Preface

The goal of this text is to provide a self-contained guide to Monte Carlo and quasi–Monte Carlo sampling methods. These two classes of methods are based on the idea of using sampling to study mathematical problems for which analytical solutions are unavailable. More precisely, the idea is to create samples that can be used to derive approximations about a quantity of interest and its probability distribution. In the former case, random sampling is used, while in the latter, *low-discrepancy sampling* is used.

Quasi–Monte Carlo sampling methods are typically used to provide approximations for multivariate integration problems defined over the unit hypercube. They do so by creating sets or sequences of vectors $(u_1, \ldots, u_s)$, with each u_j taking values between 0 and 1, that sample the s-dimensional unit hypercube more regularly than random samples do, hence mimicking in a better way — with less *discrepancy* — the uniform distribution over that space. For this reason, most of the theory that underlies these constructions has been developed for problems that can be described as integration problems over the s-dimensional unit hypercube.

On the other hand, random sampling — via the use of Monte Carlo methods — has been developed and used in a variety of situations that do not necessarily fit the formulation above, which makes use of a function defined over the unit hypercube. In particular, stochastic simulation models are usually constructed using random variables defined over the real numbers, the nonnegative integers, or other domains that are not necessarily the unit interval between 0 and 1. However, the computer implementation of such models always relies, at its lowest level, on a source of (pseudo)random numbers that are uniformly distributed between 0 and 1. Therefore, at least in principle, it is always possible to reformulate a simulation model using a vector of input variables defined over the s-dimensional unit hypercube.

Being able to perform this "translation" — between the more intuitive simulation formulation and the one viewing the simulation program as a function f transforming input numbers $u_1, \ldots, u_s$ into an observation of the output quantity of interest — is extremely important when we want to successfully replace random sampling by quasi-random sampling in such problems. For this reason, we will be discussing this translation throughout the book, referring to it as the "integration versus simulation" formulation, with the understanding that by "integration" we mean the formulation of the problem using a function defined over the unit hypercube.

Because integration is the main area for which quasi-random sampling has been used so far, a large part of this text is devoted to this topic. In addition, simulation studies are often designed to estimate the mathematical expectation of some quantity of interest. In such cases, the translation of this goal into the formulation that uses a function f, as described in the preceding paragraph, means we wish to estimate the integral of that function. Hence these problems also fit within the integration framework.

A number of books have been written on the Monte Carlo method and its applications (especially in finance) [120, 121, 137, 145, 165, 211, 236, 293, 314, 386, 391, 418, 424], stochastic simulation [45, 175, 217, 218, 243, 389], and quasi–Monte Carlo methods [128, 308, 339, 441]. The purpose of this text is to present all these topics together in one place in a unified way, using the "integration versus simulation" formulation to help tie everything together. After reading this book, the reader should be able to apply random sampling to a wide range of problems and understand how to correctly replace it by quasi-random sampling. The selection of topics has been done in that perspective, and I certainly do not claim to be covering all aspects of Monte Carlo and quasi–Monte Carlo methods or surveying all possible applications for which these methods have been used. A very good source of information that contains the most recent advances in this field is the biannual *Monte Carlo and Quasi–Monte Carlo Methods* conference proceedings by Springer.

This book is organized as follows. The first chapter introduces the Monte Carlo method as a tool for multivariate integration and describes the integration versus simulation formulation using several examples. The more general use of Monte Carlo as a way to approximate a distribution is also studied. The second chapter gives an overview of different methods that can be used to generate random variates from a given probability distribution, a task that needs to be done extensively in any simulation study. This material comes early in the text because of its relevance in understanding the integration versus simulation formulation. Chapter 3 contains information on random number generators, which are essential for using random sampling on a computer. Methods for improving the efficiency of the Monte Carlo method that fall under the umbrella of *variance reduction techniques* are discussed in Chapter 4. A description of quasi–Monte Carlo constructions and the quality measures that can be used to assess them is done in Chapter 5. Several connections with random number generators are done in that chapter, which is the reason why their presentation precedes our discussion of quasi–Monte Carlo methods. Chapter 6 discusses the use of quasi–Monte Carlo methods in practice, including randomized quasi–Monte Carlo and ANOVA decompositions. The last two chapters are devoted to applications, with Chapter 7 focused on financial problems and Chapter 8 discussing more complex problems than those typically tackled by quasi–Monte Carlo methods.

This text can be used for a graduate course on Monte Carlo and quasi–Monte Carlo methods aimed either at statistics, applied mathematics, computer science, engineering, or operations research students. It may also be useful to researchers and practitioners familiar with Monte Carlo methods who want to learn about quasi–Monte Carlo methods.

The level of this text should be accessible to graduate students with varied backgrounds, as long as they have a basic knowledge of probability and statistics. There is an appendix at the end explaining a few key concepts in algebra required to understand some of the quasi–Monte Carlo constructions. Problem sets are provided at the end of each chapter to help the reader put in practice the different concepts discussed in the text.

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