

Chapter 1

Introducing Optimization

1.1. A tank design problem

In an optimization problem we seek values for certain *design* or *control variables* which minimize (or sometimes maximize) an *objective function*. A good example is the problem of finding the dimensions of a rectangular open-topped tank in order to obtain the smallest surface area which encloses a given volume, V^* . (The purpose of such a design might be to minimize heat loss through the sides.) We denote the height by x_1 and the lengths of the edges of the base by x_2 and x_3 . The volume and surface area are then given by

$$V = x_1x_2x_3 \quad \text{and} \quad S = 2x_1x_2 + 2x_1x_3 + x_2x_3.$$

Hence the design problem can be posed as

$$\text{Minimize } S = 2x_1x_2 + 2x_1x_3 + x_2x_3 \quad \text{subject to } x_1x_2x_3 = V^*. \quad (1.1.1)$$

This is a three-variable optimization problem which includes an *equality constraint*. Methods for solving problems of this kind are discussed in Chapters 16–18; but an alternative *unconstrained* formulation can be obtained by eliminating one of the unknowns. Because $x_3 = V^*x_1^{-1}x_2^{-1}$ we can also seek the optimum tank dimensions by solving

$$\text{Minimize } S = 2x_1x_2 + 2V^*x_2^{-1} + V^*x_1^{-1}. \quad (1.1.2)$$

The solution of problems of this kind is discussed in Chapters 5–11.

The optimal tank dimensions can be found by solving either (1.1.1) or (1.1.2). However an important factor has been omitted from both of them. If any two of the x_i have negative values then the constraint

on volume can still be satisfied but the surface area may be negative. Because a negative value for S is necessarily less than a positive one, a solution with, say, $x_1 < 0$ and $x_2 < 0$ might seem “better” than a solution with all the x_i positive. Of course, negative dimensions have no practical meaning and so the problem formulation should explicitly exclude them. We can do this by adding *inequality constraints*, as in

$$\text{Minimize } 2x_1x_2 + 2x_1x_3 + x_2x_3 \text{ s.t. } x_1x_2x_3 = V^*, \quad x_i \geq 0, \quad i = 1, 2, 3. \quad (1.1.3)$$

or

$$\text{Minimize } 2x_1x_2 + 2V^*x_2^{-1} + V^*x_1^{-1} \text{ s.t. } x_i \geq 0, \quad i = 1, 2. \quad (1.1.4)$$

(The abbreviation “s.t.” is often used instead of “subject to”.) Methods for dealing with problems such as (1.1.3) and (1.1.4) are considered in Chapters 20–23.

In this chapter and the next we restrict ourselves to unconstrained problems involving only one variable. We can obtain such a problem from the tank design example by adding an extra requirement that the base must be square; that is, $x_2 = x_3$. Now the expressions for volume and surface area become

$$V = x_1x_2^2 \quad \text{and} \quad S = 4x_1x_2 + x_2^2.$$

Using the constraint on V to eliminate x_1 , we get S in terms of x_2 only; that is,

$$S = 4V^*x_2^{-1} + x_2^2. \quad (1.1.5)$$

Figure 1.1 shows S as a function of x_2 when $V^* = 5$. In this case the minimum occurs when $x_2 \approx 2.2$.

Figure 1.1 illustrates the well-known fact that, at the minimum of a *differentiable* function, the slope – that is, the first derivative – is zero. Hence, for this rather simple problem, we can obtain the minimum surface area by solving

$$\frac{dS}{dx_2} = -4V^*x_2^{-2} + 2x_2 = 0$$

which gives $x_2 = (2V^*)^{1/3}$. Hence, when $V^* = 5$, the optimum square base has edges of length 2.1544.

Not all optimization problems are as easy as the minimization of (1.1.5). Some objective functions are hard to differentiate; and, even when the first derivative has been found, the equation obtained by setting it to zero may be difficult to solve. This book describes some of the computational methods used by engineers and scientists to deal with

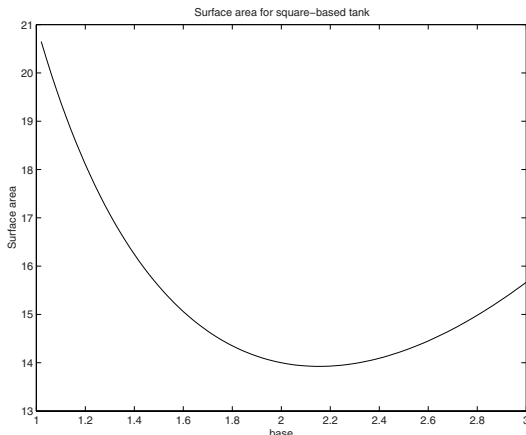


Figure 1.1. Tank surface area as a function of x_2 .

optimization problems which do not have an analytical solution. Such problems occur in many situations, for example, finding a formula which gives the closest match to some experimental data, choosing the shortest route which avoids a number of obstacles, or devising a maintenance schedule which gives the least operating cost. Such case studies are used later in the book as a basis for the practical comparison of different optimization methods.

Exercises

1. What happens to the surface area (1.1.5) as $x_2 \rightarrow 0$? What is the minimum value of S if x_2 lies in the range $-1 < x_2 < 0$?
2. If $x_2 = x_3$, reformulate (1.1.2) as an unconstrained minimization problem involving x_1 only. Using the value $V^* = 5$, plot a graph of the objective function in the range $1 \leq x_1 \leq 3$. Hence deduce the minimum surface area. What happens to the surface area as $x_1 \rightarrow 0$?
3. Formulate the problem of finding the maximum volume that can be enclosed by a rectangular open tank with a fixed surface area and then estimate a solution when the base of the tank is square and the fixed surface area is 8. (Note that maximizing a function $F(x)$ is equivalent to minimizing $-F(x)$.)

1.2. Least squares data-fitting

Suppose that a laboratory experiment produces a record of measured temperatures, θ , ($^{\circ}\text{C}$) against time t (minutes), as in Table 1.1. Suppose also that we believe the underlying relationship between θ and t is linear, of the form $\theta = at$, for some unknown coefficient a . The data points

Measurement i	1	2	3	4
Time t_i	1.0	2.0	3.0	4.0
Temperature θ_i	2.3	5.1	7.2	9.5

Table 1.1. Experimental data for temperature versus time.

do not, in fact, lie on a straight line (perhaps because of experimental errors). Hence, out of all the straight lines which pass near the data points, we wish to find the one which gives the best approximation, in the sense that the discrepancies between the data and the straight line *model* are as small as possible.

Figure 1.2 shows the errors $at_i - \theta_i$ as vertical lines PP' .

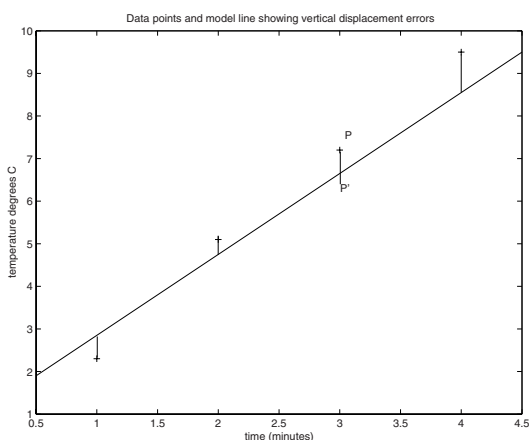


Figure 1.2. Data points and model line showing vertical errors.

A common way to find the best approximation is to choose a to minimize the sum of squares of these vertical errors; that is, we want a to solve the problem

$$\text{Minimize } F(a) = \sum_{i=1}^4 (at_i - \theta_i)^2. \quad (1.2.1)$$

At a minimum of F , the first derivative $F'(a)$ is zero. Hence the optimum value of a satisfies

$$\frac{dF}{da} = 2 \sum_{i=1}^4 (at_i - \theta_i)t_i = 0. \quad (1.2.2)$$

This leads to

$$a \sum_{i=1}^4 t_i^2 = \sum_{i=1}^4 \theta_i t_i.$$

Substituting for t_i and θ_i from Table 1.1 we get $30a = 72.1$ and so $a \approx 2.4033$.

This simple problem is an example of the *least squares* approach to approximating a set of data points by a model function. The approach can be extended (as shown in later chapters) to models with more than one unknown coefficient.

The data-fitting problem we have just solved is extremely easy because F is a quadratic function of the variable a . This means that equation (1.2.2) is linear and yields a unique answer. We now show that some optimization problems are not so straightforward by considering another way to minimize discrepancies between the data and the model. Rather than dealing with just the vertical error at a data point, we take account of the *total displacement* given by the perpendicular distance of (t_i, θ_i) from the line $\theta = at$ as shown in Figure 1.3.

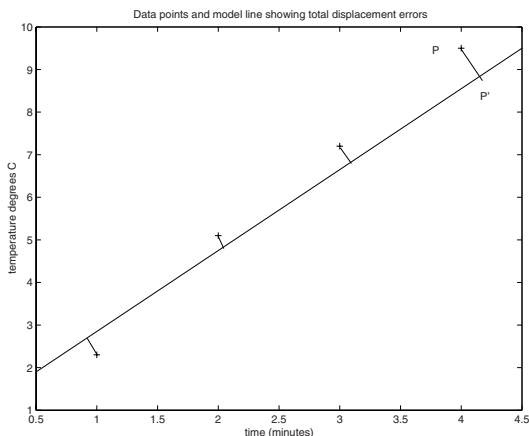


Figure 1.3. Data points and model line showing total displacement errors.

We can determine the perpendicular distance between point and line as follows. A typical point, P' , on the line $\theta = at$ has coordinates (t, at) and the slope of the line joining P' to the data point P with coordinates (t_i, θ_i) is

$$m = \frac{at - \theta_i}{t - t_i}.$$

We want to find the value of t which makes $m = -a^{-1}$ because the line PP' will then be perpendicular to $\theta = at$ and P' will be the model point which is closest to the data point P . The value of t at the *footpoint* P' is found by solving

$$-a^{-1}(t - t_i) = at - \theta_i.$$

Hence P' is defined by

$$t = \tau_i = \frac{a^{-1}t_i + \theta_i}{a + a^{-1}} = \frac{t_i + a\theta_i}{a^2 + 1}. \quad (1.2.3)$$

The total displacement PP' is then $\sqrt{(t_i - \tau_i)^2 + (\theta_i - a\tau_i)^2}$ and to get the optimum straight line $\theta = at$ we must find a by solving

$$\text{Minimize } \hat{F}(a) = \sum_{i=1}^8 \phi_i^2 \quad (1.2.4)$$

where $\phi_i = (t_i - \tau_i)$ and $\phi_{i+4} = (\theta_i - a\tau_i)$ for $i = 1, \dots, 4$. (1.2.5)

Of course, τ_1, \dots, τ_4 and ϕ_1, \dots, ϕ_8 are functions of a . If we substitute the known values of t_i and θ_i we see that (1.2.4) is a more complicated expression than the corresponding function (1.2.1) in the vertical least-squares problem. From (1.2.3) we get

$$\tau_1 = \frac{1 + 2.3a}{a^2 + 1}; \quad \tau_2 = \frac{2 + 5.1a}{a^2 + 1}; \quad \tau_3 = \frac{3 + 7.2a}{a^2 + 1}; \quad \tau_4 = \frac{4 + 9.5a}{a^2 + 1}.$$

Hence

$$\phi_1 = \left(1 - \frac{1 + 2.3a}{a^2 + 1}\right), \quad \phi_5 = \left(2.3 - \frac{a + 2.3a^2}{a^2 + 1}\right)$$

with similar expressions for the remaining ϕ_i .

It is now clear that the function (1.2.4) is not quadratic and its first derivative is not linear. Hence, forming and solving the equation $\hat{F}'(a) = 0$ is more difficult than for the vertical least-squares problem. In practice, we would normally minimize a function such as $\hat{F}(a)$ by using iterative methods of the kind described in the next chapter. More information about total least squares and the footpoint problem is given in [27].

We can, of course, estimate the minimum of $\hat{F}(a)$ by plotting a graph, as shown in Figure 1.4. In this case, the best straight line approximation in the *total* least squares sense is very similar to the approximation based on vertical least squares with slope $a \approx 2.4$.

Exercises

1. Using vertical displacements, find the straight line $y = mx$ to give a least-squares approximation to the data points (3, 7), (4, 8), (6, 11).
2. Show that the footpoint P' could have been found by putting $\tau_i = t_f$ where $t = t_f$ solves the problem

$$\text{Minimize } (t_i - t)^2 + (\theta_i - at)^2.$$

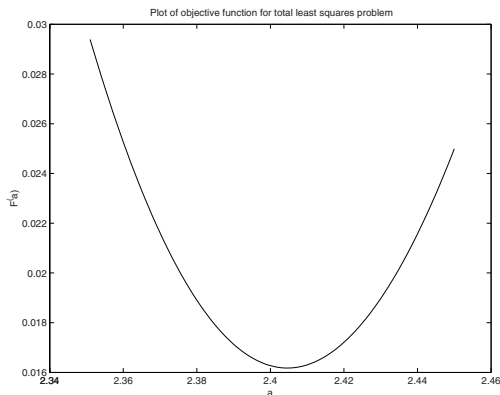


Figure 1.4. Plot of $\hat{F}(a)$ the total least squares error function.

3. Use the data in the worked example to find expressions for ϕ_2, \dots, ϕ_4 and ϕ_6, \dots, ϕ_8 ; hence complete the expression for $\hat{F}(a)$ in (1.2.4) and obtain the expression for $\hat{F}'(a)$. Plot a graph to estimate the solution of $\hat{F}'(a) = 0$.
4. Consider the data in Table 1.1 and suppose θ_4 is changed to 14.2. Calculate a model line $\theta = at$ using both vertical and total least squares. Comment on the difference between the two solutions.

1.3. A routing problem

Suppose a robot vehicle starts at the origin and is required to proceed to a point P , as shown in Figure 1.5. It must move initially along the x -axis and then turn towards P at some point Q . The circle represents a “no-go” area which the vehicle must avoid. The point Q is to be chosen to minimize a combination of the total distance travelled and the length of the route that lies within the circle.

If the line from Q to P cuts the circle at R and S then we can define the optimum route as the one which minimizes

$$F = \text{distance } OQ + \text{distance } QP + \rho(\text{distance } RS)$$

where ρ is a positive constant. This form of function penalizes the portion of the route inside the no-go region. If ρ is large we expect little or none of the optimum route to pass through the circle. On the other hand, as $\rho \rightarrow 0$, the optimum route will come closer to the straight line OP .

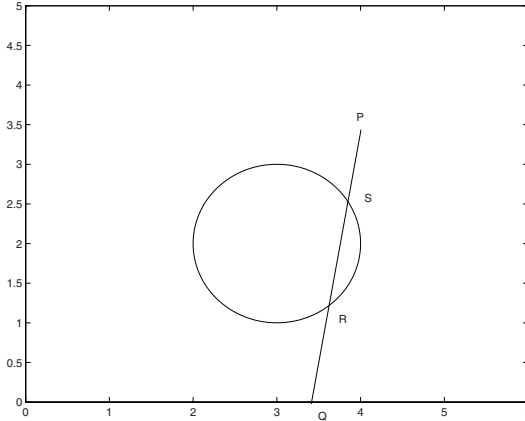


Figure 1.5. A routing problem.

If P has coordinates (x_p, y_p) and if x is the (unknown) distance OQ then the total length of the route is

$$d(x) = x + \sqrt{(x - x_p)^2 + y_p^2}. \quad (1.3.1)$$

(We are assuming that P and the circle are in the positive quadrant and hence that x is positive.) We now need to determine the points of intersection (if any) of line QP and the circle. We assume that P is outside the no-go area and also that the circle does not cut the x -axis; it follows that the line segment QP will either cut the circle twice or not at all. The coordinates of any point on the line between Q and P are

$$(x + \lambda(x_p - x), \lambda y_p) \quad \text{where} \quad 0 \leq \lambda \leq 1.$$

If the no-go area has centre (x_c, y_c) and radius r then points of intersection with QP occur when λ satisfies

$$(x + \lambda(x_p - x) - x_c)^2 + (\lambda y_p - y_c)^2 - r^2 = 0.$$

This simplifies to $\alpha\lambda^2 + \beta\lambda + \gamma = 0$, where the coefficients are given by

$$\alpha = (x - x_p)^2 + y_p^2, \quad \beta = 2[(x_p - x)(x - x_c) - y_p y_c] \quad (1.3.2)$$

$$\text{and} \quad \gamma = (x - x_c)^2 + y_c^2 - r^2. \quad (1.3.3)$$

We let $\delta = \beta^2 - 4\alpha\gamma$. If $\delta \leq 0$ there are no points of intersection with the circle and so the distance RS is zero. On the other hand, if $\delta > 0$ the intersection points are given by

$$\lambda_1 = \frac{-\beta + \sqrt{\delta}}{2\alpha}, \quad \lambda_2 = \frac{-\beta - \sqrt{\delta}}{2\alpha}. \quad (1.3.4)$$

The distance RS is then given by $|\lambda_1 - \lambda_2| \times (\text{distance } QP)$. Hence the optimum route is obtained by minimizing

$$F(x) = d(x) + \rho v(x) \quad (1.3.5)$$

where $d(x)$ is given by (1.3.1) and

$$v(x) = |\lambda_1 - \lambda_2| \sqrt{(x - x_p)^2 + y_p^2}. \quad (1.3.6)$$

Note that α, β and γ are functions of x because of (1.3.2) and (1.3.3). Hence (1.3.4) implies that λ_1 and λ_2 also depend on x . It is possible – but not trivial – to differentiate $F(x)$ but it will not be possible to find an analytical solution to the equation $F'(x) = 0$.

If we take the target point (x_p, y_p) as $(5, 4)$ and define the no-go region by $x_c = y_c = 2$, $r = 1$ then we can plot the function (1.3.5), as shown in Figure 1.6. It is clear that the optimum value of x is about 1.62. The solution path leaves the x -axis at a point Q such that PQ is a tangent to the circular boundary of the no-go region.

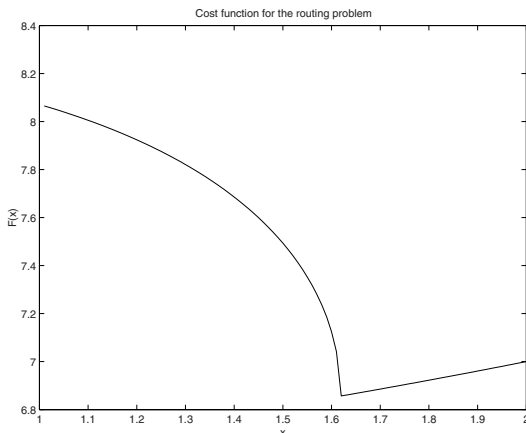


Figure 1.6. Nonsmooth cost function for the routing problem.

Figure 1.6 shows that the minimum corresponds to a “kink” in $F(x)$; that is, the slope of (1.3.5) is not zero at the optimum but instead has a discontinuity. This is due to the presence of the square root in (1.3.6). The function (1.3.5) is said to be *nonsmooth*.

Most of the optimization methods described in this book are intended for use with smooth (i.e., continuously differentiable) functions. We can formulate the routing problem in terms of a function which is smooth if we choose to minimize

$$\Phi(x) = d(x) + \rho v(x)^3 \quad (1.3.7)$$

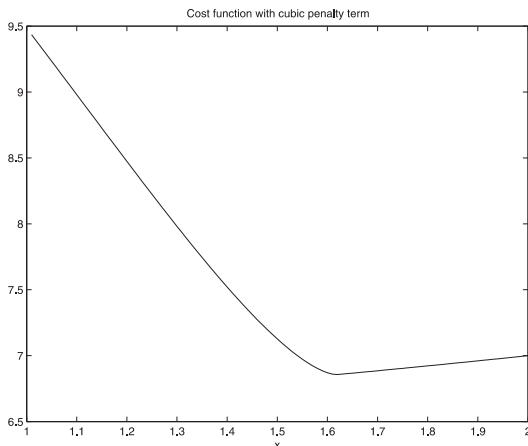


Figure 1.7. Smooth cost function for the routing problem.

whose graph is given in Figure 1.7 (for $\rho = 1$). The minimum of (1.3.7) occurs at approximately the same place as that of the nonsmooth function (1.3.5). The use of functions such as (1.3.7) in some real-life routing problems is described in [65, 9].

Exercises

1. Calculate expressions for the first derivatives of (1.3.1) and (1.3.6).
2. Using the sample data $x_p = 5$, $y_p = 4$, $x_c = y_c = 2$, $r = 1$, plot graphs to determine the minima of (1.3.5) and (1.3.7) when $\rho = 0.5, 0.05$ and 0.005 . Comment on any differences you observe.
3. Use the data $x_p = 4$, $y_p = 8$, $x_c = 4$, $y_c = 2$, $r = 2$ to plot graphs of (1.3.5) and (1.3.7) with $\rho = 1$ in the range $0 \leq x \leq 10$. Comment on what you observe.



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