It was not until the late 1930s that a proof that the cyclic method would always produce a value of $Q_i = 1$ was produced by Ayyangar.[53] He noted that this process could be represented as the expansion of $\sqrt{D}$ into a type of semiregular continued fraction which would always be periodic.

We note that if (as is certainly the case for $n = 0$)

$$P_n B_{n-1} \equiv A_{n-1} \pmod{Q_n} \ ,$$

then, by (2.28),

$$\begin{aligned} P_{n+1}B_n - A_n &= B_{n-1}(P_{n+1}^2 - D)/Q_n \\ &\equiv 0 \pmod{Q_{n+1}} \ . \end{aligned}$$

Thus, by induction we may assume that $Q_n \mid (P_n B_{n-1} - A_{n-1})$. Since $Q_n \mid (P_{n+1}B_{n-1} + A_{n-1})$ by construction and $(Q_n, B_{n-1}) \mid (A_{n-1}, B_{n-1})$, we get $(Q_n, B_{n-1}) = 1$ and

$$P_{n+1} \equiv -P_n \pmod{Q_n} \ .$$

Hence,

$$P_{n+1} = q_n Q_n - P_n \tag{2.30}$$

for some $q_n \in \mathbb{Z}$. If we now begin with $n = 0$ and define

$$\phi_i = \frac{P_i + \sqrt{D}}{Q_i} (> 0) \quad (i = 0, 1, 2, \dots) \ ,$$

$$\eta_{i+1} = \operatorname{sign}(P_{i+1}^2 - D) \ ,$$

we get

$$\phi_{i+1} = \frac{P_{i+1} + \sqrt{D}}{Q_{i+1}} = \frac{\eta_{i+1} Q_i}{\sqrt{D} - P_{i+1}} \ .$$

By (2.30),

$$\frac{\sqrt{D} - P_{i+1}}{Q_i} = \phi_i - q_i \ ;$$

hence,

$$\phi_{i+1} = \frac{\eta_{i+1}}{\phi_i - q_i} \ . \tag{2.31}$$

We now investigate the problem of the value of q_n.

Theorem 2.2. *If we put* $q = \lfloor (P_n + \sqrt{D})/Q_n \rfloor$, *then* $0 < q \le q_n \le q + 1$.

Proof. Put $P = qQ_n - P_n$, $P' = (q+1)Q_n - P_n$ and note that $P \equiv P' \equiv P_{n+1} \pmod{Q_n}$. By definition of q, we have $P < \sqrt{D}$ and $P' > \sqrt{D}$.

If $q_n < q$, then

$$0 < P_{n+1} = q_n Q_n - P_n < P < \sqrt{D}\,.$$

Hence, $|D - P_{n+1}^2| = D - P_{n+1}^2$, $|D - P^2| = D - P^2$. Since $P_{n+1} < P$, we get $D - P_{n+1}^2 > D - P^2$, which is impossible by selection of P_{n+1}.

If $q_n > q + 1$, then

$$P_{n+1} = q_n Q_n - P_n > P' > \sqrt{D}\,.$$

In this case, $|D - P_{n+1}^2| = P_{n+1}^2 - D$, $|D - {P'}^2| = {P'}^2 - D$, and $P_{n+1}^2 - D > {P'}^2 - D$, which is also impossible. □

By Theorem 2.2 and (2.31), we see that

$$\phi_{i+1} > 1 \quad (i = 0, 1, 2, \dots)\,.$$

This means that the expression (2.31) can be used to give us

$$\sqrt{D} = q_0 + \cfrac{\eta_1}{q_1 + \cfrac{\eta_2}{q_2 + \cfrac{\eta_3}{q_3 + \ddots}}}\,, \tag{2.32}$$

a semiregular[54] continued fraction expansion of $\sqrt{D}$.

A number of misconceptions continue to circulate concerning the cyclic method. One of these is that it was rediscovered by Lagrange. This, as Selenius has pointed out, is not the case. Lagrange made use of simple continued fractions, which would not necessarily be the same as the semiregular continued fractions implicitly employed by the cyclic method. Often the algorithm is attributed to Bhaskara II, but as mentioned by Shankar Shukla,[55] Bhaskara made no claim to being the originator of the method, and as Jayadeva, who worked in the 10th century or earlier, had discovered a variant of the technique, it seems that it must have been developed much earlier than the time of Bhaskara. Finally, there is the belief, perhaps due to Tannery,[56] that the cyclic method derives from Greek influences. There seems, in spite of Tannery's analysis, to be little solid evidence in support of this. The simple fact is that, as mentioned earlier, we do not really know what the Greeks knew about the Pell equation. What we do know, however, is that the Indian methods display a history of steady development and refinement up to and including the discovery of the cyclic method, and this very strongly suggests that Hankel's[57] position that the Indians evolved the technique by themselves is the correct one.

2.4 Fermat and His Successors

The story of the Pell equation resumes with the challenge[58] issued in 1657 to Frénicle in particular and mathematicians in general by Fermat. Fermat had most likely, through his research, come to recognize the fundamental nature of the Pell equation. He asks for a proof of the following statement:

> Given any [positive] number [D] whatever that is not a square, there are also given an infinite number of squares such that, if the square is multiplied into the given number and unity is added to the product, the result is a square.

It next requests a general rule by which solutions of the problem could be determined and, as examples, asks for solutions when $D = 109, 149, 433$.

The story of how the second part of this challenge was answered by Brouncker and Wallis has been very well told by Weil[59] and Mahoney[60] and needs no elaboration here. Instead, we will content ourselves with giving a somewhat different account from that provided by Weil[61] concerning Brouncker's technique for solving the Pell equation. We emphasize that, although Brouncker's method is equivalent to what we will describe, he did not think about it in quite this way.

Let $P, Q, R \in \mathbb{Z}$, where $Q \neq 0$,

$$P^2 - QR = D > 0 \ ,$$

and D is not an integral square. Put

$$F(X, Y) = QX^2 - 2PXY + RY^2 \tag{2.33}$$

and let ρ and ρ' denote the zeros of $F(x, 1)$. Since D is not a square, we know that $\rho, \rho' \notin \mathbb{Q}$. Brouncker seems to have used the following result, although he provides no proof of it.

Proposition 2.3. *Suppose $\rho > 1$ and $\rho' < 0$. If $F(X, Y) = 1$, where $X, Y \in \mathbb{Z}$ and $X > Y > 1$, then $\lfloor \rho \rfloor < X/Y < \lfloor \rho \rfloor + 1$.*

Proof. Since $F(X, Y) = 1$, we may assume that $X = qY + Z$, where $0 < Z < Y$. Also,

$$|Q||X - \rho' Y||X - \rho Y| = 1 \ . \tag{2.34}$$

Since $\rho' < 0$, we get $|X - \rho' Y| = X - \rho' Y > X > 1$. Also, $X - \rho Y = (q - \rho)Y + Z$; thus, if $q - \rho < -1$, then $X - \rho Y < -Y + Z \leq -1$, and if $q - \rho > 0$, then $X - \rho Y > Z \geq 1$. In either case, $|X - \rho Y| > 1$, which is impossible by (2.34). It follows that $\rho - 1 < q < \rho$ or $q = \lfloor \rho \rfloor$. □

If we substitute $X = qY + Z$ in (2.33), we get

$$F'(Y, Z) = Q'Y^2 - 2P'YZ + R'Z^2 \ ,$$

where $Q' = q^2 Q - 2qP + R$, $P' = P - qQ$, $R' = Q$, and

$$P'^2 - Q'R' = D \,. \tag{2.35}$$

It is easy to show that

$$\frac{P' - \sqrt{D}}{Q'} = \frac{1}{\frac{P+\sqrt{D}}{Q} - q}, \qquad \frac{P' + \sqrt{D}}{Q'} = \frac{1}{\frac{P-\sqrt{D}}{Q} - q} \,.$$

Thus, if τ and τ' are the zeros of $F'(x,1)$, then $\tau = 1/(\rho - q)$, $\tau' = 1/(\rho' - q)$. If $\rho > 1$, $\rho' < 0$, and $q = \lfloor \rho \rfloor$, then $\tau > 1$, $\tau' < 0$.

With these preliminary observations, we can now go on to describe Brouncker's very ingenious technique. We suppose T, U is a solution of $T^2 - DU^2 = 1$ and put $Q_0 = 1$, $P_0 = 0$, $R_0 = -D$, $X_0 = T$, and $X_1 = U$. We have $F_0(X_0, X_1) = Q_0 X_0^2 - 2P_0 X_0 X_1 + R_0 X_1^2 = 1$ and $\rho_0 = \sqrt{D}$, $\rho_0' = -\sqrt{D}$ are the zeros of $F_0(x, 1)$. Putting $q_0 = \lfloor \rho_0 \rfloor$ and substituting $q_0 X_1 + X_2$ for X_0 in $F_0(X_0, X_1)$ we get $F_1(X_1, X_2) = 1$ $(0 < X_2 < X_1)$. Here,

$$Q_1 = q_0^2 Q_0 - 2q_0 P_0 + R_0, \quad P_1 = P_0 - q_0 Q_0, \quad R_1 = Q_0 \,.$$

We put $\rho_1 = 1/(\rho_0 - q_0)$, $q_1 = \lfloor \rho_1 \rfloor$, and $X_1 = q_1 X_2 + X_3$ $(0 < X_3 < X_2)$ and compute $F_2(X_2, X_3)$ $(= 1)$, etc. In fact, if $F_i(X_i, X_{i+1}) = 1$ $(0 < X_{i+1} < X_i)$, we put

$$\rho_i = \frac{1}{\rho_{i-1} - q_{i-1}} \,, \tag{2.36}$$

$q_i = \lfloor \rho_i \rfloor$, and

$$X_i = q_i X_{i+1} + X_{i+2} \tag{2.37}$$

in F_i to obtain $F_{i+1}(X_{i+1}, X_{i+2}) = 1$ with

$$Q_{i+1} = (P_{i+1}^2 - D)/Q_i, \quad P_{i+1} = P_i - q_i Q_i, \quad R_{i+1} = Q_i \,,$$

by (2.35).

As the sequence $\{X_i\}$ is a strictly decreasing (for increasing i) sequence of positive integers, this process must come to a halt with $X_j = 1$, $X_{j+1} = 0$ for some $j \geq 0$. To find T and U, all that is necessary is to proceed backward using (2.37) once all the values of $q_0, q_1, q_2, \ldots, q_{j-1}$ have been determined.

We will now exemplify[62] the process for the case of

$$T^2 - 13U^2 = 1 \,. \tag{2.38}$$

Here,

$$F_0(X_0, X_1) = X_0^2 - 13X_1^2 \,, \qquad q_0 = \left\lfloor \sqrt{13} \right\rfloor = 3 \,;$$

$$F_1(X_1, X_2) = -4X_1^2 + 6X_1X_2 + X_2^2 \,, \qquad q_1 = \left\lfloor \frac{3 + \sqrt{13}}{4} \right\rfloor = 1 \,;$$

$$F_2(X_2, X_3) = 3X_2^2 - 2X_2X_3 - 4X_3^2\,, \qquad q_2 = \left\lfloor \frac{1+\sqrt{13}}{3} \right\rfloor = 1\,;$$

$$F_3(X_3, X_4) = -3X_3^2 + 4X_3X_4 + 3X_4^2\,, \qquad q_3 = \left\lfloor \frac{2+\sqrt{13}}{3} \right\rfloor = 1\,;$$

$$F_4(X_4, X_5) = 4X_4^2 - 2X_4X_5 - 3X_5^2\,, \qquad q_4 = \left\lfloor \frac{1+\sqrt{13}}{4} \right\rfloor = 1\,;$$

$$F_5(X_5, X_6) = -X_5^2 + 6X_5X_6 + 4X_6^2\,, \qquad q_5 = \left\lfloor \frac{3+\sqrt{13}}{1} \right\rfloor = 6\,;$$

$$F_6(X_6, X_7) = 4X_6^2 - 6X_6X_7 - X_7^2\,, \qquad q_6 = \left\lfloor \frac{3+\sqrt{13}}{4} \right\rfloor = 1\,;$$

$$F_7(X_7, X_8) = -3X_7^2 + 2X_7X_8 + 4X_8^2\,, \qquad q_7 = \left\lfloor \frac{1+\sqrt{13}}{3} \right\rfloor = 1\,;$$

$$F_8(X_8, X_9) = 3X_8^2 - 4X_8X_9 - 3X_9^2\,, \qquad q_8 = \left\lfloor \frac{2+\sqrt{13}}{3} \right\rfloor = 1\,;$$

$$F_9(X_9, X_{10}) = -4X_9^2 + 2X_9X_{10} + 3X_{10}^2\,, \qquad q_9 = \left\lfloor \frac{1+\sqrt{13}}{4} \right\rfloor = 1\,;$$

$$F_{10}(X_{10}, X_{11}) = X_{10}^2 - 6X_{10}X_{11} - 4X_{11}^2\,.$$

We observe that $F_{10}(X_{10}, X_{11}) = 1$ can be easily achieved with $X_{10} = 1$ and $X_{11} = 0$. We can now find

$$\begin{array}{lll} X_9 = q_9X_{10} + X_{11} = 1\,, & X_8 = q_8X_9 + X_{10} = 2\,, & X_7 = 3\,, \\ X_6 = 5\,, & X_5 = 33\,, & X_4 = 38\,, \\ X_3 = 71\,, & X_2 = 109\,, & X_1 = 180\,, \\ X_0 = 649\,, & & \end{array}$$

and put $T = 649$, $U = 180$ as a solution of (2.38).

Brouncker used his method to find solutions of several difficult Pell equations, including $x^2 - 433y^2 = 1$. This was a major feat of calculation, as the value of y is a number of 19 digits. However, neither he nor Wallis nor Frénicle was able to provide a proof that the Pell equation could always be solved (non-trivially) for any positive non-square value of D. Fermat[63] took notice of this and stated that he had such a proof "by means of *descente* duly and appropriately applied." Unfortunately, Fermat provided no further information concerning his proof than this. Hofmann[64] and, with greater success, Weil[65] have attempted to reconstruct what Fermat's method might have been. While we may never really know what this was, it is nevertheless very likely that Fermat did have a proof. The fact that he selected 109, 149, and

433 for values of D as challenge examples is particularly suggestive because the corresponding Pell equations have large values of t and u.

The method of Brouncker was modified and extended by Euler, who realized that, as is apparent from (2.36), continued fractions could be used to provide an efficient algorithm for solving the Pell equation. However, even through he had devised all of the important tools, he just fell short of proving that his method would work for any non-square D. As mentioned earlier, the development of such a technique was first done by Lagrange in a rather clumsy work, which he later improved. For further information on this particularly interesting part of mathematical history, the reader is referred to Weil's book. In the next chapter we will describe Lagrange's method of using simple continued fractions to solve the Pell equation.[66]

Notes and References

[1][Kon01].
[2][Whi12].
[3][Dic19], Vol. II, Ch. 12.
[4][Les73], pp. 421–446. A more accessible source for some of this is [Les97], p. 100.
[5][Fra72], Vol. II, p. 587, note 243.
[6][Fra72], Vol. I, p. 409.
[7][Tho80], Vol. II, p. 203–206.
[8]That is, a fifth and sixth of both of the males and of the females.
[9][Arc99] and [Len02].
[10][Kru80], p. 124; [Fra72], Vol. II, p. 588, note 245; [Dij87], p. 398, note 2.
[11][Kru80].
[12][Fra72], Vol. I, pp. 407–408.
[13]See [Dic19], Vol. II, pp. 342–343.
[14]See [Amt80], p. 155ff or [Hea12], p. 320.
[15]For the original Greek version of the scholium and a French translation, see [Arc71], pp. 171–173.
[16]See, for example, [Sch93] and [Wat95].
[17][Wur30]. For a more easily accessible version, see [Hea12], pp. 319–323.
[18][Dij87], p. 399, note 3.
[19][Pau96b], II. 1, p. 534–535; [Hea12], p. xxxv.
[20][Fra72], Vol. I, p. 409.
[21][Dij87], p. 399; [Fra72], Vol. II, p. 590, note 256.
[22][Kno86], p. 295.
[23][Kru80], p. 124. See also [Fra72], Vol. II, p. 590, note 257.
[24][Hom46], Book XII, lines 127–130, p. 192.
[25][Str61], 6.2.1; [Thu92], Book VI, 2.
[26][Amt80].
[27][Bei64], p. 249.
[28][Var98].
[29][Fra72], p. 409.
[30]For a lengthy treatment of this, see [Kno75b], Ch. II. A translation of the relevant work of Theon is in [Fow87], p. 58.
[31][Fow87], pp. 101–102.
[32][Kno75a], p. 137.
[33][Hea12], pp. 91–98.
[34]See [Hea12], 1xxx–xcix; [Kno75a], pp. 136–139.
[35][Kno75b].
[36][Fow87].
[37][Kno75b], pp. 255–261; [Fow87], Ch. 2.
[38][Kno75b], p. 258.
[39][Hea81], Vol. I, pp. 202–212; [vdW54], pp. 165–179.
[40][Fow87], p. 45.
[41][Fow87], Section 2.3(b).
[42][Kno75a], p. 138; [Fow87], pp. 42–44.
[43][Fow87], Ch. 3.
[44][Fow87], p. 50, pp. 54–55.
[45][Tan84].

[46] [Ses82], Part I.
[47] Two useful sources for this material are [DS62] and [Sri67].
[48] [Hea64], p. 281.
[49] [Col17], p. 363.
[50] [Sel63] and [Sel75].
[51] This is never explicitly stated, but it seems to be implicit in the *kuttaka* process that would be used to find P_{n+1}.
[52] [Han65], p. 202.
[53] See [Ayy40].
[54] See §38 of Vol. I of [Per57].
[55] [Shu54], p. 1 and p. 20.
[56] [Tan37], p. 240ff.
[57] [Han65], pp. 203–204.
[58] [Fer12], pp. 333–335. An English version can be found in [Hea64], pp. 285–286.
[59] [Wei84].
[60] [Mah94].
[61] [Wei84], pp. 92–97.
[62] This example of Brouncker's can be found in [Fer12], Vol. III, p. 480. It is also reprinted in [Whi12], pp. 53–55.
[63] [Fer12], p. 433.
[64] [Hof94].
[65] [Wei79]; [Wei84], Section XIII.
[66] For another perspective on this see Edwards [Edw05], pp. 65–112.

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